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SOME PROPERTIES OF THE FIXED POINTS SET FOR MULTIFUNCTIONS

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Many fixed point theorems can be extended for multifunctions, elementary examples showing that in this case the unicity part is lost. We study the transfer of some properties of the sets that are values of a multifunction to the set of fixed points.

Let (X,d) be a metric space and $S(X) = \{A \in \mathcal{P}(X) \mid A \neq \varnothing,\ A = \overline{A}\}$. We denote by H the Hausdorff metric on $S(X)$, and for $x \in X$, $D(x,A) = \inf_{y\in A} d(x,y)$.

Definition 1. The multifunction $f : X \to S(X)$ is *contractive* if

$$H(f(x), f(y)) < d(x,y),\ \forall x,y \in X,\ x \neq y.$$

Definition 2. Let $\varphi : \mathbb{R}^5_+ \to \mathbb{R}_+$ be a function satisfying the following properties:

a) $\varphi(r) \leq \varphi(s)$, for $r \leq s,\ r,\ s \in R^5_+$

b) φ is continuous

c) $\varphi(r,r,r,r,r,) < r$, for $r > 0$

d) $r - \varphi(r,r,r,r,r,) \to +\infty$, for $r \to +\infty$.

The multifunction $f : X \to S(X)$ is a *φ-contraction* if

$$H(f(x)f(y)) \leq \varphi(d(x,y), D(x,f(x)), D(y,f(y)), D(x,f(y)), D(y,f(x))),$$
$$\forall x,y \in X.$$

We consider the case of the multifunctions defined on the real axis $\mathbb{R}$ or on some subsets of $\mathbb{R}$.

Theorem 1 *Let $f : [a,b] \to \mathcal{P}([a,b])$ be a contractive multifunction with $f(x)$ non-void, compact and convex, $\forall x \in [a,b]$. Then the fixed points set is non-void, compact and convex.*

Proof. The only non-void, compact and convex sets in $\mathbb{R}$ being the closed intervals, we have $f(x) = [m(x), M(x)]$, where m and M are functions which are defined on $[a,b]$ with values in the same interval. We obtain $H(f(x), f(y)) =$

$\max\{|M(x) - M(y)|\,,\, |m(X) - m(y)|\}$. The multifunction f being contractive, we have for $x \neq y$

$$|M(x) - M(y)| < |x - y|$$
$$|m(x) - m(y)| < |x - y|\,.$$

We can apply the theorem of Edelstein [1], following which the functions m and M have a unique fixed point. *Let* x_m and x_M be the fixed points, which are also fixed points for f. It is easy to show that $x_m \leq x_M$ and that on the left of x_m and on the right of x_M there are not other fixed points. Any point x in $]x_m,\, x_M[$ is also in $[m(x), M(x)]$, so it is a fixed point.

We have proved that the fixed points set is $[x_m, x_M]$, so it is non-void, compact and convex. ■

Remark. The statement of Theorem 1 is not true for $f : \mathbb{R} \to S(\mathbb{R})$, as the following example shows.

Let $g : \mathbb{R} \to \mathbb{R}$ be a function defined by $g(x) = \begin{cases} \frac{5}{2}, \text{ for } x \in]-\infty, 2[\\ x + \frac{1}{x}, \text{ for } x \in]2, +\infty[\end{cases}$ and $f : R \to S(\mathbb{R})$, $f(x) = [0, g(x)]$.

f satisfies the condition

$$H(f(x), f(y)) < |x - y|\,,\ for\ x \neq y,$$

so it is contractive. It also has as values non-void, compact and convex sets, but the fixed points set is $[0, +\infty[$ and it is not compact.

Theorem 2 *Let $\varphi : \mathbb{R}_+^5 \to \mathbb{R}_+$ be a function as in Definition 2, and $f : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ a φ-contraction with $f(x)$ non-void, compact and convex, $\forall x \in \mathbb{R}$. In this case the fixed points set is non-void, compact and convex.*

Proof. We have $f(x) = [m(x), M(x)]$ and $H(f(x), f(x, y)) = \max\{|m(x) - m(y)|\,,$ $|M(x) - M(y)|\}$. Because f is a φ-contraction, and φ satisfies the condition a), it follows that

$$|m(x) - m(y)| \leq \varphi(d(x, y), D(x, f(x)), D(y, f(y)), D(x, f(y)), D(y, f(x))) \leq \varphi(d(x, y), d(x, m(x)), d(y, m(y)), d(x, m(y)), d(y, m(x))).$$

We obtain similarly

$$|M(x) - M(y)| \leq \varphi(d(x, y), d(x, M(x)), d(y, M(y)), d(x, M(y)), d(y, M(x))).$$

By theorem 1 in [2] it follows that $m : \mathbb{R} \to \mathbb{R}$ has a unique fixed point x_m and $M : \mathbb{R} \to \mathbb{R}$ a unique fixed point x_M. There are not other fixed points on the left on x_m and on the right of x_M; any x in $[x_m, x_M]$ is a fixed point, so the fixed points set is compact and convex.

If we consider some other metric spaces, the properties mentioned above may not be true. In [3] there are given examples of multifunctions in $\mathbb{R}^2$ having as values convex and compact sets, whose fixed points set is however not convex; those multifunctions satisfy conditions of the type $H(f(x), f(y)) \leq kd(x, y)$, with $0 < k < 1$, so they are a special case of the multifunctions considered here. ■

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References

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[2] Rus, I. A., *Some metrical fixed point theorems*, Studia, Univ. Babeş-Bolyai **24**, 1 (1979), 73-77.

[3] Schirmer, H., *Properties of the fixed point set of contractive multifunction*, Canad. Math. Bull.**13**, 2 (1970), 169-173.

UNELE PROPRIETĂŢI ALE MULŢIMII PUNCTELOR FIXE
PENTRU APLICAŢII MULTIVOCE
(Rezumat)

În lucrare se studiază cazul în care proprietăţi ale imaginilor unei aplicaţii multivoce (convexitatea şi compactitatea) se transmit la mulţimea punctelor fixe. Rezultatele le generalizează pe cele obţinute de H. Schirmer pentru contracţii, aplicaţiile considerate aici fiind φ-contracţii sau contractive, definite pe $\mathbb{R}$, respectiv pe intervale compacte.