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POINT-TO-SET MAPPINGS. CONTINUITY

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INTRODUCTION

This paper contains some continuity properties for the point-to-set mappings; there is not yet a general theory of continuous point-to-set mappings, but there are different definitions, given by many authors for specific purposes. The definitions are not generally equivalent and it is important to know in what cases they mean the same thing.

The continuity for point-to-set mappings was defined by generalizing some equivalent definitions of the continuity of functions; thus there were obtained various definitions for the notions of semicontinuity. We list here the definition for lower semicontinuity, upper semicontinuity and upper semicompactness and the relations between them. One obtains different definitions of continuity combining two types of semicontinuity.

Another way to define continuity for a point-to-set mapping is to reduce the problem to the continuity of functions. This demands to introduce an adequate topology on a family of subsets of the range; a point-to-set mapping can be regarded as a function having values in that family of subsets, any subset being now a point in the new topological space.

The first chapter of the paper contains a background in topology; there are listed some topological properties necessary in the following sections. Chapter II includes the definition of the point-to-set mappings and their algebraic properties. Chapter III presents the notions of semicontinuity and the relations between them. We give there a list of examples. Chapter IV is dedicated to the study of the continuity of the point-to-set mappings. It contains various ways to topologize some families of subsets of the range to obtain suitable definitions for the continuity of the point-to-set mappings regarded as functions. At the end of this chapter there are two examples showing the natural way in which point-to-set mappings arise in mathematical programming and optimal control.

To make the difference between a function defined on the set X with values in Y and a point-to-set mapping, we use for functions the notation $f : X \to Y$ and for the point-to-set mappings $F : X \multimap Y$.

The paper refers only to the basic continuity properties for point-to-set mappings and the relations between then. The study could be extended to other properties of the point-to-set mappings.

CONTENTS

CHAPTER I

ELEMENTS OF TOPOLOGY

1 Topological Spaces. Definitions

Let X a set and $\mathcal{P}(X) = \{A |\, A \subset X\}$ the family of the subsets of X.

Definition 1.1 *A* topological structure *(or a* topology*) on the set X is a system $\mathcal{T} \subset \mathcal{P}(X)$ which satisfies the following conditions*

$(T1)$ *$\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$;*

$(T2)$ *any union of elements in $\mathcal{T}$ is an element in $\mathcal{T}$;*

$(T3)$ *any finite intersection of elements in $\mathcal{T}$ is an element in $\mathcal{T}$.*

Definition 1.2 *The pair $(X, \mathcal{T})$ formed of a set X and a topology $\mathcal{T}$ on X is called a* topological space*. The elements of the set X are called* points *of the topological space, and the elements of $\mathcal{T}$ are called* open sets*.*

Example 1.1 *Let $\mathbb{R}^n = \{(x_1, ..., x_n) \,|\, x_i \in \mathbb{R}, i = \overline{1, n}\}$ and $\mathcal{T}$ the system formed of $\emptyset$ and the subsets G of $\mathbb{R}^n$ that satisfy the following condition:*

for any $x \in \mathbb{R}^n$, there is $I = (a_1, b_1) \times ... \times (a_n, b_n)$, with (a_i, b_i) open intervals, $i = \overline{1, n}$ and $x \in I \subset G$.

Then $(\mathbb{R}^n, \mathcal{T})$ is a topological space and $\mathcal{T}$ is the usual (natural) topology on $\mathbb{R}^n$. From now on $\mathbb{R}^n$ will be considered with the usual topology, unless there is another specification.

Example 1.2 *For $X \neq \emptyset$, $\mathcal{T} = \{\emptyset, X\}$ satisfies the conditions $(T1) - (T3)$, so $\mathcal{T}$ is a topology on X; it is called the indiscrete topology on X.*

Example 1.3 *For $X \neq \emptyset$, $\mathcal{T} = \mathcal{P}(X)$ satisfies the conditions $(T1) - (T3)$; the topology is called discrete.*

Example 1.4 *For $X \neq \emptyset$, $\mathcal{T} = \{A \subset X \,|\, \complement A \text{ finite}\} \cup \{\emptyset\}$ is a topology named the topology of the finite complements.*

Definition 1.3 *In the topological space $(X,\ \mathcal{T})$ a neighbourhood of the point $x \in X$ is any subset V of X which includes an open set containing x.*

Example 1.5 a) *X is a neighbourhood for all its points.*

b) *In* $(\mathbb{R}^n, \mathcal{T})$ *a set* $V \subset R^n$ *is a neighbourhood of* x *iff there is an* n *- dimensional open interval* I *such that* $x \in I \subset V$.

The following theorems give a characterization of the open sets and some important properties of the system $\mathcal{V}(x)$ of all the neighbourhoods of x.

Theorem 1.1 *A set* M *is open in the topological space* $(X, \mathcal{T})$ *iff it is a neighbourhood of any* $x \in M$.

Theorem 1.2 *For any point* $x \in X$ *the system* $\mathcal{V}(x)$ *of all the neighbourhoods of* x *has the following properties*

$(N1)$ $\forall V \in \mathcal{V}(x),\ x \in V$

$(N2)$ $V \in \mathcal{V}(x),\ V \subset U \Rightarrow U \in \mathcal{V}(x)$

$(N3)$ $\forall n \in N,\ \forall V_1, ..., V_n \in \mathcal{V}(x) \Rightarrow \cap_{i=1}^{n} V_i \in \mathcal{V}(x)$

$(N4)$ $\forall V \in \mathcal{V}(x), \exists U \in \mathcal{V}(x)$ *such that* $\forall y \in U,\ V \in \mathcal{V}(y)$.

Definition 1.4 *A set of neighbourhoods* $\mathcal{V}'(x)$ *of the point* x *having the property that for any* $V \in \mathcal{V}(x)$ *there is a* $V' \in \mathcal{V}'(x)$ *such that* $V' \subset V$ *is called a* fundamental system of neighbourhoods of the point x.

Example 1.6 *In* $(\mathbb{R}^n, \mathcal{T})$ *the system of all the open* n*-dimensional intervals which contain the point* x *is a fundamental system of neighbourhoods of* x.

Definition 1.5 *A set* $F \subset X$ *is called a* closed set *if its complement* $\complement F$ *is an open set.*

One can easily prove

Theorem 1.3 *The closed sets of a topological space* $(X, \mathcal{T})$ *have the following properties*

$(C1)$ X *and* $\emptyset$ *are closed sets*

$(C2)$ *any intersection of closed sets is a closed set*

$(C3)$ *any finite union of closed sets is a closed set.*

Definition 1.6 *Let* $M \subset X$, X *being endowed with the topology* $\mathcal{T}$. *A point* $x \in X$ *is called an* adherent point *of the set* M *if for any* $V \in \mathcal{V}(x)$, $M \cap V \neq \emptyset$.

The set of the adherent points of M *is called the* adherence (closure) *of the set* M*; it is denoted by* $\overline{M}$.

We have the following theorem

Theorem 1.4 *A set $M \subset X$ is closed iff $M = \overline{M}$.*

Definition 1.7 *Let $X \subset M$, X being endowed with the topology $\mathcal{T}$. A point $x \in X$ is called a* cluster (accumulation) point *of the set M if for any $V \in \mathcal{V}(x)$ we have $V \cap (M \setminus \{x\}) \neq \emptyset$.*

The set of the accumulation points of the set M is denoted by M' and it is called the cluster (derived) set *of M.*

Remark 1.1 *The following equality is true: $\overline{M} = M \cup M'$.*

Theorem 1.5 *A set $M \subset X$ is closed iff it contains all its cluster points.*

Definition 1.8 *A point $x \in M$ is called an* interior point *of the set M if M is a neighbourhood of x.*

The set of the interior points of the set M is denoted by $\operatorname{Int} M$ *and it is called the* interior *of the set M.*

Theorem 1.6 *A set M is open iff* $\operatorname{Int} M = M$.

Definition 1.9 *A set $M \subset X$ is called* dense *in X if $\overline{M} = X$.*

Example 1.7 *The set of all rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$ endowed with the usual topology.*

2 Basis. Subbasis. The Axiom of Countability

In this section $X = (X, \mathcal{T})$ denotes a topological space, unless there is another specification.

Definition 2.1 *A family $\mathcal{B} \subset \mathcal{T}$ is called a* basis *for the topology $\mathcal{T}$ if any open set $G \in \mathcal{T}$ is a union of elements from $\mathcal{B}$.*

Remark 2.1 *We consider that $\bigcup\limits_{i \in \emptyset} B_i = \emptyset$.*

Theorem 2.1 *Let X be a topological space, $\mathcal{B}$ a basis for the topology on X and $u(\mathcal{B})$ the family of unions of elements from $\mathcal{B}$. Then $\mathcal{T} = u(\mathcal{B})$.*

We have the following characterization of a basis.

Theorem 2.2 *Let X be a set; $\mathcal{B} \subset \mathcal{P}(X)$ is a basis for a topology on X iff* *a)* $X = \bigcup_{B \in \mathcal{B}} B$

b) for any B_1*,* $B_2 \in \mathcal{B}$*,* $B_1 \cap B_2$ *is a union of elements from* $\mathcal{B}$*.*

Proof.

1. Let $\mathcal{B}$ be a basis. X being an open set, $X = \bigcup_{B \in \mathcal{B}} B$. For B_1, $B_2 \in \mathcal{B} \subset \mathcal{T}$, $B_1 \cap B_2 \in \mathcal{T}$, so $B_1 \cap B_2$ is a union of elements from $\mathcal{B}$.

2. Let $\mathcal{B} \subset \mathcal{P}(X)$ having the properties $a)$ and $b)$. We show that $u(\mathcal{B})$ is a topology.

We have obviously $\emptyset, X \in u(\mathcal{B})$.

If G_1 and G_2 are in $u(\mathcal{B})$, $G_1 = \bigcup_{i \in I} B_i$ and $G_2 = \bigcup_{j \in J} B_j$, hence $G_1 \cap G_2 = \bigcup_{i \in I} \bigcup_{j \in J} (B_i \cap B_j)$; by the condition $b)$ it following that $G_1 \cap G_2 \in u(\mathcal{B})$.

We have proved that $u(\mathcal{B})$ is a topology, $\mathcal{B}$ being a basis for it. If $(X, \mathcal{T})$ is a topological space, the theorem 1.1 shows that $u(\mathcal{B}) = \mathcal{T}$. ■

There is a connection between the basis for a topology and the fundamental system of neighbourhoods of the points $x \in X$.

Theorem 2.3 *A family of open sets* $\mathcal{B} \subset \mathcal{T}$ *forms a basis iff for any* $x \in X$ *the set* $\mathcal{B}(x) = \{B \in \mathcal{B} | x \in B\}$ *is a fundamental system of neighbourhoods of* x*.*

Example 2.1 *The family of the* n*-dimensional open intervals in* $\mathbb{R}^n$ *forms a basis for the usual topology in* $\mathbb{R}^n$*.*

Definition 2.2 *A family* $\mathcal{S} \subset \mathcal{P}(X)$ *composed of some subsets of the topological space* X *is called a* subbasis *for the topology* $\mathcal{T}$ *if the family of all finite intersections of elements in* $\mathcal{S}$ *is a basis for* $\mathcal{T}$*.*

Remark 2.2 *If* $\mathcal{S}$ *is a subbasis for a topology* $\mathcal{T}$*, any open set* $G \in \mathcal{T}$ *is a union of finite intersections of elements in* $\mathcal{S}$*.*

Definition 2.3 *A topological space* X *satisfies* the first countability axiom *if any point* $x \in X$ *has a countable fundamental system of neighbourhoods.*

Example 2.2 **a)** $\mathbb{R}$ *with the usual topology satisfies the first countability axiom.*

b) *Let* $X = \mathbb{R} \cup \{\omega\}$ *where* ω *is not a real number. Let* $\mathcal{T}$ *be the family of all the open sets in the usual topology on* X *and of all the sets* D *which contain* ω *and have finite complements. Then* $\mathcal{T}$ *is a topology on* X *and* $(X, \mathcal{T})$ *does not satisfy the first countability axiom.*

Remark 2.3 *If X satisfies the first countability axiom, then any point $x \in X$ has a countable fundamental system of neighbourhoods, such that $B_j \subset B_i$ if $j \geq i$, $i, j \in \mathbb{N}$. Indeed, if $\{V_n | n \in \mathbb{N}\}$ is a countable fundamental system of neighbourhoods, the system $B_1 = V_1$, $B_2 = V_1 \cap V_2$,... has the required property.*

Definition 2.4 *A topological space X satisfies* the second countability axiom *if it has a countable basis. Then X is also called a* space with a countable basis.

Definition 2.5 *A topological space X is called* separable *if there is a countable subset of X which is dense in X.*

The next theorems show the connection between these notions.

Theorem 2.4 *Any space with a countable basis is separable.*

Theorem 2.5 *Any space with a countable basis satisfies the first countability axiom.*

We give now an example of separable space with no countable basis.

Example 2.3 *Let X be an uncountable set and $\mathcal{T}$ the topology of the finite complements. An infinite set D is dense in $(X, \mathcal{T})$ because it intersects any non-void open set. The intersection of all the open sets which contain a point $x_0 \in X$ is the set $\{x_0\}$, because the complement of any set $\{x\} \neq \{x_0\}$ is an open set containing x_0. If X has a countable basis $\mathcal{B}$, the intersection of the sets in $\mathcal{B}$ that contain x_0 will be $\bigcap_{i \in I} B_i = \{x_0\}$. The complement $\complement\{x_0\} = \bigcup_{i \in I} \complement B_i$ is a countable union of finite sets $\complement B_i$, so it will be countable; this contradicts the fact that X is an uncountable set.*

Example 2.4 *A space which satisfies the first countability axiom, but does not satisfy the second one.*

Let X be an uncountable set with the discrete topology. For any $x \in X$, the set $\{x\}$ forms a finite fundamental system of neighbourhoods of x, but the space has not a countable basis. X is also not separable.

Definition 2.6 *A* covering *of a set M is a family of sets $\mathcal{A} = \{ A_i | i \in I\}$ such that $M \subset \bigcup_{i \in I} A_i$. The covering is called* open *if any set A_i, $i \in I$ is open. If the set I is finite (countable), the covering $\mathcal{A}$ is called* finite *(*countable*). A* subcovering *of $\mathcal{A}$ is a covering $\mathcal{A}'$ such that $\mathcal{A}' \subset \mathcal{A}$.*

Definition 2.7 *A topological space X is called* Lindelöf *if any open covering has a countable subcovering.*

Theorem 2.6 *(Lindelöf) Any open covering of a subset $M \subset X$, X being a topological space with a countable basis, has a countable subcovering.*

Proof. Let $\mathcal{B}$ be a countable basis of $\mathcal{T}$ and $\mathcal{A}$ an open covering of M. For any $A \in \mathcal{A} \subset \mathcal{T}$ we have $A = \bigcup_{i \in I} B_i$, where I is at most a countable set and $B_i \in \mathcal{B}$. It follows that M is covered by the subfamily $\mathcal{B}' \subset \mathcal{B}$ of the sets B_i that appear in the union which describes the sets $A \subset \mathcal{A}$. For any $B' \subset \mathcal{B}'$ we consider a set $A' \in \mathcal{A}$ such that $B' \subset A'$. The family of the sets A' determines a subcovering $\mathcal{A}' \subset \mathcal{A}$ of the set M. ■

Remark 2.4 *Th. 2.6 means that any space with a countable basis is a Lindelöf space.*

The next theorem shows a way to construct a basis for a topology on a cartesian product of topological spaces.

Theorem 2.7 *If $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ are two topological spaces, then the family $\mathcal{B} = \{\mathcal{A} \subset X_1 \times X_2 | \, A = G_1 \times G_2, \ G_1 \in \mathcal{T}_1, \ G_2 \in \mathcal{T}_2\}$ is a basis for a topology $\mathcal{T}$ on the set $X = X_1 \times X_2$.*

Definition 2.8 *The topology $\mathcal{T}$ from Th. 2.7 on $X = X_1 \times X_2$ is called the* product topology. *The spaces $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ are the* coordinate spaces *of the product space X. The product topology is also denoted by $\mathcal{T} = \mathcal{T}_1 \times \mathcal{T}_2$. One can define is a similar way the product topology for a finite number of topological spaces.*

Remark 2.5 *The system $\mathcal{S}$ of the sets $S_{1,G_1} = \{x \in X_1 \times X_2 | \, x_1 \in G_1\}$ and $S_{2,G_2} = \{x \in X_1 \times X_2 | \, x_2 \in G_2\}$ where $G_1 \in \mathcal{T}_1$ and $G_2 \in \mathcal{T}_2$ forms a subbasis for the product topology on $X_1 \times X_2$.*

3 Separation Properties

Definition 3.1 *A topological space X is called a T_0-*space *if for any x, $y \in X$, $x \neq y$, there is an open set G which contains one point but does not contain the other one.*

*X is called a T_1-*space *if for any x, $y \in X$, $x \neq y$, there are two open sets G_1 and G_2, $x \in G_1$, $y \neq G_1$ and $y \in G_2$, $x \neq G_2$.*

*X is called a T_2-*space *(a Hausdorff space) if for any x, $y \in X$, $x \neq y$, there are two open sets G_1 and G_2 with $x \in G_1$ and $y \in G_2$, $G_1 \cap G_2 = \emptyset$.*

X is called a T_3-space *if for any closed set A and $x \notin A$, there are two open sets G_1 and G_2, $x \in G_1$, $A \subset G_2$ and $G_1 \cap G_2 = \emptyset$.*

A topological space which is T_1 and T_3 is called a regular space.

X is called a T_4-space *if for any closed disjoint sets A, B there are two open and disjoint sets G_1 and G_2, $A \subset G_1$ and $B \subset G_2$.*

A topological space which is T_1 and T_4 is called a normal space.

X is called a T_5-space *if for any A, $B \subset I$ with $A \cap \overline{B} = \emptyset$, $\overline{A} \cap B = \emptyset$ there are two open disjoint sets G_1 and G_2 with $A \subset G_1$ and $B \subset G_2$.*

A topological space which is T_1 and T_5 is called completely normal.

Example 3.1 *[31]*

1) *Topological spaces which are T_0.*

   a) $X = \{a, b\}$, $\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}\}$.

   b) $X = [-1, 1]$ with the topology generated by a subbasis consisting of the sets $[-1, b)$ for $b > 0$ and $(a, 1]$ for $a < 0$ (the overlapping interval topology).

2) *Let X be a countable space with the topology of finite complements; X is then a T_1-space.*

3) *Topological spaces which are T_2.*

   a) $(\mathbb{R}, \mathcal{T})$ with the usual topology is a T_2- space.

   b) Let $X = \{(x, y) \in Q^2 |\, y \geq 0\}$ and an irrational number θ. The irrational slope topology $\mathcal{T}$ on X is generated by ε- neighbourhoods of the form $N_\varepsilon(x, y) = \{(x, y)\} \cup B_\varepsilon(x + y/\theta) \cup B_\varepsilon(x - y/\theta)$, where $B_\varepsilon(z) = \{r \in \mathbb{Q} |\, |r - z| < \varepsilon\}$. Each $N_\omega(x, y)$ consists of the point (x, y) and two intervals on the rational x-axis centered at the two irrational points $x \pm y/\theta$; the lines joining these point to (x, y) have slope $\pm\theta$. The topological space X is T_2 (so it is also T_1 and T_0), but has not other separation properties.

4) *Let $X = \bigcup\limits_{i=0}^{\infty} L_i$ be the union of lines in the plane, where $L_0 = \{(x, 0) |\, x \in (0, 1)\}$ and for $i \geq 1$, $L_i = \{(x, 1/i) |\, x \in [0, 1)\}$. If $i \geq 0$, each point of L_i except for $(0, 1/i)$ is an open set; the basis neighbourhoods of $(0, 1/i)$ are the subsets of L_i with finite complements. Similarly, the sets $U_i(x, 0) = \{(x, 0)\} \cup \{(x, 1/n) |\, n > i\}$ form a basis for the points in L_0.*

   The space X with this topology is T_3, but not T_4.

5) *Let X be the set of the real numbers; for each irrational x we choose a sequence $(x_i)_{i\in\mathbb{N}}$ of rationals converging to it in the usual topology. The rational sequence topology $\mathcal{T}$ is then defined by declaring each rational open, and selecting the sets $U_n(x) = \{x_i\}_{i=n}^{\infty} \cup \{x\}$ as a fundamental system of neighbourhoods for the irrational point x.*

   The topological space X is then regular, but not normal.

6) *Let $X = \{x \in \mathbb{Z}_+ |\, x \geq 2\}$ together with the topology generated by the sets of the form $U_n = \{x \in X |\, x \text{ divides } n\}$ for $n \geq 2$. The space X with the divisor topology is T_4. It is also T_0, but has not other separation properties.*

7) *Let X be the closed unit square $[0,1] \times [0,1]$; for the points $p = (s,t)$ which are not on the diagonal $\triangle = \{(x,x) |\, x \in [0,1]\}$ a fundamental system of neighbourhoods is formed by the intersection of $X - \triangle$ with an open vertical line segment centered at p, $\mathbb{N}_\varepsilon(s,t) = \{(s,t) \in X - \triangle |\, |t-y| < \varepsilon\}$. For the points $(x,x) \in \triangle$ the neighbourhoods are the intersection of X with the open horizontal stripes less a finite number of vertical lines: $M_\varepsilon(s,s) = \{(x,y) \in X |\, |y-s| < \varepsilon,\ x \neq x_0, x_1, ..., x_n\}$. This topological space, named the Alexandroff square, is normal, but not T_5.*

8) *The indiscrete topology is T_5, but not completely normal.*

9) *$X = \mathbb{R}$ with $\mathcal{T}$ determined by the basis $\mathcal{B} = \{(a,b] |\, a < b\}$ (the upper limit topology) is completely normal.*

We give now some results that we shall use in the following sections.

Theorem 3.1 *A topological space X is a T_1-space iff any one-point set is closed.*

Theorem 3.2 *Any neighbourhood of a cluster point of the infinite set M in a T_1-space contains infinitely many points of the set.*

Theorem 3.3 *In a T_2-space any convergent sequence $(x_n)_{n\in\mathbb{N}}$ has a unique limit.*

Definition 3.2 *The sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to the point x if any neighbourhood $V \in \mathcal{V}(x)$ contains all the terms of the sequence, except for a finite number of them.*

4 Relativization

In many cases it is necessary to topologize subsets of a topological space, the topology being related to the initial topology. Let $(X, \mathcal{T})$ be a topological space and $M \subset X$ a subset. In the most natural way, the open sets in M will be intersections with M of the open sets in X.

Theorem 4.1 *The system $\mathcal{T}_M = \{G \cap M | G \in \mathcal{T}\}$ is a topology on M.*

We can give now

Definition 4.1 *The topology $\mathcal{T}_M = \{G \cap M | G \in \mathcal{T}\}$ is called the* relative topology *of M, M being a subset of X.*

Remark 4.1 *An open set in the relative topology is not necessarily open in the total space. So, if we consider in $\mathbb{R}$ with the usual topology the set $M = [-1, 1]$, the set $(0, 1] = M \cap (0, 2)$ is open in M, but not in $\mathbb{R}$.*

However, we have the following result

Theorem 4.2 *Any set $G \in \mathcal{T}_M$ is open in X iff M is open in X.*

The next theorems establish relations between the closed sets of X and M.

Theorem 4.3 *A set $F \subset M$ is closed in M iff $F = H \cap M$, H being a closed set in X.*

Theorem 4.4 *Any closed set in M is closed in X iff M is closed in X.*

The neighbourhoods in $(M, \mathcal{T}_M)$ are related to those in $(X, \mathcal{T})$ by

Theorem 4.5 *A set $V \subset M$ is a neighbourhood of x in the space $(M, \mathcal{T}_M)$ iff $V = U \cap M$, where U is a neighbourhood of x in $(X, \mathcal{T})$.*

We obtain from Th. 4.5.

Theorem 4.6 *Any neighbourhood of x in $(M, \mathcal{T}_M)$ is a neighbourhood of x in $(X, \mathcal{T})$ iff M is a neighbourhood of x in $(X, \mathcal{T})$.*

The next theorems refer to adherence.

Theorem 4.7 *A point $x \in M$ is an adherent point of the set $A \subset M$ in $(M, \mathcal{T}_M)$ iff it is an adherent point for A in $(X, \mathcal{T})$.*

Theorem 4.8 *The adherence of a set A in $(M, \mathcal{T}_M)$ is the intersection of M with the adherence of A in $(X, \mathcal{T})$, that is $\overline{A}_M = \overline{A} \cap M$.*

5 Nets

Definition 5.1 *Let $D \neq \emptyset$ a set and $\geq$ a binary relation having the properties*

1) $a \geq b$ *and* $b \geq c \Rightarrow a \geq c$ *(transitivity)*

2) $a \geq a$ *(reflexivity)*

3) $\forall a, b \in D$, *there exists* $c \in D$ *such that* $c \geq a$ *and* $c \geq b$.

The pair $(D, \geq)$ *is a* directed set.

Example 5.1 *a)* $(\mathbb{R}, \geq)$ *and* $(\mathbb{N}, \geq)$ *with the natural order are directed sets.*

b) $(\mathcal{V}(x), \subset)$ *is a directed set, where* $\mathcal{V}(x)$ *is the family of all neighbourhoods of the point* x *and* $A \subset B$ *means that* B *includes* A.

c) $(\mathcal{P}_f, \supset)$ *is also a directed set where* $\mathcal{P}_f$ *is the family of all finite subsets of a non-void set* M *and* $A \supset B$ *means that* A *includes* B.

Definition 5.2 *A* net *in* X *is a function* $s : (D, \geq) \rightarrow X$, *where* $(D, \geq)$ *is a directed set. We write* $s(d) = s_d$.

Definition 5.3 *A net* $s : (D, \geq) \rightarrow X$ *is* in the set A *if* $s_d \in A$, $\forall d \in D$. *The set is* eventually in A *if there is an element* $d_0 \in D$ *such that for each* $d \geq d_0$, $s_d \in A$; *the net is* frequently in A *if for each* $d_0 \in D$, *there is* $d \geq d_0$ *such that* $s_d \in A$.

If s is frequently in A, the set $E = \{d \in D | \, s_d \in A\}$ has the property that for each $d \in D$, there is $d' \in E$ such that $d' \geq d$. Such subsets of D are called cofinal. A cofinal subset of D is also directed by $\geq$, because for $a, b \in E$ there is $c \in D$ such that $c \geq a$, $c \geq b$; but for $c \in D$ there is $d \in E$, $d \geq c$, so $d \geq a$ and $d \geq b$.

We have the following obvious property.

Theorem 5.1 *A net* s *is frequently in* A *iff a cofinal subset of* D *maps in* A*; this happens iff the net is not eventually in the complement of* A.

Definition 5.4 *A net* s *in a topological space* $(X, \mathcal{T})$ converges to x *in the topology of* X *if it is eventually in any neighbourhood of* x.

Example 5.2 *a) If* X *is a discrete space,* s *converges to* x *iff* s *is eventually in* $\{x\}$*; that is that there is an element* $d_0 \in D$ *such that for each* $d \geq d_0$, $s_0 = x$.

b) If X is an indiscrete space, any net converges to any point of X. It follows that a net may converge to more than one point.

Theorem 5.2 *In a topological space X, a point $x \in X$ is an accumulation point for the subset A of X iff there is a net in $A \setminus \{x\}$ which converges to x.*

Proof.

1. Let x be an accumulation point for the subset A; then for each $U \in \mathcal{V}(x)$, there is $s_U \in A \cap (U \setminus \{x\})$. The family $\mathcal{V}(x)$ of the neighbourhoods of x is directed by $\subset$, so we obtain a net s which is in $A \setminus \{x\}$. We show that s is eventually in any neighbourhood of x. Let $V \in \mathcal{V}(x)$ and $V' \subset V$; it results that $s_{V'} \in V' \subset V$, so that $s_{V'} \in V$ for each $V' \subset V$.

2. If there is a net in $A \setminus \{x\}$ which converges to x, then it has values in any neighbourhood of x, so $A \setminus \{x\}$ intersects any neighbourhood of x. ■

The next theorems have similar proofs.

Theorem 5.3 *A point $x \in X$ is in A iff there is a net in A which converges to x.*

Theorem 5.4 *A subset $A \subset X$ is closed iff there is not a net in A which converges to a point of $X \setminus A$.*

The following theorem gives a characterization of Hausdorff spaces.

Theorem 5.5 *A topological space is Hausdorff iff any net converges to at most one point.*

Proof.

1. Let X be a Hausdorff space and $x \neq y$. Then there are $U \in \mathcal{V}(x)$ and $V \in \mathcal{V}(y)$, $U \cap V = \emptyset$. But a net cannot be eventually in both sets, and it follows that a net in X cannot converge both to x and y.

2. We assume that X is not a Hausdorff space; let $x \neq y$ be two points such that any neighbourhood of x intersects any neighbourhood of y. Let $(\mathcal{V}(x), \subset)$ and $(\mathcal{V}(y), \subset)$ be directed sets; we define an order on the cartesian product $\mathcal{V}(x) \times \mathcal{V}(y)$ by setting $(U, V) \geq (U', V')$ if $U \subset U'$ and $V \subset V'$. The cartesian product is obvious directed by $\geq$.

For any $(U, V) \in \mathcal{V}(x) \times \mathcal{V}(y)$ we have $U \cap V \neq \emptyset$; let $s_{(U,V)} \in U \cap V$. If $(U', V') \geq (U, V)$, then $s_{(U\prime,V\prime)} \in U' \cap V' \subset U \cap V$ and it follows that the net $\left(s_{(U,V)}\right)_{(U,V) \in \mathcal{V}(x) \times \mathcal{V}(y)}$ converges to both x and y. It remains that if any net converges to at most one point, the space is Hausdorff. ■

Definition 5.5 *Let $s : (D, \geq) \to X$ a net and $s' : (D', \geq') \to D$ a net in D that satisfies the condition*

$$(1) \qquad \forall a \in D, \exists\ a' \in D' \text{ such that } [b' \in D',\ b' \geq' a' \Rightarrow s'(b') \geq a].$$

The net $s \circ s' : (D', \geq') \to X$ is called a subnet *of s.*

Remark 5.1 *The way in which we defined the subnet $s \circ s'$ implies that if s is eventually in a set A, the subnet $s \circ s'$ is also eventually in A.*

Example 5.3 *a) Let $E \subset D$ a cofinal subset, directed by the induced relation and $s : (D, \geq) \to X$ a net. If $s' : (E, \geq) \to D$ is the identical function on E, $s \circ s'$ will be subnet of s.*

b) Another way of obtaining subnets is the following. Let $(D', \geq')$ be a directed set and $s' : (D' \geq') \to (D, \geq)$ an isotone function ($s'(a) \geq s'(b)$ if $a \geq' b$) such that $\operatorname{Im} s'$ is a cofinal subset in D. Then $s \circ s'$ will be a subnet of s. This way of constructing subnets is used in Lemma 5.1.

Definition 5.6 *A point $x \in X$ is a* cluster point *of the net s if s is frequently in any neighbourhood of x.*

Example 5.4 *a) A net with no cluster point. The sequence $(n)_{n \in N}$ considered as a net has no cluster point in the usual topology of $\mathbb{R}$.*

b) A net with infinitely many cluster points. For the sequence of all rational numbers considered as a net any real number is a cluster point.

Remark 5.2 *If a net converges to a point, this is obviously a cluster point. But the converse is not true. For the sequence $-1, 1, -1, -2, -1, 3, \ldots$ the point -1 is the unique cluster point, but the sequence fails to converge to -1.*

Theorem 5.6 *A point x is a cluster point of the net s iff s has a subnet which converges to x.*

In the proof we need the following

Lemma 5.1 *Let s be a net, $\mathcal{A}$ a family of subsets of X directed by $\subset$ such as s is frequently in any member of $\mathcal{A}$. Then there is a subnet of s which is eventually in any member of $\mathcal{A}$.*

Proof. Let $s : (D, \geq) \to X$ the net which is frequently in any member of $\mathcal{A}$ and $D' = \{(d, A) | \, d \in D, \, A \in \mathcal{A}, \, s_d \in A\}$. Then D' is directed by the relation $(d, A) \geq' (e, B)$ if $d \geq e$ and $A \subset B$; indeed, for (d, A) and (e, B) there is a $C \in \mathcal{A}$, $C \subset A$ and $C \subset B$, and $c \in D$, $c \geq d$, $c \geq e$ with $s_c \in C$. Then $(c, C) \in D'$ and $(c, C) \geq' (d, A)$ and $(c, C) \geq' (e, B)$.

We define $s' : (D', \geq') \to D$ by $s' (d, A) = d$. The function s' is obviously isotone and $\operatorname{Im} s'$ is a cofinal set in D, because s is frequently in any member of $\mathcal{A}$. It follows that $s \circ s'$ is a subnet of s. The subnet $s \circ s'$ is eventually in any member of $\mathcal{A}$. ■

We give now the proof of Th. 5.6

Proof.

1. Let x be a cluster point for s and $\mathcal{A} = \mathcal{V}(x)$. In this case Lemma 5.1 applies and we obtain a subnet of s which is eventually in any member of $\mathcal{A}$, that is it converges to x.

2. If x is not a cluster point for s, there is a neighbourhood V of x such that s is not frequently in V; it follows that s is eventually in $\complement V$. Then any subnet of s is (by Remark 5.1) eventually in $\complement V$, and cannot converge to x. It follows that if there is a subnet of s which converges to x, then x is a cluster point for s. ■

The next theorem gives a characterization of cluster points by means of closure.

Theorem 5.7 *Let s be a net; for any $d \in D$ we consider the set $A_d = \{s_e | \, e \geq d\}$. Then x is a cluster point for s iff $x \in \overline{A}_d$ for any $d \in D$.*

The connection between subsequences and subnets is given in

Theorem 5.8 *Any subsequence of a sequence is a subnet of the sequence considered as a net. It is not true that any subnet of a sequence is a subsequence.*

Proof. The first assertion is obvious. To prove the second we consider the next example.

Let $(N, \geq)$ be the set of all natural numbers with the usual order and $s : \mathbb{N} \to \mathbb{R}$ defined by $s(n) = n$, $n \in \mathbb{N}$. The net $s^* \circ s : \mathbb{N} \to \mathbb{R}$ given by $s^* \circ s(n) = n - \left[\frac{n}{4}\right]$ is not a subsequence of s, but it is a subnet. ■

6 Continuous functions

Definition 6.1 *Let X and Y be two sets. A* function *defined on X, with values in Y is a subset f of the cartesian product $X \times Y$, having the property that for any $x \in X$, there is a $y \in Y$ and only one such that $(x, y) \in f$.*

We denote a function by $f : X \to Y$. Instead of $(x, y) \in f$, one usually writes $y = f(x)$.

Definition 6.2 *The* inverse of the function $f : X \to Y$ *is a subset* f^{-1} *of the cartesian product* $Y \times X$*, given by* $f^{-1} = \{ (y, x) \in Y \times X | \, (x, y) \in f \}$*.*

Remark 6.1 *Usually* f^{-1} *is not a function,* $f^{-1}(y)$ *being not formed by a unique point.*

Definition 6.3 *If* $A \subset X$*, the set* $f(A) = \{ f(x) | \, x \in A \}$ *is called the* image through f *of the set* A*. If* $B \subset Y$*, the set* $f^{-1}(B) = \{ x | \, f(x) \in B \}$ *is called the* counter image through f *of the set* B*.*

Definition 6.4 *The function* f *is* surjective*, if* $f(X) = Y$ *and is* one-to-one *if*

$$f(x) = f(x') \Rightarrow x = x',$$

that is if card $f^{-1}\{y\} \leq 1$, $\forall y \in Y$.

The function f *is* bijective *if it is both surjective and one-to-one. If* f *is bijective,* $f^{-1} : Y \to X$ *is also a function.*

Definition 6.5 *Let* $f : X \to Y$ *and* $g : Y \to Z$ *be two functions. The function* $g \circ f : X \to Z$ *given by* $g \circ f(x) = g(f(x))$ *is called a* function of a function *(*composed function*).*

Theorem 6.1 *Let* $f : X \to Y$ *and* $g : Y \to Z$ *functions,* A*,* B*,* A_i*,* $i \in I$ *subsets of* X *and* M*,* N*,* M_j*,* $j \in J$ *subsets of* Y*. The following properties are true*

(1) $A \subset B \Rightarrow f(A) \subset f(B)$

(1′) $M \subset N \Rightarrow f^{-1}(M) \subset f^{-1}(N)$

(2) $f\left(\bigcup_{i \in I} A_i\right) = \bigcup_{i \in I} f(A_j)$

(2′) $f^{-1}\left(\bigcup_{j \in J} M_j\right) = \bigcup_{j \in J} f^{-1}(M_j)$

(3) $f\left(\bigcap_{i \in I} A_i\right) \subset \bigcap_{i \in I} f(A_i)$

(3′) $f^{-1}\left(\bigcap_{j \in J} M_j\right) = \bigcap_{j \in J} f^{-1}(M_j)$

(4) $f(A \setminus B) \supset f(A) \setminus f(B)$*; it follows that* $f\left(\complement_X A\right) \supset f(X) \setminus f(A)$

(4′) $f^{-1}(M \setminus N) = f^{-1}(M) \setminus f^{-1}(M)$; *it follows that* $f^{-1}\left(\complement_Y M\right) = \complement_X f^{-1}(M)$

(5) $f^{-1}(f(A)) \supset A$

(5′) $f(f^{-1}(M)) \subset M$

(6) $f(f^{-1}(M) \cap A) = M \cap f(A)$; *if* $M \subset f(X)$, $f(f^{-1}(M)) = M$

(7) $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C)), \forall C \subset Z$.

Definition 6.6 *Let* $A \subset X$ *and* $f : X \to Y$*. The* restriction of f *to the set* A *is a function* $f|_A : A \to Y$ *such that* $f|_A(x) = f(x), \forall x \in A$. f *is called then an* extension of $f|_A$.

Definition 6.7 *Let* $(X, \mathcal{T}_X)$ *and* $(Y, \mathcal{T}_Y)$ *be two topological spaces. The function* $f : X \to Y$ *is called* continuous at $x \in X$ *if for any neighbourhood* U *of* $f(x)$ *there is a neighbourhood* V *of* x *such that* $f(V) \subset U$.

Remark 6.2 *The function* f *is continuous at* x *iff in the above condition* U *and* V *are members of a fundamental system of neighbourhoods of* $f(x)$ *and* x.

From now on we shall denote $X = (X, \mathcal{T}_X)$ and $Y = (Y, \mathcal{T}_Y)$. It is easy to establish the following test of continuity.

Theorem 6.2 *The function* $f : X \to Y$ *is continuous at* $x \in X$ *iff for any neighbourhood* U *of* $f(x)$, $f^{-1}(U)$ *is a neighbourhood of* x.

Definition 6.8 *The function* $f : X \to Y$ *is called* continuous on X *if it is continuous at any point of* X.

The next theorem and its corollary give a characterization of the continuity on X.

Theorem 6.3 *For the function* $f : X \to Y$*, the following statements are equivalent*

(1) f *is continuous on* X

(2) *for any* $A \subset X$, $f\left(\overline{A}\right) \subset \overline{f(A)}$

(3) *if* M *is an open set in* Y*, then* $f^{-1}(M)$ *is open in* X

(4) *if* M *is a closed set in* Y*, then* $f^{-1}(M)$ *is closed in* X.

Corollary 6.1 *Let $f : X \to Y$ be a function and $\mathcal{S}$ a subbasis of the topology on Y; f is continuous on X iff for any $S \in \mathcal{S}$, $f^{-1}(S)$ is open in X.*

The next theorem is related to the continuity of the composed functions and of some restrictions; its proof is simple using the test (4) of Th. 6.3.

Theorem 6.4 *Let X, Y and Z be topological spaces. Then the following statements are true.*

(1) *If $f : X \to Y$ and $g : Y \to Z$ are continuous, $g \circ f$ is continuous.*

(2) *If $f : X \to Y$ is continuous and $A \subset X$ is endowed with the relative topology then $f|_A : A \to Y$ is continuous.*

(3) *If $f : X \to Y$ is continuous and $f(X)$ is endowed with the relative topology, then $f_1 : X \to f(X)$, $f_1(x) = f(x)$ is continuous on X.*

Remark 6.3 *The image of a closed (open) set through a continuous function f is not necessary a closed (open) set.*

7 Compact spaces

Definition 7.1 *A topological space X is called a* compact space *if every system of open sets of X which covers X contains a finite subsystem also covering X.*

Example 7.1 *$\mathbb{R}$ with the usual topology is not compact, but any closed interval $[a, b] \subset \mathbb{R}$ with the relative topology is compact.*

It is well-known the following characterization of compactness.

Theorem 7.1 *In a topological space X, the following statements are equivalent*

(1) *X is a compact space*

(2) *Any system of closed subsets of X having the finite intersection property (every finite subsystem has a non-void intersection) has a non-void intersection.*

Corollary 7.1 *If in the compact space X we have $F_1 \supset F_2 \supset ... \supset F_n \supset ...$, where F_i are closed and non-void sets, $i \in \mathbb{N}$, then $\bigcap\limits_{i=1}^{\infty} F_i \neq \emptyset$.*

Definition 7.2 *A topological space is called* sequential compact *if every sequence of points of the space contains a convergent subsequence.*

Theorem 7.2 *In a topological space which satisfies the second countability axiom, the sequential compactness implies the compactness.*

Theorem 7.3 *In a topological space which verifies the first countability axiom, the compactness implies the sequential compactness.*

By Th. 2.5 we also obtain

Corollary 7.2 *In a topological space which satisfies the second countability axiom, the sequential compactness is equivalent to the compactness.*

Generally speaking, in a topological space the sequential compactness and the compactness are distinct notions. The compactness can be characterized in terms of nets.

Theorem 7.4 *A topological space is compact iff any net has a cluster point.*

Using Th. 5.6 we obtain

Corollary 7.3 *A topological space is compact iff any net has a convergent subnet.*

We have also the following result given by Alexander.

Theorem 7.5 *Let X be a topological space and $\mathcal{S}$ a subbasis of its topology. If every covering of X formed by member of the subbasis $\mathcal{S}$ contains a finite subcovering, then the space is compact.*

We can define a notion of compactness for subsets of a topological space.

Definition 7.3 *A subset M of the topological space $(X, \mathcal{T})$ is compact if $(M, \mathcal{T}_M)$, where $\mathcal{T}_M$ denotes the relative topology, is compact.*

It is easy to prove

Theorem 7.6 *A subset M of the topological space is compact iff any open covering of M with open subsets of $(X, \mathcal{T})$ contains a finite subcovering.*

Theorem 7.7 *If M_1 and M_2 are compact sets, $M_1 \cup M_2$ is also compact.*

Remark 7.1 *It follows from Th. 7.7 that any finite union of compact sets is compact.*

The next theorems give relations between compactness, closure and separation properties of X.

Theorem 7.8 *Any closed subset of a compact space is compact.*

Theorem 7.9 *Any compact subset of a Hausdorff space is closed.*

Remark 7.2 *In a compact Hausdorff space, a subset is compact iff it is closed.*

Theorem 7.10 *A compact Hausdorff space is T_3 and T_4.*

The continuous functions defined on compact spaces have special properties described below.

Theorem 7.11 *If $f : X \to Y$ is a continuous function on the compact space X, then $f(X)$ is compact.*

Remark 7.3 *The compactness is an invariant of continuity.*

Theorem 7.12 *Let X be a compact space and Y a Hausdorff space. If $f : X \to Y$ is continuous, then f is closed.*

At the end of this section, we give a theorem due to Tychonoff, which can be proved using Th. 7.11, Remark 2.5 and Th. 7.5.

Theorem 7.13 *A topological product space is compact iff the spaces of coordinates are compact.*

Remark 7.4 *The theorem of Tychonoff is true for any product space (the product being not necessary finite).*

8 Metric spaces

Definition 8.1 *For a given set X, we call a* distance *or a* metric *on X a function $d : X \times X \to \mathbb{R}$ having the properties*

(1) $d(x, y) = 0 \Leftrightarrow x = y$

(2) $d(x, y) = d(y, x)\,,\ \forall x, y \in X$

(3) $d(x, z) \leq d(x, y) + d(y, z)\,, \forall x, y, z \in X$ *(the triangular inequality).*

Remark 8.1 *From (1) - (3) it follows that $d(x, y) \geq 0$, $\forall x,\ y \in X$.*
The couple (X, d) is called a metric space.

Example 8.1 1) $(\mathbb{C}, d)$ *where $\mathbb{C}$ denotes the set of all complex numbers and $d(z_1, z_2) = |z_1 - z_2|$ is a metric space.*

2) $(\mathbb{R}^n, d)$ *with $d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$ is a metric space; d is called the euclidian metric.*

3) $C[a, b]$, *the set of all continuous functions defined on $[a, b]$ with real values, with $d(f, g) = \max_{x \in [a,b]} |f(x) - g(x)|$ (the Tchebycheff metric) is a metric space.*

4) *Let (X, d) be a metric space and 2^X the family of the non-void bounded closed subsets of X and $D(A, B) = \max \left\{ \sup\limits_{x \in A} \inf\limits_{y \in B} d(x, y), \sup\limits_{y \in B} \inf\limits_{x \in A} d(x, y) \right\}$, $\forall A,\ B \in 2^X$ (the Pompeiu-Hausdorff metric). Then $\left(2^X, D\right)$ is a metric space (see §1, Ch. IV).*

Definition 8.2 *Let (X, d) be a metric space. The number $d(x, y)$ is called the* distance between x and y. The distance from the point x to the set $A \subset X$, $A \neq \emptyset$ *is the number $d(x, A) = \inf\{ d(x, a) |\, a \in A\}$. The* distance between the non-void sets A and B *is the number $d(A, B) = \inf\{ d(a, b) |\, a \in A,\ b \in B\}$.*

Definition 8.3 *The* diameter *of the non-void set A is $d(A) = \sup\{ (d(x, y) |\, x,\ y \in A\}$. The set A is called a* bounded set *if $d(A)$ is finite.*

Definition 8.4 *An* open (closed) ball of center x and radius r *($x \in X$, $r > 0$) is the set $B(x, r) = \{y |\, d(x, y) < r\}$ (respectively $\overline{B}(x, r) = y |\, d(x, y) \leq r\}$. A* sphere of center x and radius r *is the set $S(x, r) = \{ y |\, d(x, y) = r\}$.*

We will topologize now the metric space with a topology determined by the metric d.

Definition 8.5 *A set $G \subset X$ is called* open *if $G = \emptyset$ or if for any $x \in G$, there is a positive number r such that $B(x, r) \subset G$.*

Theorem 8.1 *The family $\mathcal{T}$ of the open sets defined above determines on X a topological structure.*

Remark 8.2 1) *The family $\mathcal{B}$ of all the open balls of the metric space (X, d) is the basis of a topology on X.*

2) *The system $\{B(x, r) | \, r > 0\}$ is a fundamental system of neighbourhoods of x in the topology determined by d.*

3) *All metric spaces satisfy the first countability axiom, because $\{B(x, q) | \;\; q \in \mathbb{Q}, q > 0\}$ is a fundamental countable system of neighbourhoods for x.*

4) *For $A \subset X$, $A \neq \emptyset$, we have $d(x, A) = 0$ iff $x \in \overline{A}$.*

The distance function (the metric) has the following continuity properties.

Theorem 8.2 *The function $d : X \times X \to \mathbb{R}$ is continuous.*

Theorem 8.3 *If $M \subset X$ is a fixed non-void set, the function $d(\cdot, M) : X \to \mathbb{R}$ is continuous.*

The continuity of the functions defined between metric spaces can be characterized in the following way.

Theorem 8.4 *(Heine) Let $f : (X, d) \to (X', d')$ be a function; f is called* continuous at the point $x \in X$ *iff for any sequence $(x_n)_{n \in \mathbb{N}}$ which converges to x, the sequence $(f(x_n))_{n \in \mathbb{N}}$ converges to $f(x) \in Y$.*

The metric spaces have important separation properties, like the following ones.

Theorem 8.5 *Any metric space is T_1.*

Theorem 8.6 *Any metric space is T_2.*

Theorem 8.7 *Any metric space is normal.*

We have also

Theorem 8.8 *A metric space which is separable has a countable basis.*

It follows by Th. 2.4.

Corollary 8.1 *A metric space* (X, d) *is separable iff it has a countable basis.*

The compact metric spaces have all the properties of compact spaces, but in this case the compactness can be characterized in a peculiar way.

Definition 8.6 *In the metric space* (X, d) *an* ε-net *is a finite set* $N_\varepsilon \subset X$ *having the property that* $d(x, N_\varepsilon) < \varepsilon$, $\forall x \in X$.

It follows easily

Theorem 8.9 *If the metric space* (X, d) *has an* ε*-net, then* X *is bounded.*

Definition 8.7 *A metric space is* totally bounded *if for any* $\varepsilon > 0$ *the space possesses an* ε*-net.*

Remark 8.3 *From Th. 8.9 it follows that any totally bounded space is bounded.*

The following theorems contain a characterization of totally boundedness.

Theorem 8.10 *A metric space* (X, d) *is totally bounded iff for any* $\varepsilon > 0$ *there is a finite covering of the space with sets of diameter smaller than* ε.

Theorem 8.11 *Any totally bounded metric space is separable.*

We have by Th. 8.8.

Corollary 8.2 *Any totally bounded metric space satisfies the second countability axiom.*

There is the next relation between totally boundedness and sequential compactness.

Theorem 8.12 *A sequential compact metric space is totally bounded.*

Definition 8.8 *The metric space* (X, d) *has the* Bolzano - Weierstrass property *if any infinite set has an accumulation point.*

The next theorem gives some tests of compactness for metric spaces, which are joined to those more general from Th. 7.1.

Theorem 8.13 *In a metric space (X, d) the following conditions are equivalent*

(1) *The space is compact.*

(2) *The space is sequential compact.*

(3) *The space has the Bolzano-Weierstrass property.*

An important class of metric spaces in the class of complete spaces.

Definition 8.9 *A sequence $(x_n)_{n \in \mathbb{N}}$ in a metric space (X, d) is called a* fundamental sequence *(or* Cauchy sequence*) if for any $\varepsilon > 0$ there is a natural number n_ε such that $d(x_m, x_n) < \varepsilon$, $\forall m, n > n_\varepsilon$.*

Remark 8.4 *A sequence is fundamental iff for any $\varepsilon > 0$ there is a $n_\varepsilon \in \mathbb{N}$ such that $d(x_n, x_{n_\varepsilon}) < \varepsilon$, $\forall n > n_\varepsilon$.*

We have the following

Theorem 8.14 *Any convergent sequence in a metric space X is fundamental.*

Remark 8.5 *The converse of Th. 8.14 is not true. Indeed, if in (X, d) we have a sequence converging to x_0, in the space $(X \setminus \{x_0\}, d')$, d' being the related metric, this sequence does not converge. But it is a fundamental sequence.*

We give now

Definition 8.10 *A metric space in which any fundamental sequence is convergent is called a* complete space*.*

Example 8.2 $\mathbb{C}$, $\mathbb{R}^n$ *and $C[a, b]$ with the metrics of Ex. 8.1 are complete. The set of rational numbers with the related metric is a non-complete space.*

The totally boundedness and compactness of a metric space can be characterized using the fundamental sequences and the completeness.

Theorem 8.15 *A metric space is totally bounded iff any sequence has a fundamental subsequence.*

Theorem 8.16 *A metric space is compact iff it is complete and totally bounded.*

Corollary 8.3 *A closed subset of a complete space is compact iff it is totally bounded.*

Definition 8.11 *A function $f : (X, d) \to (Y, d')$ is called* uniformly continuous *on X if for any $\varepsilon > 0$ there is $\eta > 0$ such that for any x,$y \in X$ with $d(x, y) < \eta$, we have $d'(f(x), f(y)) < \varepsilon$.*

Theorem 8.17 *If (X, d) is a compact metric space and $f : (X, d) \to (Y, d')$ is continuous, then f is uniformly continuous.*

9 Connectedness

Definition 9.1 *A topological space is called* connected *if it cannot be represented as a union of two closed, non-void, disjoint sets. A space which is not connected is called* disconnected. *A subset $M \subset X$ is called a* connected set *if $(M, \mathcal{T}_M)$ is connected topological space, $\mathcal{T}_M$ being the relative topology on M.*

Definition 9.2 *The sets A and B are called* separated *if $J(A, B) = \left(\overline{A} \cap B\right) \cup \left(A \cap \overline{B}\right) = \emptyset$.*

The connectedness can be characterized in the following way.

Theorem 9.1 *A set $M \subset X$ is connected iff it cannot be represented as a union of two non-void and separated sets.*

The connected sets have also the following property.

Theorem 9.2 *If a connected set M is included in the union of two separated sets, then M is included in one of those sets.*

CHAPTER II

POINT-TO-SET MAPPINGS. GENERALITIES

1 Point-to-set mappings, semi-univocal mappings, functions

The notion of function, which requires for any element of the domain an element and only one of the range, is too restrictive and it excludes some of the most frequent correspondences in mathematics. Thus, if we want to establish a correspondence between a complex number and the n-root of that number $(n \geq 2,\ n \in \mathbb{N})$, we observe that to one number there correspond more than one number of $\mathbb{C}$. Precisely the correspondence $\sqrt[n]{} : \mathbb{C} \to \mathcal{P}(\mathbb{C}),\ \sqrt[n]{z} = \{t \in \mathbb{C} |\, t^n = z\}$ associates to one element of $\mathbb{C}$ a subset of $\mathbb{C}$, not a single element. We call then $\sqrt[n]{\cdot}$ a point-to-set mapping. Another reason which determines a study of the point-to-set mapping is the asymmetry between functions and their inverses; one knows that the inverse of a function is not a function, but an object of another nature - and it will be called a point-to-set mapping. Of course, the function will be a special case of point-to-set mappings, where the image of any element of the domain is a set which contains one element and only one. So, the point-to-set mappings are a natural generalization of functions. The algebraic properties of functions are similar to those of point-to-set mappings; they are the subject of this chapter.

Let X and Y be two sets, $X \neq \emptyset$.

Definition 1.1 *A* point-to-set mapping *(multifunction) or, briefly, a mapping defined on X with values in Y is a function $F : X \to \mathcal{P}(Y)$, where $\mathcal{P}(Y) = \{A |\, A \subset Y\}$ is the family or all the subsets of Y. It is denoted by $F : X \multimap Y$. The set $F(x)$ is called the* image *through F of x.*

The effective domain *of F is $D(F) = \{x \in X |\, F(x) \neq \emptyset\}$, and the* range *of F is $R(F) = \bigcup\limits_{x \in X} F(x)$.*

Definition 1.2 *The set $\Gamma(F) = \{(x, y) \in X \times Y |\, y \in F(x)\}$ is called the* graph *of the mapping $F : X \multimap Y$.*

Definition 1.3 *If the point-to-set mapping $F : X \multimap Y$ satisfies the condition* card $F(x) = 1$ *for any $x \in X$, the mapping is called a single-valued mapping or a* function*. We will denote the mappings by capital letters, and the functions by small ones.*

Definition 1.4 *A point-to-set mapping is called* semiunivocal *if $F(x) \cap F(x') = \emptyset$ implies that $F(x) = F(x')$.*

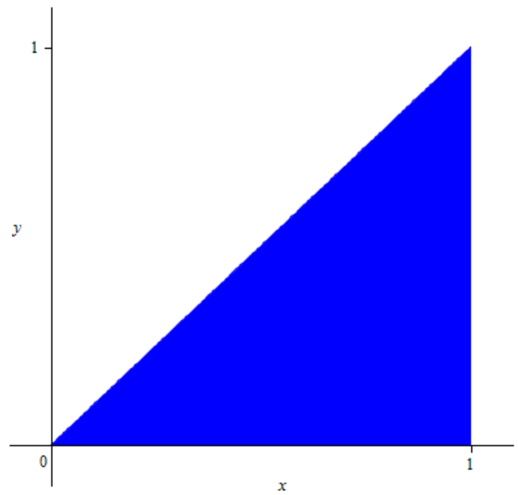

Fig. 1

It is obvious that a function is also a semi-univocal mapping.

Example 1.1 *A semi-univocal mapping which is not single-valued.*
Let $F : \mathbb{R}^n \setminus \{0\} \multimap \mathbb{R}^n$, $F(x) = \{\lambda x | \lambda > 0\}$.

Definition 1.5 *A mapping* $F : X \multimap Y$ *is called* one-to-one *if for any* $x, x' \in X$, $x \neq x'$ *we have* $F(x) \cap F(x') = \emptyset$; F *is* surjective (onto) *if* $R(F) = Y$.

Any one-to-one mapping is a semi-univocal one.

Example 1.2 *A semi-univocal mapping which is not injective.*
Let $F : X \multimap Y$, $F(x) = A$, *where* $A \neq \emptyset$ *is a fixed subset of* Y.

Definition 1.6 *The* inferior inverse *(or, briefly, the inverse) of the point-to-set mapping* $F : X \multimap Y$ *is the mapping denoted by* $F^- : Y \multimap X$ *and given by* $F^-(y) = \{x \in X | y \in F(x)\}$.
The effective domain of F^- *is* $R(F)$.
For $B \subset Y$, $B \neq \emptyset$ *we denote* $F^-(B) = \{x \in X | F(x) \cap B \neq \emptyset\}$. *We admit that* $F^-(\emptyset) = \emptyset$.

Example 1.3 *Let* $F : [0,1] \multimap [0,1]$ *given by* $F(x) = [0,x]$. *Then* $F^-(y) = [y,1]$, $\forall y \in [0,1]$. *We have also* $F^-\left(\left[\frac{1}{4}, \frac{3}{4}\right]\right) - \left[\frac{1}{4}, 1\right]$ *and* $F^-\left(\left[0, \frac{1}{4}\right]\right) = [0,1]$. *The graph of* F *is given in Fig.1.*

Definition 1.7 *The* superior inverse *of the mapping* $F : X \multimap Y$ *is* $F^+ : \mathcal{P}(Y) \to \mathcal{P}(X)$ *given by* $F^+(B) = \{x \in D(F) | F(x) \subset B\}$. *We put conventionally* $F^+(\emptyset) = \emptyset$.

Example 1.4 *For F in example 1.3 we have $F^+\left(\left[\frac{1}{4},\frac{3}{4}\right]\right)=\emptyset$ and $F^+\left(\left[0,\frac{1}{4}\right]\right)=\left[0,\frac{1}{4}\right]$.*

Remark 1.1 *It is true that $F^+(B)\subset F^-(B)$, $\forall B\subset Y$.*

Remark 1.2 *For the function $f:X\to Y$ we have $f^-(B)=f^+(B)=f^{-1}(B)$.*

2 Properties of the point-to-set mappings related to set operations

We will prove in this section some results for point-to-set mappings similar to those of Th. 1.6, Ch. I.

In the following we will consider a point-to-set mapping $F:X\multimap Y$, some subsets of X denoted by A, B, A_i, $i\in I$ and some subsets of Y denoted by M, N, M_j, $j\in J$.

Theorem 2.1 (1) $A\subset B\Rightarrow F(A)\subset F(B)$

(1′) $M\subset N\Rightarrow F^-(M)\subset F^-(N)$ *and* $F^+(M)\subset F^+(N)$.

Proof. (1) Let $y\in F(A)$: it means that there is an element $x\in A$ such that $y\in F(x)$. Because of $A\subset B$, we have $x\in B$ and $y\in F(x)$, i.e. $y\in F(B)$.

(1′) Let $x\in F^-(M)$: then $F(x)\cap M\neq\emptyset$. Because of $M\subset N$ we obtain $F(x)\cap N\neq\emptyset$ and $x\in F^-(N)$. If $x\in F^+(M)$, we have $x\in D(F)$ and $F(x)\subset M$; it follows that $x\in D(F)$ and $F(x)\subset N$, hence $x\in F^+(N)$. ∎

Theorem 2.2 (2) $F\left(\bigcup\limits_{i\in I}A_i\right)=\bigcup\limits_{i\in I}F(A_i)$

(2′) $F^-\left(\bigcup\limits_{j\in J}M_j\right)=\bigcup\limits_{j\in J}F^-(M_j)$ *and* $F^+\left(\bigcup\limits_{j\in J}M_j\right)\supset\bigcup\limits_{j\in J}F^+(M_j)$.

Proof. (2) $y\in F\left(\bigcup\limits_{i\in I}A_i\right)\Leftrightarrow\exists x\in\bigcup\limits_{i\in I}A_i,\ y\in F(x)\Leftrightarrow\exists i_0\in I,$

$\exists x\in A_{i_0},\ y\in F(x)\Leftrightarrow\exists i_0\in I,\ y\in F(A_{i_0})\Leftrightarrow y\in\bigcup\limits_{i\in I}F(A_i)$

(2′) $x\in F^-\left(\bigcup\limits_{j\in J}M_j\right)\Leftrightarrow F(x)\cap\left(\bigcup\limits_{j\in J}M_j\right)\neq\emptyset\Leftrightarrow\exists j_0\in J,\ F(x)\cap M_{j_0}\neq\emptyset\Leftrightarrow\exists j_0\in J,\ x\in F^-(M_{j_0})\Leftrightarrow x\in\bigcup\limits_{j\in J}F^-(M_j)$.

$x\in F^+(M_j)\Leftrightarrow\exists j_0\in J,\ x\in F^+(M_{j_0})\Leftrightarrow\exists j_0\in J,\ x\in D(F),$ $F(x)\subset M_{j_0}\Rightarrow x\in D(F),\ F(x)\subset\bigcup\limits_{j\in J}M_j\Leftrightarrow x\in F^+\left(\bigcup\limits_{j\in J}M_j\right)$. ∎

Remark 2.1 *The inclusion for F^+ in (2') is generally strict. Indeed, if for F from example 1.3, we denote $M_1 = \left[0, \frac{1}{4}\right]$ and $M_2 = \left[\frac{1}{2}, \frac{3}{4}\right]$, we obtain $F^+(M_1) = \left[0, \frac{1}{4}\right]$ and $F^+(M_2) = \emptyset$. We have $F^+(M_1) \cup F^+(M_2) = \left[0, \frac{1}{4}\right]$.*

But $M_1 \cup M_2 = \left[0, \frac{3}{4}\right]$ and $F^+\left(\left[0, \frac{3}{4}\right]\right) = \left[0, \frac{3}{4}\right] \supset_{\neq} \left[0, \frac{1}{4}\right]$.

Theorem 2.3 (3) $F\left(\bigcap\limits_{i\in I} A_i\right) \subset \bigcap\limits_{i\in I} F(A_i)$

$(3')$ $F^-\left(\bigcap\limits_{j\in J} M_j\right) \subset \bigcap\limits_{j\in J} F^-(M_j)$ *and* $F^+\left(\bigcap\limits_{j\in J} M_j\right) = \bigcap\limits_{j\in J} F^+(M_j)$.

Proof. (3) $y \in F\left(\bigcap\limits_{i\in I} A_i\right) \Leftrightarrow \exists x \in \bigcap\limits_{i\in I} A_i,\ y \in F(x) \Leftrightarrow \forall i \in I,\ \exists x \in A_i,$ $y \in F(x) \Rightarrow \forall i \in I,\ y \in F(A_i) \Leftrightarrow y \in \bigcap\limits_{i\in I} F(A_j)$.

$(3')$ $x \in F^-\left(\bigcap\limits_{j\in J} M_j\right) \Leftrightarrow F(x) \cap \left(\bigcap\limits_{j\in J} M_j\right) \neq \emptyset \Rightarrow \forall j \in J,\ F(x) \cap M_j \neq \emptyset \Leftrightarrow \forall j \in J,\ x \in F^-(M_j) \Leftrightarrow x \in \bigcap\limits_{j\in J} F^-(M_j)$.

$x \in F^+\left(\bigcap\limits_{j\in J} M_j\right) \Leftrightarrow x \in D(F),\ F(x) \subset \bigcap\limits_{j\in J} M_j \Leftrightarrow x \in D(F)$ and $\forall j \in J,\ F(x) \subset M_j \Leftrightarrow \forall j \in J,\ x \in F^+(M_j) \Leftrightarrow x \in \bigcap\limits_{j\in J} F^+(M_j)$. ■

Remark 2.2 *a) The inclusion in (3) is generally strict.*

Let $F : [0,5] \multimap [0,5]$ given by

$$F(x) = \begin{cases} [0,x],\ x \in [0,2] \cup [3,5] \\ [0,1],\ x \in (2,3) \end{cases},$$

whose graph is given in Fig.2.

For $A_1 = [0,3)$ and $A_2 = (2,5]$ we have $F(A_1) = [0,2]$ and $F(A_2) = [0,5]$, hence $F(A_1) \cap F(A_2) = [0,2]$. But $A_1 \cap A_2 = (2,3)$ and $F((2,3)) = [0,1] \subset_{\neq} [0,2]$.

b) The inclusion in $(3')$ related to F^- is also strict.

For F given in a) and $M_1 = \left[\frac{1}{2}, \frac{3}{2}\right]$, $M_2 = (1,2]$ we have $F^-(M_1) = \left[\frac{1}{2}, 5\right]$ and $F^-(M_2) = (1,2] \cup [3,5]$; $F^-(M_1) \cap F^-(M_2) = (1,2] \cup [3,5]$.

But $M_1 \cap M_2 = (1, \frac{3}{2}]$ and $F^-\left((1, \frac{3}{2}]\right) = (1, \frac{3}{2}] \cup [3,5] \subset_{\neq} F^-(M_1) \cap F^-(M_2)$.

Theorem 2.4 (4) $F(A \setminus B) \supset F^-(A) \setminus F(B)$

$(4')$ $F^-(M \setminus N) \supset F^-(M) \setminus F^-(N)$ *and* $F^-(M \setminus N) \subset F^+(M) \setminus F^+(N)$.

We obtain from $(4')$ that $F\left(\complement_X A\right) \supset F(X) \setminus F(A)$, $F^-\left(\complement_Y M\right) \subset D(F) \setminus F^-(M)$ and $F^+\left(\complement_Y M\right) \subset D(F) \setminus F^+(M)$.

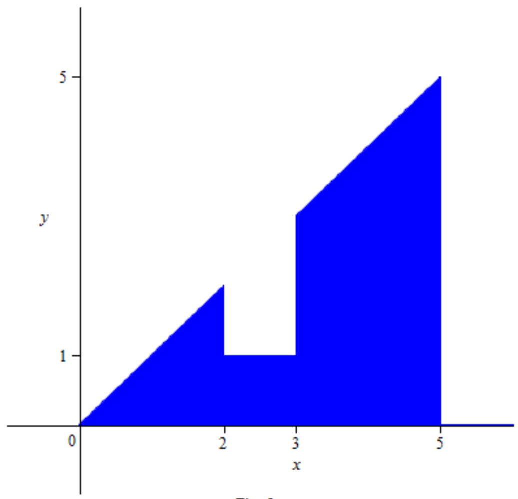

Fig. 2

Proof. (4) $y \in F(A) \setminus F(B) \Leftrightarrow y \in F(A)$ and $y \notin F(B) \Rightarrow \exists x \in A$, $y \in F(x)$, $x \notin B \Leftrightarrow \exists x \in A \setminus B$, $y \in F(x) \Rightarrow y \in F(A \setminus B)$.

(4′) $x \in F^-(M) \setminus F^-(N) \Leftrightarrow F(x) \cap M \neq \emptyset$ and $F(x) \cap N = \emptyset \Leftrightarrow F(x) \cap M \neq \emptyset$ and $F(x) \subset \complement_Y N \Leftrightarrow F(x) \cap M \neq \emptyset$ and $F(x) = F(x) \cap \complement_Y N \Rightarrow F(x) \cap \left(M \cap \complement_Y N\right) \neq \emptyset \Leftrightarrow F(x) \cap (M \setminus N) \neq \emptyset \Leftrightarrow x \in F^-(M \setminus N)$.

$x \in F^+(M \setminus N) \neq \emptyset \Leftrightarrow x \in D(F)$, $F(x) \subset M \setminus N \Leftrightarrow x \in D(F)$, $F(x) \subset M$, $F(x) \subset \complement_Y N \Rightarrow x \in F^+(M) \setminus F^+(N)$. ■

Remark 2.3 *All the inclusions in Th. 2.4 are strict.*

a) For F defined in Remark 2.2 a), $A = (2,5]$ and $B = [0,3)$ we have $F(A) = [0,5]$, $F(B) = [0,2]$ and $F(A) \setminus F(B) = (2,5]$. But $A \setminus B = [3,5]$ and $F(A \setminus B) = [0,5] \supset_{\neq} F(A) \setminus F(B)$.

b) For F defined in Example 1.3, $M = \left[0, \frac{1}{4}\right]$ and $N = \left[\frac{1}{4}, \frac{3}{4}\right]$ we have $F^-(M) = [0,1]$ and $F^-(N) = \left[\frac{1}{4}, 1\right]$, hence $F^-(M) \setminus F^-(N) = [0, \frac{1}{4})$. But $M \setminus N = [0, \frac{1}{4})$ and $F^-(M \setminus N) = [0,1] \supset_{\neq} F^-(M) \setminus F^-(N)$.

For the same F, we have $F^+(M) = \left[0, \frac{1}{4}\right]$ and $F^+(N) = \emptyset$; $F^+(M \setminus N) = [0, \frac{1}{4}) \subset_{\neq} F^+(M) \setminus F^+(N)$.

Remark 2.4 *If F is surjective, then $F\left(\complement_X A\right) \supset \complement_Y F(A)$; if it is bijective, $F\left(\complement_X A\right) = \complement_Y F(A)$.*

Theorem 2.5 (5) $A \subset F^-(F(A)) : A \cap D(F) \subset F^+(F(A))$

(5′) $F(F^-(M)) \supset M \cap R(F)$; $F(F^+(M)) \subset M$.

Proof. (5) $x \in A \Rightarrow F(x) \cap F(A) \neq \emptyset \Leftrightarrow x \in F^-(F(A))$

$x \in A \cap D(F) \Leftrightarrow x \in D(F)$, $x \in A \Rightarrow x \in D(F)$, $F(x) \subset F(A) \Leftrightarrow x \in F^+(F(A))$

(5′) $y \in M \cap R(Y) \Leftrightarrow \exists x \in X,\ y \in M \cap F(x) \Rightarrow \exists x \in F^{-}(M)$, $y \in F(x) \Leftrightarrow y \in F(F^{-}(M))$.

$y \in F(F^{+}(M)) \Leftrightarrow \exists x \in F^{+}(M),\ y \in F(x) \Leftrightarrow \exists x \in D(F),\ F(x) \subset M,\ y \in F(x) \Rightarrow y \in M$. ■

Remark 2.5 *All the inclusions in Th. 2.5 are generally strict.*

a) For F defined in Example 1.3 and $A = \left[\frac{1}{4}, \frac{3}{4}\right]$ we have $F(A) = \left[0, \frac{3}{4}\right]$; $F^{-}(F(A)) = [0,1] \supset_{\neq} A$; $F^{+}(F(A)) = \left[0, \frac{3}{4}\right] \supset_{\neq} A \cap D(F) = A$.

b) For the same F and $M = \left[\frac{1}{4}, \frac{3}{4}\right]$ we have $F^{-}(M) = \left[\frac{1}{4}, 1\right]$ and $F(F^{-}(M)) = [0,1]$, $M \cap R(F) = \left[\frac{1}{4}, \frac{3}{4}\right]$.

Now for F defined in Remark 2.2 a) and $M = \left[0, \frac{5}{2}\right]$, we obtain $F^{+}(M) = [0,3)$ and $F(F^{+}(M)) = [0,2] \subset_{\neq} M$.

Theorem 2.6 (6) *$F(F^{-}(M) \cap A) \supset M \cap F(A)$; $F(F^{+}(M) \cap A) \subset M \cap F(A)$.*

Proof. $y \in M \cap F(A) \Leftrightarrow \exists x \in A,\ y \in F(x),\ y \in M \Rightarrow \exists x \in A$, $y \in F(x),\ F(x) \cap M \neq \emptyset \Leftrightarrow \exists x \in A,\ x \in F^{-}(M),\ y \in F(x) \Leftrightarrow \exists x \in F^{-}(M) \cap A,\ y \in F(x) \Leftrightarrow y \in F(F^{-}(M) \cap A)$.

$y \in F(F^{+}(M) \cap A) \Leftrightarrow \exists x \in F^{+}(M) \cap A,\ y \in F(x) \Leftrightarrow \exists x \in A \cap D(F)$.

$F(x) \subset M,\ y \in F(x) \Rightarrow y \in M$ and $y \in F(A) \Leftrightarrow y \in M \cap F(A)$. ■

Remark 2.6 *The two inclusions in Th. 2.6 are strict.*

For F defined in Example 1.3, $M = \left[\frac{1}{4}, \frac{3}{4}\right]$, $A = \left[0, \frac{3}{4}\right]$ we have $F(A) = \left[0, \frac{3}{4}\right]$, hence $M \cap F(A) = \left[\frac{1}{4}, \frac{3}{4}\right]$.

But $F^{-}(M) = \left[\frac{1}{4}, 1\right]$ and $F^{-}(M) \cap A = \left[\frac{1}{4}, \frac{3}{4}\right]$; we obtain $F(F^{-}(M) \cap A) = \left[0, \frac{3}{4}\right] \supset_{\neq} M \cap F(A)$.

For F defined in Remark 2.2 a), $M = \left[0, \frac{5}{2}\right]$, $A = [2,4]$ we have $F(A) = [0,4]$ and $M \cap F(A) = \left[0, \frac{5}{2}\right]$. We obtain similarly $F^{+}(M) = [0,3)$ and $F^{+}(M) \cap A = [2,3)$, hence $F(F^{+}(M) \cap A) = [0,2] \subset_{\neq} M \cap F(A)$.

Remark 2.7 *The statements (1)-(6) in Th. 1.6, Ch. I are consequences of the theorems proved here, because of Remark 1.2.*

3 Operations with point-to-set mappings, properties

The images through a point-to-set mapping being sets, we can define operations with the point-to-set mappings according to the operations which can be done with the images through the respective mappings.

Definition 3.1 *Let F_1, $F_2 : X \multimap Y$ point-to-set mappings.*

The union *of the mappings F_1 and F_2 is a mapping denoted by $F_1 \cup F_2 : X \multimap Y$ given by $(F_1 \cup F_2)(x) = F_1(x) \cup F_2(x)$.*

The intersection *of F_1 and F_2 is a mapping denoted by $F_1 \cap F_2 : X \multimap Y$ given by $(F_1 \cap F_2)(x) = F_1(x) \cap F_2(x)$.*

The cartesian product *of F_1 and F_2 is a mapping denoted by $F_1 \times F_2 : X \multimap Y \times Y$ given by $(F_1 \times F_2)(x) = F_1(x) \times F_2(x)$.*

The composed *of $F : X \multimap Y$ and $G : Y \multimap Z$ is a mapping denoted by $G \circ F : X \multimap Z$ given by $(G \circ F)(x) = G(F(x))$.*

Theorem 3.1 *For F, F_1, $F_2 : X \multimap Y$, $G : Y \multimap Z$ and $A \subset X$ we have*

(1) $(F_1 \cup F_2)(A) = F_1(A) \cup F_2(A)$

(2) $(F_1 \cap F_2)(A) \subset F_1(A) \cap F_2(A)$

(3) $(F_1 \times F_2)(A) \subset F_1(A) \times F_2(A)$

(4) $(G \circ F)(A) = G(F(A))$

Proof.

(1) $y \in (F_1 \cup F_2)(A) \Leftrightarrow \exists x \in A,\ y \in (F_1 \cup F_2)(x) \Leftrightarrow \exists x \in A,\ y \in F_1(x) \cup F_2(x) \Leftrightarrow y \in F_1(A) \cup F_2(A)$

(2) $y \in (F_1 \cap F_2)(A) \Leftrightarrow \exists x \in A,\ y \in (F_1 \cap F_2)(x) \Leftrightarrow \exists x A,\ y \in F_1(x) \cap F_2(x) \Rightarrow y \in F_1(A) \cap F_2(A)$

(3) $(y_1, y_2) \in (F_1 \times F_2)(A) \Leftrightarrow \exists x \in A,\ (y_1, y_2) \in (F_1 \times F_2)(x) \Leftrightarrow$

$\Leftrightarrow \exists x \in A,\ y_1 \in F_1(x),\ y_2 \in F_2(x) \Rightarrow y_1 \in F_1(A),\ y_2 \in F_2(A) \Leftrightarrow$

$\Leftrightarrow (y_1, y_2) \in F_1(A) \times F_2(A)$

(4) $(G \circ F)(A) = \bigcup\limits_{x \in A} (G \circ F)(x) = \bigcup\limits_{x \in A} G(F(x)) = G \bigcup\limits_{x \in A} (F(x)) = G(F(A))$.

■

Remark 3.1 *The inclusions* (2) *and* (3) *are generally strict.*

Let F_1, $F_2 : [0,1] \multimap [0,1]$ given by $F_1(x) = [0, x]$ and $F_2(x) = [0, 1-x]$, $\forall x \in [0,1]$. Then

$$(F_1 \cap F_2)(x) = \begin{cases} x, x \in \left[0, \frac{1}{2}\right] \\ 1 - x, x \in \left(\frac{1}{2}, 1\right] \end{cases}.$$

For $A = \left[\frac{1}{4}, \frac{3}{4}\right]$ we have $F_1(A) = F_2(A) = \left[0, \frac{3}{4}\right]$, so $F_1(A) \cap F_2(A) = \left[0, \frac{3}{4}\right]$; on the other side, $(F_1 \cap F_2)(A) = \left[0, \frac{1}{2}\right] \subset_{\neq} F_1(A) \cap F_2(A)$. On the same conditions, $\left(\frac{3}{4}, \frac{3}{4}\right) \in F_1(A) \times F_2(A)$ but $\left(\frac{3}{4}, \frac{3}{4}\right) \notin (F_1 \times F_2)(A)$, so $(F_1 \times F_2)(A) \subset_{\neq} F_1(A) \times F_2(A)$.

Remark 3.2 *If we apply* (4) *of Th. 3.1 for functions, we obtain* (7) *of Th. 6.1, Ch. I; so this theorem is entirely a consequences of the results for point-to-set mappings applied in the special case of functions.*

Definition 3.2 *A mapping* $F : X \multimap Y$ *is called* constant *if* $F(x) = C$, $\forall x \in X$, *where* C *is a fixed subset of* Y.

A constant mapping F satisfies the condition $(F \cap G)(A) = F(A) \cap G(A)$, $\forall G : X \multimap Y$ a point-to-set mapping and $A \subset X$.

Definition 3.3 *The point-to-set mapping* $\mathrm{id}_X : X \multimap X$, $\mathrm{id}_X(x) = \{x\}$ *is called the identical mapping of the set* X.

The next two theorems show that the intersection and the cartesian product of mappings preserve the properties of semi-univocal or one-to-one mappings.

Theorem 3.2 *If* $F_1, F_2 : X \multimap Y$ *are semi-univocal, then* $F_1 \cap F_2$ *and* $F_1 \times F_2$ *are also semi-univocal.*

Proof.

$(F_1 \cap F_2)(x) \cap (F_1 \cap F_2)(x') \neq \emptyset \Rightarrow F_1(x) \cap F_1(x') \neq \emptyset$ and $F_2(x) \cap F_2(x') \neq \emptyset \Rightarrow F_1(x) = F_1(x')$ and $F_2(x) = F_2(x') \Rightarrow F_1(x) \cap F_2(x) = F_1(x') \cap F_2(x') \Rightarrow (F_1 \cap F_2)(x) = (F_1 \cap F_2)(x')$.

$(F_1 \times F_2)(x) \cap (F_1 \times F_2)(x') \neq \emptyset \Rightarrow (F_1(x) \times F_2(x)) \cap (F_1(x') \times F_2(x')) \neq \emptyset \Rightarrow F_1(x) \cap F_1(x') \neq \emptyset$ and $F_2(x) \cap F_2(x') \neq \emptyset \Rightarrow F_1(x) = F_1(x')$ and $F_2(x) = F_2(x') \Rightarrow F_1(x) \times F_2(x) = F_1(x') \times F_2(x') \Rightarrow (F_1 \times F_2)(x) = (F_1 \times F_2)(x')$. ■

Theorem 3.3 *If one of the mappings* $F_1, F_2 : X \multimap Y$ *is one-to-one, the mappings* $F_1 \cap F_2$ *and* $F_1 \times F_2$ *are also one-to-one.*

Proof. Let F_1 be one-to-one and $x \neq x'$ two points of X. F_1 being one-to-one, $F_1(x) \cap F_1(x') = \emptyset$, so $F_1(x) \cap F_1(x') \cap F_2(x) \cap F_2(x') = \emptyset$, i.e. $(F_1 \cap F_2)(x) \cap (F_1 \cap F_2)(x') = \emptyset$. It follows that $F_1 \cap F_2$ is one-to-one.

Similarly, we obtain from $F_1(x) \cap F_1(x') = \emptyset$ that $(F_1(x) \cap F_1(x')) \times (F_2(x) \cap F_2(x')) = \emptyset$, and then $(F_1(x) \times F_2(x)) \cap (F_1(x') \times F_2(x')) = \emptyset$. It follows that $(F_1 \times F_2)(x) \cap (F_1 \times F_2)(x') = \emptyset$, and $F_1 \times F_2$ is a one-to-one mapping. ■

The two inverses of a mapping have the properties mentioned in the following theorems, where $F : X \multimap Y$ is a mapping having $D(F) = X$.

Theorem 3.4 *For* $M \subset Y$ *we have*

(5) $\complement_X F^-(M) = F^+(\complement_Y M)$; $\complement_X F^+(M) = F^-(\complement_Y M)$.

Proof.

$x \in \complement_X F^-(M) \Leftrightarrow x \in X,\ F(x) \cap M = \emptyset \Leftrightarrow x \in D(F) = X,$ $F(x) \subset \complement_Y M \Leftrightarrow x \in F^+(\complement_Y M)$

$x \in \complement_X F^+(M) \Leftrightarrow x \in X,\ x \notin F^+(M) \Leftrightarrow x \in X,\ \exists y \in F(x),$ $y \notin M \Leftrightarrow x \in X,\ \exists y \in F(x),\ y \in \complement_Y M \Leftrightarrow x \in X,\ F(x) \cap \complement_Y M \neq \emptyset \Leftrightarrow$ $x \in F^-(\complement_Y M)$. ∎

Theorem 3.5 *The sets $M \subset Y$ for which $F^+(M) = F^-(M)$ are called* pure *sets and form a complemented lattice $\mathcal{M}$.*

Proof.

If $M \in \mathcal{M}$ we have also $\complement_Y M \in \mathcal{M}$ because applying Th. 3.4 we obtain $F^+(\complement_Y M) = \complement_X F^-(M) = \complement_X F^+(M) = F^-(\complement_Y M)$.

For $M_1,\ M_2 \in \mathcal{M}$, we have $F^-(M_1 \cup M_2) = F^-(M_1) \cup F^-(M_2) = F^+(M_1) \cup F^+(M_2) \subset F^+(M_1 \cup M_2)$ (by Th. 2.2) and $F^+(M_1 \cup M_2) \subset F^-(M_1 \cup M_2)$, hence $F^-(M_1 \cup M_2) = F^+(M_1 \cup M_2)$. It follows that $M_1 \cup M_2 \in \mathcal{M}$.

We have also $M_1 \cap M_2 \in \mathcal{M}$, because $\complement_Y(M_1 \cap M_2) = \complement_Y M_1 \cup \complement_Y M_2$. ∎

We have proved in Th. 2.5 that generally $A \subset F^-(F(A))$. The next theorem establishes the properties of the sets for which the relation takes place with equality.

Theorem 3.6 *The subsets $A \subset X$ for which $F^-(F(A)) = A$ are called* stable *and form a complemented lattice $\mathcal{A}$.*

Proof. We first prove that $\complement_X A \in \mathcal{A}$, if $A \in \mathcal{A}$. We have obviously $\complement_X A \subset F^-(F(\complement_X A))$. Let now $x \in F^-(F(\complement_X A))$, so $F(x) \cap F(\complement_X A) \neq \emptyset$. It follows that there is $x' \in \complement_X A$ with $F(x) \cap F(x') \neq \emptyset$. It we had $x \in A$, then $F(x') \cap F(A) \neq \emptyset$ and $x' \in F^-(F(A)) = A$, which is a contradiction. We obtain that $x \notin A$, hence $x \in \complement_X A$ and the inverse inclusion is also proved.

Let now $A_1,\ A_2 \in \mathcal{A}$; $F^-(F(A_1 \cup A_2)) = F^-(F(A_1) \cup F(A_2)) = F^-(F(A_1)) \cup F^-(F(A_2)) = A_1 \cup A_2$, so $A_1 \cup A_2 \in \mathcal{A}$ (we used Th. 2.2).

It follows easily that for $A_1,\ A_2 \in \mathcal{A}$ we have also $A_1 \cap A_2 \in \mathcal{A}$. ∎

Theorem 3.7 *The function $f : \mathcal{P}(X) \to \mathcal{P}(X)$, $f(A) = F^+(F(A))$ is a* closure function*, in the sense that the following conditions are fulfilled*

(6) $A \subset f(A)$

(7) $A \subset B \Rightarrow f(A) \subset f(B)$

(8) $f(f(A)) = f(A)$.

Proof.

(6) follows from Th. 2.5, because $X = D(F)$.

(7) follows from Th. 2.1.

To prove (8), we apply (6) and (7), so $A \subset f(A) \Rightarrow f(A) \subset f(f(A))$. For the inverse inclusion, let $x \in f(f(A))$, but $x \notin f(A)$. Because $x \notin f(A)$, there is $y \in F(x)$, $y \notin F(A)$. From $x \in f(f(A))$ we obtain $F(x) \subset F(f(A))$; but $y \in F(x)$ and $y \in F(f(A))$. There is then $t \in f(A)$ with $y \in F(t)$, so $F(t) \subset F(A)$ and $y \in F(t)$, hence $y \in F(A)$, which is a contradiction. It remains that our assumption was false and $x \in f(A)$, hence $f(f(A)) \subset f(A)$. From the two inclusions we obtain $f(f(A)) = f(A)$. ■

CHAPTER III

SEMICONTINUITY OF THE POINT-TO-SET MAPPINGS

If $F : X \multimap Y$ is a point-to-set mapping and X, Y have a topological structure, it is natural to try to find some continuity notions for F. Continuity ideas for point-to-set mappings appeared since 1926, in the papers of W. A. Wilson [35], L. S. Hill [12] and W. Hurewicz [14], for some special cases. In the years 1932-33, K. Kuratowski [16] and G. Bouligand [5] gave more general definitions. There followed other definitions given by many authors to be most adequate to the problems they studied. A historical exposition of the development of the theory of point-to-set mappings in the first half of our century was done by B. Mc. Allister [19].

It is not yet established a unitary terminology for the notions of continuity and semi-continuity. In recent times, some papers as [30] study the relation between the various definitions given already and they propose new names and new types of continuity. In the following we will take into account the trials made in time by the mathematicians for elaborating more adequate definitions and discovering the connections with the earlier ones [1, 6, 28, 29, 32].

The definitions admitted here are those assumed by Berge [1] which pretend very few conditions on the mapping F. For example, it is permitted to exist points where the image through F is the void set. The notion of upper semicompactness is that given by W. Sobieszek and P. Kowalski [30].

1 Lower semicontinuity

Let X and Y be topological spaces and $F : X \multimap Y$ a point-to-set mapping.

Definition 1.1 *The mapping F is called* lower semicontinuous (l.s.c.) *at $x_0 \in X$ if for any open set $U \subset Y$ with $U \cap F(x_0) \neq \emptyset$ there is a neighbourhood V of x_0 such that $F(x) \cap U \neq \emptyset$ for any $x \in V$.*

With the notations given in Ch. II, the condition in the definition of l.s.c. at x_0 is equivalent to the fact that for any open set $U \subset Y$ with $x_0 \in F^-(U)$ there is a neighbourhood V of x_0 such that $V \subset F^-(U)$.

If $F(x_0) = \emptyset$, then F is l.s.c. at x_0.

K. Kuratowski gave the next definition using sequences.

Definition 1.1' *The mapping F with $D(F) = X$ is lower semicontinuous at $x_0 \in X$ if for any sequence $(x_n)_{n\in\mathbb{N}}$ convergent to the* *limit x_0, and for any $y \in F(x_0)$ there is a sequence $(y_n)_{n\in\mathbb{N}}$ convergent to y and satisfying the condition $y_n \in F(x_n)$, $\forall n \in N$.*

The two definitions are not generally equivalent. The next theorems study the relation between these definitions. In Th. 1.1 - Th. 1.4 F is supposed to satisfy the condition $D(F) = X$.

Theorem 1.1 *If Y satisfies the first countability axiom a mapping l.s.c. at x_0 in the sense of D.1.1 is also l.s.c in the sense of D.1.1'.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be convergent to $x_0 \in X$ and $y \in F(x_0)$. We will obtain the sequence $(y_n)_{n\in\mathbb{N}}$ required in D.1.1'. By the first countability axiom, there is a countable fundamental system of neighbourhoods of y, with $U_{k+1} \subset U_k$ (Remark 2.3, Ch. I). We have obviously $U_k \cap F(x_0) \neq \emptyset$, $\forall k \in \mathbb{N}$. Using D 1.1 we obtain for every U_k a $V_k \in \mathcal{V}(x_0)$ such that $U_k \cap F(x) \neq \emptyset$, $\forall x \in V_k$.

Because $x_n \to x_0$ $(n \to \infty)$, there is $\{N_k\}_{k\in\mathbb{N}}$ with $N_{k+1} > N_k$ such that for any k we have $x_n \in V_k$ for $n > N_k$.

Let now $k = 1$. For $n \in \{1, 2, ..., N_1\}$ we choose arbitrarily $y_n \in F(x_n)$, and for $n \in \{N_1+1, N_1+2, ..., N_2\}$ we choose $y_{N_1+i} \in F(x_{N_1+i}) \cap U_{1,i}$, $i \in \{1, 2, ..., N_2 - N_1\}$. An analogous procedure applied for any k leads us to the sequence $(y_n)_{n\in\mathbb{N}}$ with $y_n \in F(x_n)$. We prove now that $y_n \to y$ $(n \to \infty)$.

Let W be a neighbourhood of y; there is then $k \in \mathbb{N}$ such that $U_k \subset W$. From the way of obtaining $(y_n)_{n\in\mathbb{N}}$ we have $y_n \in U_k \subset W$, $\forall n > N_k$, and the convergence of $(y_n)_{n\in\mathbb{N}}$ is proved. ■

Theorem 1.2 *If X satisfies the first countability axiom, then a mapping l.s.c. at x_0 in the sense of D.1.1' is also l.s.c. in the sense of D.1.1.*

Proof. We suppose that F is l.s.c. at x_0 in the sense of D.1.1' but not in the sense of D.1.1. There is then an open set $U \subset Y$, $U \cap F(x_0) \neq \emptyset$ such that for any neighbourhood V of x_0 there is $x \in V$ with $F(x) \cap U = \emptyset$. Let V_n a countable fundamental system of neighbourhoods of x_0, with $V_{n+1} \subset V_n$, $\forall n \in N$. By the assumption we made, for any $n \in \mathbb{N}$ there is $x_n \in V_n$ such that $F(x_n) \cap U = \emptyset$. We have of course $x_n \to x_0$ $(n \to \infty)$. Let $y \in F(x_0) \cap U$. By D.1.1' there is $y_n \in F(x_n)$, $y_n \to y$ $(n \to \infty)$, which contradicts the fact that $F(x_n) \cap U = \emptyset$, $\forall n \in \mathbb{N}$. It follows that our assumption was false, and the l.s.c. at x_0 in the sense of D.1.1' implies the l.s.c. in the sense of D.1.1. ■

Corollary 1.1 *If X and Y satisfy the first countability axiom, the two definitions of l.s.c. at. x_0 are equivalent.*

Remark 1.1 *If F is a single-valued mapping, D.1.1 represents the continuity in the sense of Cauchy, and D.1.1' that in the sense of Heine.*

Theorem 1.3 *[32] A point-to-set mapping F is l.s.c. at x_0 in the sense of D.1.1 iff the following condition is satisfied:*
(1) for any net $x : (D, \geq) \to X$ convergent to x_0 and any $y \in F(x_0)$ and $U \in \mathcal{V}(y)$, there is $d_U \in D$ such that for $d \geq d_U$, $F(x_d) \cap U \neq \emptyset$.

Proof. 1. Let F be l.s.c. at x_0 in the sense of D.1.1 and x a net convergent to x_0. Let $y \in F(x_0)$ and U an open neighbourhood of y. Because F is l.s.c. at x_0, there is $V \in \mathcal{V}(x_0)$ such that $F(x) \cap U \neq \emptyset$, $\forall x \in V$. x being convergent to x_0, there is $d_U \in D$ such that $x_d \in V$, $\forall d \geq d_U$. Therefore, we have $F(x_d) \cap U \neq \emptyset$, $\forall d \geq d_U$.

2. We suppose now that F is not l.s.c. at x_0. It follows that there are $y \in F(x_0)$ and $U \in \mathcal{V}(y)$ such that for any $V \in \mathcal{V}(x_0)$ there is $x_V \in V$ with $F(x_V) \cap U = \emptyset$. We consider the net $x : (\mathcal{V}(x_0), \supset) \to X$, where we choose $x_V \in V$ with $F(x_V) \cap U = \emptyset$; it is obvious that x converges to x_0. For the net x, the point $y \in F(x_0)$ and $U \in \mathcal{V}(y)$, the condition (1) is not fulfilled. Then the condition (1) implies the l.s.c. at x_0. ■

Remark 1.2 *If $F : X \multimap$ fulfils the condition*
(2) for any net $x : (D, \geq) \to X$ convergent to x_0 and $y_0 \in F(x_0)$ there is a net $y : (D, \geq) \to Y$ convergent to y_0, $y_d \in F(x_d)$, $\forall d \in D$,
then the condition (1) is also true.

Indeed, for any $U \in \mathcal{V}(y_0)$, from the convergence of the net y to y_0, we obtain that there is a $d_U \in D$ such that $y_d \in U$, $\forall d > d_U$. But $y_d \in F(x_d)$, hence $F(x_d) \cap U \neq \emptyset$.

We have then

Theorem 1.4 *If the mapping $F : X \multimap Y$ satisfies the condition (2), then F is l.s.c. at x_0 in the sense of D.1.1.*

We will consider in the following l.s.c. mappings in the sense of the definition with neighbourhoods.

Definition 1.2 *The mapping $F : X \multimap Y$ is called* lower semicontinuous (l.s.c.) *if it is l.s.c. at any point $x_0 \in X$.*

The next two theorems gives characterizations for the global l.s.c.

Theorem 1.5 *The mapping $F : X \multimap Y$ is l.s.c. iff for any open set $G \subset Y$, the set $\{x \in X |\, F(x) \cap G \neq \emptyset\} = F^-(G)$ is open in X.*

Proof. 1. Let F be l.s.c. and $G \subset Y$ an open set. If $F^{-}(G) = \emptyset$, it is obviously open; let now $F^{-}(G) \neq \emptyset$ and $x_0 \in F^{-}(G)$. We have $F(x_0) \cap G \neq \emptyset$ and by the definition of l.s.c. at x_0, there is $V \in \mathcal{V}(x_0)$ with $F(x) \cap G \neq 0$, $\forall x \in V$. It follows that $V \subset F^{-}(G)$, so $F^{-}(G)$ is a neighbourhood of every point of it, so it is an open set.

2. Let now $F^{-}(G)$ be an open set for any open set $G \subset X$. If $x_0 \in X$ is a point for which $F(x_0) = \emptyset$, F is l.s.c. at x_0. If $F(x_0) \neq \emptyset$, we consider an open set $U \subset Y$ such that $U \cap F(x_0) \neq \emptyset$, so $x_0 \in F^{-}(U)$. The set $F^{-}(U)$ is open and $F(x) \cap U \neq \emptyset$, $\forall x \in F^{-}(U)$ and it follows that F is l.s.c. at x_0. Because x_0 was chosen arbitrary in X, F is l.s.c. ■

Theorem 1.6 *The mapping $F : X \multimap Y$ is l.s.c. iff for any closed set $H \subset Y$, the set $\{x \in X |\, F(x) \subset H\}$ is closed in X.*

Proof. The proof is obvious using Th. 1.4 and the fact that $\complement_X(F^{-}(G)) = \{x \in X |\, F(x) \cap G = \emptyset\} = \{x \in X |\, F(x) \subset \complement_Y G\}$. ■

The property of l.s.c of mappings is related to the selection theorems.

Definition 1.3 *Let $F : X \multimap Y$ be a mapping with $D(F) = X$. A* selection of the mapping F *on the set X is a continuous function $f : X \to Y$ satisfying $f(x) \in F(x)$, $\forall x \in X$.*

E. Michael gives in [10] the following result.

Theorem 1.7 *If $F : X \multimap Y$ is a point-to-set mapping for which for any $x_0 \in X$ and $y_0 \in F(x_0)$ there are a neighbourhood $V \in \mathcal{V}(x_0)$ and a selection f for $F|_V$ with $f(x_0) = y_0$, then F is l.s.c.*

Proof. Let $x_0 \in X$ and $U \subset Y$ an open set with $U \cap F(x_0) \neq \emptyset$; hence there is $y_0 \in U \cap F(x_0)$. For x_0 and y_0 we find a neighbourhood $V \in \mathcal{V}(x_0)$ and a selection for $F|_V$, with $f(x_0) = y_0$. Because of the continuity of f, the set $f^{-}(U) = \{x \in X |\, f(x) \in U\}$ is open. Let $V' = V \cap f^{-}(U)$; V' is non-void $(x \in V')$ and it is a neighbourhood of x_0. Let now $x \in V'$ be an arbitrary chosen element; we have $x \in V$ and $f(x) \in U$. Because $x \in V$ we obtain $f(x) \in F(x)$, and it follows that $U \cap F(x) \neq \emptyset$, $\forall x \in V'$ and the definition of the l.s.c. is satisfied. ■

2 Upper semicontinuity and upper semicompactness

Let X and Y be topological spaces and $F : X \multimap Y$ a point-to-set mapping.

Definition 2.1 *The point-to-set mapping F is called* upper semicontinuous (u.s.c.) at $x_0 \in X$ *if for any open set $U \subset Y$ with $F(x_0) \subset U$ there is a neighbourhood V of x_0 such that $F(x) \subset U$ for any $x \in V$.*

Remark 2.1 *If $F(x_0) = \emptyset$, F is u.s.c. at x_0 iff there is $V \in \mathcal{V}(x_0)$ such that $F(V) = \emptyset$.*

Definition 2.2 *The mapping F is called* upper semicompact (u.s.co.) at $x_0 \in X$ *if F is u.s.c. at x_0 and $F(x_0)$ is compact.*

Definition 2.2' *[29] The mapping $F : X \multimap Y$ with $D(F) = X$ is u.s.co. at $x_0 \in X$ if for the sequence $(x_n)_{n\in\mathbb{N}}$ convergent to x_0 and $y_n \in F(x_n)$ there is a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ convergent to a point $y_0 \in F(x_0)$.*

In Th. 2.1 - Th. 2.4, which establish the relation between the different definitions, we will consider $F : X \multimap Y$ with $D(F) = X$.

Theorem 2.1 *If F is u.s.co. at x_0 in the sense of D.2.2, then it is also u.s.co. at x_0 in the sense of D.2.2'.*

Proof. We suppose that F is u.s.co. in the sense of D.2.2 but not in the sense of D.2.2'. Then there is a sequence $(x_n)_{n\in\mathbb{N}}$ convergent to x_0 and there are the elements $y_n \in F(x_n)$ such that for any $y \in F(x_0)$ there is an open neighbourhood $U(y)$ of y containing at most a finite number of members of the sequence $(y_n)_{n\in\mathbb{N}}$. We have $F(x_0) \subset \bigcup\limits_{y\in F(x_0)} U(y)$.

Because $F(x_0)$ is compact, we obtain a finite covering such that $F(x_0) \subset \bigcup\limits_{i=1}^{p} U(y_i)$, $y_i \in F(x_0)$, $i = \overline{1,p}$.

It follows that only a finite number of the members of the sequence $(y_n)_{n\in\mathbb{N}}$ can be in $U = \bigcup\limits_{i=1}^{p} U(y_i)$. F being u.s.co. in the sense of D.2.2, there is $V \in \mathcal{V}(x_0)$ such that $F(x) \subset U$ for any $x \in V(x_0)$. Because $x_n \to x_0\,(n \to \infty)$ there is $n_V \in \mathbb{N}$ such that $x_n \in V$, $\forall n > n_V$, hence $F(x_n) \subset U$, $\forall n > n_V$. It follows that $y_n \in U$, $\forall n > n_V$, which is in contradiction with the fact that only a finite number of the members of $(y_n)_{n\in\mathbb{N}}$ are in U. It follows that the assumption we made was false and the theorem is proved. ■

Theorem 2.2 *If X satisfies the first countability axiom and Y the second countability axiom, then F is u.s.co. at x_0 in the sense of D.2.2 iff it is u.s.co. at x_0 in the sense of D.2.2'.*

Proof. Using the result given in Th. 2.1 we have to show that the u.s.co. at x_0 in the sense of D.2.2' implies that in the sense of D.2.2.

By Th. 7.2, Ch. I, it is sufficient to show that $F(x_0)$ is sequential compact and its compactness will then follow. Let $(y_n)_{n\in\mathbb{N}}$, $y_n \in F(x_0)$, $n \in \mathbb{N}$; for $x_n = x_0$, $\forall n \in \mathbb{N}$, we have $y_n \in F(x_n)$ and $x_n \to x_0$ $(n \to \infty)$. Applying D.2.2' we obtain that there is a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ convergent to a point $x_0 \in F(x_0)$, hence $F(x_0)$ is sequential compact.

We suppose now that F is not u.s.co. at x_0 in the sense of D.2.2, so it is not u.s.c. at x_0. Then there is an open set U with $F(x_0) \subset U$ such that for any $V \in \mathcal{V}(x_0)$ there is $x \in V$ with $F(x) \not\subseteq U$. We obtain the existence of $y \in F(x)$, $y \notin U$. If $\{B_n\}_{n\in\mathbb{N}}$ is a fundamental system of neighbourhoods for x_0 with $B_{n+1} \subset B_n$, $\forall n \in \mathbb{N}$ then we can choose $(y_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in B_n$, $y_n \in F(x_n)$, $y_n \notin U$. By Definition 2.2' from $x_n \to x_0$ $(n \to \infty)$ it follows that there is (y_{n_k}) subsequence of $(y_n)_{n\in\mathbb{N}}$ convergent to $y_0 \in F(x_0)$. But from $y_n \notin U$ we obtain $y_0 \notin U$, contradiction with $F(x_0) \subset U$. ■

We can give a characterization of u.s.co. at x_0 using nets.

Theorem 2.3 *The mapping F is u.s.co. at x_0 in the sense of the D.2.2 iff the following condition is satisfied:*
(1) *for any net $x : (D, \geq) \to X$ convergent to x_0 and for any $y_d \in F(x_d)$, there is $y_0 \in F(x_0)$ cluster point for the net $y : (D, \geq) \to Y$.*

Proof. 1. We suppose that there is $x : (D, \geq) \to X$ convergent to x_0 and $y : (D, \geq) \to Y$ with $y_d \in F(x_d)$, $\forall d \in D$ such that any $y_0 \in F(x_0)$ is not a cluster point for y. Then there is $U(y_0) \in \mathcal{V}(y_0)$ open set such that y is not frequently in $U(y_0)$, i.e. there is $d_0 \in D$ with $y_d \notin U(y_0)$ for $d \geq d_0$. We have then $F(x_0) \subset \bigcup\limits_{y_0\in F(x_0)} U(y_0)$ and let $U_1, ..., U_p$ a finite covering. There is d' such that for any $d \geq d'$, $y_d \notin U = \bigcup\limits_{l=1}^{p} U_i$. Because F is u.s.co. at x_0, there is a neighbourhood $V \in \mathcal{V}(x_0)$ such that $F(x) \subset U$, $\forall x \in V$. The net x being convergent to x_0, there is $d' \in D$ such that for any $d \geq d'$, $x_d \in V$, hence $F(x_d) \subset U$. It follows that $y_d \in U$, $\forall d \geq d'$, which is a contradiction. So the condition (1) most be fulfilled.

2. Let now suppose that condition (1) is satisfied. We prove first that $F(x_0)$ is a compact set. Let $y_d \in F(x_0)$ and $x_d = x_0$, $\forall d \in D$, where $(D, \geq)$ is a directed set. There is then $y_0 \in F(x_0)$ cluster point for y and by Th. 7.4, Ch. I, $F(x_0)$ is compact.

We suppose now that F is not u.s.c. at x_0. There is an open set U with $F(x_0) \subset U$ such that for any $V \in \mathcal{V}(x_0)$ there is $x \in V$ with $F(x) \not\subseteq U$. We construct a net $x : (\mathcal{V}(x_0), \supset) \to X$, x_V being the element of V whose existence was just proved. Because $F(x_V) \not\subseteq U$, there is $y_V \in F(x_V) \cap \complement_Y U$. Therefore, for any $V \in \mathcal{V}(x_0)$, $y_V \notin U$.

$y_0 \in F(x_0)$ being a cluster point for $y : (\mathcal{V}(x_0), \supset) \to Y$, it follows that for any $V \in \mathcal{V}(x_0)$ there is $W \subset V$, with $y_W \in U$, which is a contradiction. So F is u.s.c. at x_0. ■

Using the result of T.5.6, Ch. I, the above theorem may be reformulated like this.

Theorem 2.4 *The mapping F is u.s.co. at x_0 in the sense of D.2.2 iff for any net $x : (D, \geq) \to X$ and for any net $y : (D, \geq) \to Y$, $y_d \in F(x_d), \forall d \in D$, there is a subnet of y convergent to an element of $F(x_0)$.*

From now on we will consider the notions in the sense of D.2.1 and 2.2.

Definition 2.3 *The mapping $F : X \multimap Y$ is called* upper semicontinuous *(*upper semicompact*) and is denoted by u.s.c. (u.s.co.) if it is upper semicontinuous (semicompact) at any point $x_0 \in X$.*

Theorem 2.5 *The mapping $F : X \multimap Y$ is u.s.c. iff for any open set $G \subset Y$, the set $\{x \in X | \, F(x) \subset G\} = M$ is open in X.*

Proof. 1. Let $F : X \multimap Y$ an u.s.c. mapping and $G \subset Y$ an open set. If $M = \emptyset$ it is obviously open. We suppose now that $M \neq \emptyset$ and take $x_0 \in M$ arbitrarily. Because F is u.s.c. at x_0, there is $V \in \mathcal{V}(x_0)$ such that $F(x) \subset G, \forall x \in V$. Therefore $V \subset M$ and M contains a neighbourhood of any of its points, being an open set.

2. Let us suppose that for any open set $G \subset Y$, the set M is open in X and prove that F is u.s.c.

Let $x_0 \in X$ and $G \subset Y$ an open set, $F(x_0) \subset G$; we have then $x_0 \in M$. Because M is open, it is a neighbourhood for x_0. For any $x \in M$, we have $F(x) \subset G$, and the condition of u.s.c. at x_0 is satisfied. ■

Using Th. 2.5 we obtain obviously.

Theorem 2.6 *The mapping $F : X \multimap Y$ is u.s.co. iff it is pointwise compact ($F(x_0)$ is a compact set for any $x_0 \in X$) and for any open set $G \subset Y$, the set $\{x \in X | \, F(x) \subset G\}$ is open in X.*

The next two theorem are consequences of Th. 2.5 and Th. 2.6.

Theorem 2.7 *The mapping $F : X \multimap Y$ is u.s.c. iff for any closed set $H \subset Y$, the set $\{x \in X | \, F(x) \cap H \neq \emptyset\} = F^{-}(H)$ is closed in X.*

Theorem 2.8 *The mapping $F : X \multimap Y$ u.s.co. iff it is pointwise compact and it satisfies the condition of Th. 2.7.*

The last theorem of this section contains an important property of a u.s.co. mapping.

Theorem 2.9 *If $F : X \multimap Y$ is u.s.co., the image $F(K)$ of a compact set $K \subset X$ is also a compact set.*

Proof. Let $\{G_i | \, i \in I\}$ an open covering of the set $F(K)$. For any $x \in K$ the set $F(x)$ is compact and can be covered by the union of a finite number of G_i's, let this finite union be G_x. Then $(\{z \in X | \, F(z) \subset G_x\})_{x \in K}$ is an open covering for K. If M_x denotes the set $\{z \in X | \, F(z) \subset G_x\}$ we obtain that there is a finite covering of K, $M_{x_1}, ..., M_{x_p}$. It follows that $G_{x_1}, ..., G_{x_p}$ will form a finite covering of $F(K)$; but every $G_{x_j}, j = \overline{1, p}$ is a finite union of G_i, $i \in I$, hence $F(K)$ has a finite covering obtained of $\{G_i | \, i \in I\}$. Therefore $F(K)$ is a compact set. ■

3 Closure

The closed mappings were studied in detail, by Berge [1]; some authors, like K. Kuratowski, named these mappings upper semicontinuous.

Definition 3.1 *The mapping $F : X \multimap Y$ is called* closed at $x_0 \in X$ *if for any $y_0 \notin F(x_0)$ there are two neighbourhoods $V \in \mathcal{V}(x_0)$ and $U \in \mathcal{V}(y_0)$ such that for any $x \in Y$, $F(x) \cap U = \emptyset$.*

Remark 3.1 *If $F(x_0) = \emptyset$, then F is closed at x_0 iff there is $V \in \mathcal{V}(x_0)$ such that $F(x) = \emptyset$, $\forall x \in V$.*

Another definition of closure is given in [29].

Definition 3.1' *The mapping $F : X \multimap Y$ with $D(F) = X$ is closed at $x_0 \in X$ if for any sequence $(x_n)_{n \in \mathbb{N}}$ converging to x_0 and $(y_n)_{n \in \mathbb{N}}$ converging to y_0, where $y_n \in F(x_n)$, $\forall n \in N$, we have $y_0 \in F(x_0)$.*

Remark 3.2 *The definition 3.1' is that given by Kuratowski for u.s.c.*

Theorem 3.1 *If F is closed at x_0 in the sense of D.3.1, the set $F(x_0)$ is closed.*

Proof. We prove that $\complement_Y F(x_0)$ is open in Y. If $\complement_Y F(x_0) = \emptyset$, it is obviously open. Let now $y \in \complement_Y F(x_0)$. The condition of D.3.1 guarantees the existence of $V \in \mathcal{V}(x_0)$ and $U \in \mathcal{V}(y_0)$ such that $F(x) \cap U = \emptyset$, $\forall x \in V$. It follows that $U \subset \complement_Y F(x_0)$, hence $\complement_Y F(x_0)$ is open in Y. It follows that $F(x_0)$ is closed in Y. ■

In the Theorems 3.2 - 3.4 we suppose that $D(F) = X$.

Theorem 3.2 *If the mapping $F : X \multimap Y$ is closed at x_0 in the sense of D.3.1, then it is also closed in the sense of D.3.1'.*

Proof. Let F be closed at x_0 in the sense of D.3.1, $(x_0)_{n\in\mathbb{N}}$ a sequence converging to x_0 and $(y_n)_{n\in\mathbb{N}}$ a sequence converging to y_0, where $y_n \in F(x_n)$, $\forall n \in \mathbb{N}$. We have to prove that $y_0 \in F(x_0)$. We suppose that $y_0 \notin F(x_0)$. Then there are two neighbourhoods $V \in \mathcal{V}(x_0)$ and $U \in \mathcal{V}(y_0)$ such that $F(x) \cap Y = \emptyset$, $\forall x \in V$. Using the convergence of $(x_n)_{n\in\mathbb{N}}$ we obtain that there is $n_0 \in \mathbb{N}$ such that $x_n \in V$, $\forall n > n_0$. Then $F(x_n) \cap U = \emptyset$, $\forall n > n_0$ and it follows that $y_n \in U$, $\forall n > n_0$, contradiction with $y_n \to y_0$ $(n \to \infty)$. It follows that F is also closed at x_0 in the sense of D.3.1'. ■

Theorem 3.3 *If X and Y satisfy the first countability axiom, F is closed at x_0 in the sense of D.3.1 iff it is closed in the sense of D.3.1'.*

Proof. Using the above theorem, we have to prove only that the closure in the sense of D.3.1' implies that in the sense of D.3.1.

Let F be closed in the sense of D.3.1' but not in the sense of D.3.1. There will be $y_0 \notin F(x_0)$ such that for any neighbourhood $V \in \mathcal{V}(x_0)$ and $U \in \mathcal{V}(y_0)$ there is $x \in V$ such that $F(x) \cap U \neq \emptyset$. Let $\{B_n\}_{n\in\mathbb{N}}$ and $\{U_n\}_{n\in\mathbb{N}}$ fundamental systems of open neighbourhoods of x_0 and y_0, with $B_{n+1} \subset B_n$ and $U_{n+1} \subset U_n$, $\forall n \in \mathbb{N}$. For any $n \in \mathbb{N}$ there is $x_n \in B_n$ such that $F(x_n) \cap U_n \neq \emptyset$, hence we can find a sequence $(y_n)_{n\in\mathbb{N}}$ such that $y_n \in F(x_n) \cap U_n$. It is obvious that $x_n \to x_0$ and $y_n \to y_0$ $(n \to \infty)$. The definition 3.1 leads us to $y_0 \in F(x_0)$ which is a contradiction with our assumption. ■

The next theorem gives a characterization of closure in the terms of nets.

Theorem 3.4 *The mapping $F : X \to Y$ is closed at x_0 in the sense of D.3.1 iff the following condition is satisfied:*
(1) *for any nets $x : (D, \geq) \to X$ and $y : (D, \geq) \to Y$ converging to x_0, respectively y_0, with $y_d \in F(x_d)$, $\forall d \in D$, it follows that $y_0 \in F(x_0)$.*

Proof. 1. Let F be closed at x_0, $x : (D, \geq) \to X$ converging to x_0, $y : (D, \geq) \to Y$ converging to y_0 such that $y_d \in F(x_d)$, $\forall d \in D$, but $y_0 \notin F(x_0)$. Because F is closed at x_0, there are two neighbourhoods $V \in \mathcal{V}(x_0)$ and $U \in \mathcal{V}(y_0)$ such that for any $x \in V$, $F(x) \cap U = \emptyset$. But the net x is convergent to x_0, so there is $d_0 \in D$ such that for $d \geq d_0$, $x_d \in V$. Then, for $d \geq d_0$, we have $y_d \notin U$ (since $y_d \in F(x_d)$), contradiction with the convergence of y to y_0. It follows that (1) is satisfied.

2. We prove now that (1) implies that F is closed at x_0. We suppose that F is not closed at x_0, so there is $y_0 \notin F(x_0)$ such that for any $V \in \mathcal{V}(x_0)$ and $U \in \mathcal{V}(y_0)$ there is $x \in V$ with $F(x) \cap U \neq \emptyset$. Let $\mathcal{V}(x_0) \times \mathcal{V}(y_0)$ be directed by the inclusion. We choose $x_{V,U} \in V$ and $y_{V,U} \in F(x_{V,U}) \cap U$. The existence of $y_{V,U}$ is guaranteed by the assumption that F is not closed at x_0.

The net x is convergent to x_0. Indeed, for $W \in \mathcal{V}(x_0)$, there is $d_0 = (W, Y)$ such that for any $(V, U) \geq (W, Y)$ (i.e., $V \subset W$ and $U \subset Y$) we have $x_{V,U} \in V \subset W$.

Similarly, y is convergent to y_0; let $U' \in \mathcal{V}(y_0)$. Then there is $d_0 = (X, U')$ such that for any $(V, U) \geq (X, U')$ (i.e. $V \subset X$ and $U \subset U'$) we have $y_{V,U} \in F(x_{V,U}) \cap U \subset U \subset U'$.

The nets x and y satisfy the hypothesis of (1), so we obtain $y_0 \in F(x_0)$ which is a contradiction. It follows that F is a closed mapping. ■

Definition 3.2 *The mapping $F : X \multimap Y$ is called* closed *if it is closed at any point $x_0 \in X$.*

Theorem 3.5 *The mapping $F : X \multimap Y$ is closed (in the sense of Definition 3.1) iff the graph $\Gamma(F)$ of F is a closed set in $X \times Y$.*

Proof. The condition from Definition 3.1 is equivalent to the fact that for any (x_0, y_0) there are two neighbourhoods $V \in \mathcal{V}(x_0)$, $U \in \mathcal{V}(y_0)$ such that $V \times U \subset \complement\Gamma(F)$ and the theorem follows immediately. ■

In the following we consider closed mappings in the sense of D.3.1.

Theorem 3.6 *If $\{F_i | i \in I\}$ is a family of closed mappings $F_i : X \multimap Y$ then $F : X \multimap Y$ given by $F = \bigcap\limits_{i \in I} F_i$ is also a closed mapping.*

Proof. Let $x_0 \in X$, $y_0 \in Y \setminus F(x_0)$; from the definition of $F(x_0)$ it follows that there is an index $i \in I$ such that $y_0 \notin F_i(x_0)$. We obtain then two neighbourhoods $V \in \mathcal{V}(x_0)$ and $U \in \mathcal{V}(y_0)$ such that $F_i(V) \cap U = \emptyset$. It follows that $F(V) \cap U = \emptyset$ and F is a closed mapping. ■

The next theorems study the relation between closed mappings and u.s.co. ones.

Theorem 3.7 *If Y is a Hausdorff space, any u.s.co. mapping $F : X \multimap Y$ is closed.*

Proof. Let $F : X \multimap Y$ be u.s.co. and $y_0 \notin F(x_0)$; because $F(x_0)$ is compact and Y is a Hausdorff space, there is in Y an open set G with $F(x_0) \subset G$ and a neighbourhood $U \in \mathcal{V}(x_0)$ such that $G \cap U = \emptyset$. Because F is u.s.c. there is $V \in \mathcal{V}(x_0)$ such that for any $x \in V$ we have $F(x) \subset G$, hence $F(x) \cap U = \emptyset$. So F is a closed mapping. ■

Theorem 3.8 *Let Y be a Hausdorff space. If $F_1 : X \multimap Y$ is a closed mapping and $F_2 : X \multimap Y$ is a u.s.co. mapping, then $F = F_1 \cap F_2$ is u.s.co.*

Proof. $F(x) = F_1(x) \cap F_2(x) \subset F_2(x)$ is a compact set. We prove that F is u.s.c.

Let $x_0 \in X$ and G open set with $F(x_0) \subset G$; we prove that there is $V \in \mathcal{V}(x_0)$ such that $F(V) \subset G$. If $F_2(x_0) \subset G$, this will be surely true, because F_2 in u.s.c. If $F_2(x_0) \not\subset G$, let $K = F_2(x_0) \cap \complement_Y G \neq \emptyset$. For any $y \in K$ we consider $U(y) \in \mathcal{V}(y)$ and $V_y \in \mathcal{V}(x_0)$ such that $F(V_y) \cap U(y) = \emptyset$. The set K being compact, there will be $y_1, ..., y_n \in K$ such that $U(y_1), ..., U(y_n)$ cover K. Let $U(K) = \bigcup_{i=1}^{n} U(y_i)$; there will be a neighbourhood $V' \in \mathcal{V}(x_0)$ such that for $x \in V'$, we have $F_2(V) \subset U(K) \cup G$. Let now $V = V_{y_1}(x_0) \cap ... \cap V_{y_n}(x_0) \cap V'$. We have $F_1(V(x_0)) \cap U(K) = \emptyset$ and $F_2(V(x_0)) \subset U(K) \cup G$, hence $(F_1 \cap F_2)(V) \subset G$. This proves that F is u.s.c. ■

Corollary 3.1 *If Y is a compact Hausdorff space, a u.s.c. mapping is closed iff it is u.s.co.*

Proof. 1. Let $F : X \multimap Y$ be closed. We consider $F_0 : X \multimap Y$ given by $F_0(x) = Y$, $\forall x \in X$, which is obviously u.s.co. By Th. 3.8 above, $F = F \cap F_0$ is u.s.co.

2. Let $F : X \multimap Y$ be u.s.co. By Th. 3.7 we obtain that F is closed. ■

Theorem 3.9 *If X is a compact Hausdorff space and $F : X \multimap X$ is a u.s.co. mapping with $D(F) = X$, then there is a compact set $K \neq \emptyset$ in X such that $F(K) = K$.*

Proof. We consider the sequence X, $F(X)$, $F^2(X)$,... of compact sets (by Th. 2.8) which are also non-void. If a member of the sequence is equal to its successor, the theorem is proved. We suppose that any two consecutive members are different.

Because $X \supset F(X)$, it follows that $F(X) \supset F^2(X)$ etc. so the sequence is decreasing; by Cor. 7.1, Ch. I we have $K = \bigcap_{n=1}^{\infty} F^n(X) \neq \emptyset$. Obviously K is a compact set an $K \subset F^{n-1}(X)$ for any $n \in \mathbb{N}$, hence $F(K) \subset F^n(X) \subset K$ and $F(K) \subset K$.

To prove inverse inclusion, let $x \in K$; then there is $x_n \in F^n(X)$ with $x \in F(x_n)$, $\forall n \in \mathbb{N}$. The sequence $(x_n)_{n \in \mathbb{N}}$ regarded as a net in X will have a subnet convergent to x_0 by Cor. 7.3, Ch. I. Because only a finite number of the members of the subnet are not in $F^n(X)$, it follows that $x_0 \in F^n(X)$, $\forall n \in \mathbb{N}$, so $x_0 \in K$. By Th. 3.4 we obtain $x \in F(x_0) \subset F(K)$. So, the inclusion $K \subset F(K)$ is also proved and $F(K) = K$. ■

4 Examples

The first examples of this section are concrete and prove that the notions of semicontinuity and closure are independent. We give then some generic examples.

Example 4.1 *A u.s.co. closed mapping which is not l.s.c.[32].*
Let $X = Y = [0,1]$ *and* $F : X \multimap Y$ *given by*

$$F(x) = \begin{cases} \{\frac{1}{2}x\}, x \in \left[0, \frac{1}{2}\right] \\ \{1 - \frac{1}{2}(1-x)\},\ x \in \left[\frac{1}{2}, 1\right] \end{cases}$$

and having the graph of Fig.1.

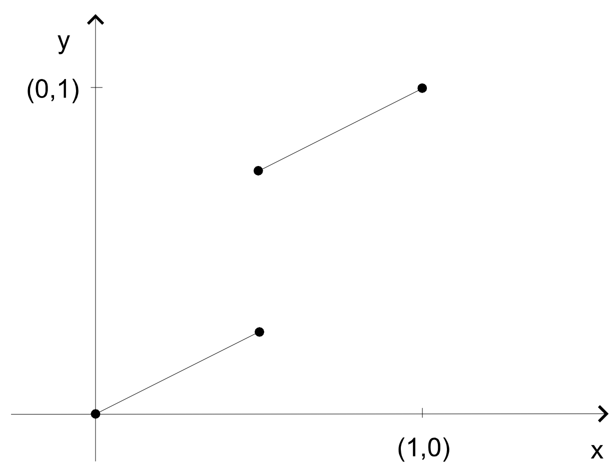

Figure 1:

F is obvious u.s.co. and closed; it is not l.s.c., because for $G = \left(\frac{1}{2}, 1\right)$ *we have* $F^-(G) = [\frac{1}{2}, 1)$*; it is not open in* X*.*

Example 4.2 *A l.s.c. closed mapping which is not u.s.c.[32].*
Let $X = Y = \mathbb{R}^2$ *and* $F : X \multimap Y$ *given by* $F(x,y) = \{(z,t) \in \mathbb{R}^2 | \, t = y\}$ *which acts like in Fig.2.*
F is not u.s.c.; we consider $U = \{(x,y) \in \mathbb{R}^2 | -1 < xy < 1\}$*, i.e. the set of points situated between the graphs of the equilateral hyperbolae* $xy = 1$ *and* $xy = -1$*. We have* $F((0,0)) \subset U$*, but any open set which contains* $(0,0)$ *contains also points* (x,y) *for which* $F(x,y) \nsubseteq U$*.*

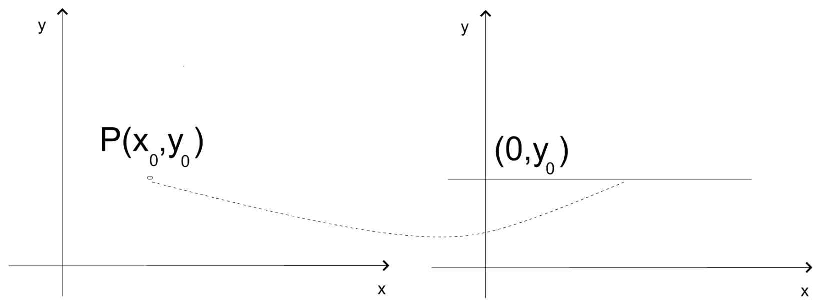

Figure 2:

Example 4.3 *A u.s.c and l.s.c. mapping which is not closed.*

Let $X = Y = \{(x, y) \in \mathbb{R}^2 | -1 \le x \le 1, -1 \le y \le 1\}$ *and* $F : X \multimap Y$ *given by*

$$F(x, y) = \{(z, t) \in \mathbb{R}^2 \big| \, z^2 + t^2 < \frac{1}{4}\}$$

The way of acting of F *is illustrated in Fig.3.*

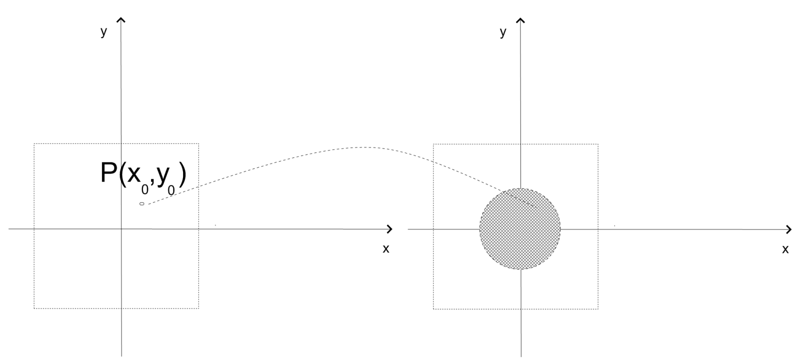

Figure 3:

F *is not closed, because* $\Gamma(F) = X \times \{(z, t) \in \mathbb{R}^2 | \, z^2 + t^2 < \frac{1}{4}\}$ *is not closed in* $X \times X$.

The next examples are general and contain in fact families of point-to-set mappings which have some of the studied properties.

Example 4.4 *Let* X,Y *be topological spaces and* $F : X \to Y$ *a continuous function. Then* $F : X \multimap Y$ *given by* $F(x) = \{f(x)\}$ *is u.s.co. and l.s.c.*

F *is obviously pointwise compact and* $F^{-}(A) = \{x \in X | \, F(x) \cap A \neq \emptyset\} = \{x \in X | \, f(x) \in A\} = f^{-1}(A)$. *Then* $f^{-1}(A)$ *is closed (open) if* A

is closed (open) (by Th. 6.3, Ch. I). Applying Th. 2.7 and Th. 1.4 it follows that F is u.s.co. and l.s.c.

If Y is a Hausdorff space, F is also closed. Let $y_0 \neq f(x_0)$ and $U \in \mathcal{V}(y_0)$, $V \in \mathcal{V}(f(x_0))$ such that $U \cap V = \emptyset$. For $V \in \mathcal{V}(f(x_0))$, there is $V' \in \mathcal{V}(x_0)$ with $f(x) \in V$ for any $x \in V'$. We have then, for any $x \in V'$, $F(x) \cap U = \emptyset$.

Example 4.5 *[26] Let X be a compact space and Y a Hausdorff one, $f : X \to Y$ a continuous function. Then the mapping $F : Y \multimap X$ given by $F(y) = \{x \in X |\, f(x) = y\}$ is u.s.co.*

Y being a Hausdorff space, the sets having one element are closed; f is continuous, so $F(y) = f^{-1}(y)$ is a closed set in the compact space X. It follows by Th. 7.8, Ch. I that $f^{-1}(y)$ is compact, so F is pointwise compact.

We prove that F is u.s.c. using Th. 2.6. Let $H \subset X$ a closed set; then $F^{-}(H) = \{y \in Y |\, F(x) \cap H \neq \emptyset\} = \{y \in Y |\, f^{-1}(y) \cap H \neq \emptyset\} = \{y \in Y |\, \exists x \in H, f(x) = y\} = f(H)$.

By Th. 7.13, Ch. I, $f(H)$ is closed and then F is u.s.c.

Example 4.6 *[1] Let $f : X \times Y \to \mathbb{R}$ a continuous function and $F : X \multimap Y$ given by $F(x) = \{y \in Y |\, f(x, y) \leq 0\}$. Then F is a closed mapping.*

If $y_0 \notin F(x_0)$, we have $f(x_0, y_0) > 0$. f being continuous, there are $V \in \mathcal{V}(x_0)$ and $U \in \mathcal{V}(y_0)$ with $f(x, y) > 0$ for any $x \in V$ and $y \in U$, so $F(V) \cap U = \emptyset$.

If (X, d) is a metric space and $\lambda : X \to \mathbb{R}$ a continuous function the map $F : X \multimap X$ given by $F(x) = \{y \in X |\, d(x, y) - \lambda(x) \leq 1\}$ is closed.

Example 4.7 *[26] Let X be a metric space and $F : X \multimap X$ a mapping whose values are non-void bounded closed sets. If F is Lipschitz in the sense that $D(f(x), f(y)) \leq \alpha d(x, y)$, $\forall x, y \in X$, then F is closed.*

X being a metric space, the first axiom of countability is satisfied and by Th.3.4 the two definitions of closed maps are equivalent. Let $x_n \to x$ and $y_n \to y$, $y_n \in F(x_n)$. We have

$$d(y, F(x)) \leq d(y, y_n) + d(y_n, F(x)) \leq d(y, y_n) + D(F(x_n), F(x)) \leq$$
$$d(y, y_n) + \alpha d(x_n, x) \overset{n\to\infty}{\to} 0$$

and it follows that $y \in \overline{F(x)} = F(x)$, hence F is closed map.

Example 4.8 *[8] Let (X, d) be a compact metric space having the fixed point property and $C(X) = \{f : X \to X |\, f \text{ continuous}\}$ endowed with* *the Tchebycheff metric. Let $F : C(X) \multimap X$ given by $F(f) = F_f$, where F_f denotes the set of the fixed points of f. The values of F are non-void and compact, and F is u.s.co.*

F_f is a closed subset of a compact space, so it is also compact. We prove that F is u.s.c. Let $U \subset X$ an open set with $F(f) \subset U$ and $\delta = \inf\{ d(x, f(x)) | \, x \in \complement_X U\}$. We have $\delta > 0$, excepting the case $U = X$, when obviously $F(g) \subset U$, $\forall g \in C(X)$. Let $B_\delta(f)$ be the open ball of center f and radius δ in $C(X)$. If $g \in B_\delta(f)$, we have $||f - g|| < \delta$. We show that $F(g) \subset U$.

Let $x^* \in F(g)$; we suppose that $x^* \notin U$. Then

$$\delta \leq d(x^*, f(x^*)) = d(g(x^*), f(x^*)) \leq ||f - g|| < \delta,$$

contradiction. It follows that $x^* \in U$ and F is u.s.c.

Example 4.9 *[27] Let (X, d) be a metric space and $F : X \multimap X$ a mapping with non-void compact values. If F is contractive in the sense that $D(F(x), F(y)) < d(x, y)$,$\forall x, y \in X$, $x \neq y$, then F is u.s.co.*

F being pointwise compact, we have to show that it is also u.s.c.

Let $x \in \overline{F^-(H)} \backslash F^-(H)$ and $(x_n)_{n\in\mathbb{N}}$ with $\lim\limits_{n\to\infty} x_n = x$, $x_n \in F^-(H)$, $x_n \neq x$ for any $n \in \mathbb{N}$. Because $x_n \in F^-(H)$, it follows $F(x_n) \cap H \neq \emptyset$; let $y_n \in F(x_n) \cap H$.

We have $d(y_n, F(x)) \leq D(F(x_n), F(x)) < d(x_n, x)$ and $\lim\limits_{n\to\infty} d(y_n, F(x)) = 0$.

But $d(y_n, F(x)) = \inf\limits_{y\in F(x)} d(y_n, y) = d(y_n, x'_n)$, with $x'_n \in F(x)$ (the infimum is taken at x'_n).

The sequence $(x'_n)_{n\in\mathbb{N}}$ is entirely in the compact $F(x)$, so it has a subsequence $\left(x'_{n_k}\right)_{k\in\mathbb{N}}$ convergent at $x_0 \in F(x)$.

We have $d(y_{n_k}, x_0) \leq d\left(y_{n_k}, x'_{n_k}\right) + d\left(x'_{n_k}, x_0\right) \overset{k\to\infty}{\rightarrow} 0$. It follows that $\lim\limits_{k\to\infty} y_{n_k} = x_0 \in F(x)$.

Because $y_{n_k} \in H$ and H is closed, $x_0 \in H$, hence $F(x) \cap H \neq \emptyset$. We have then $x \in F(H)$, contradiction with our hypothesis. It follows that $\overline{F^-(H)} = F^-(H)$ and $F^-(H)$ is a closed set.

Remark 4.1 *If F is not pointwise compact it is possible that F is not u.s.c., even if it is contractive.*

Indeed, let $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x, y) = \{(z, t) \in \mathbb{R}^2 | \, t = \frac{1}{2}y\}$. A similar proof as at Ex. 4.2 shows that F is not u.s.c., but

$$D(F(x, y), F(z, t)) = \frac{1}{2}|y - t| < \sqrt{(x - z)^2 + (y - t)^2} \text{ for } (x, y) \neq (z, t).$$

5 Properties of semicontinuous and closed point-to-set mappings

The first properties of this section are related to the operations with mappings. For a family $F_i : X \multimap Y$, $i \in I$ of mappings, let $\bigcup\limits_{i\in I} F_i : F \multimap Y$ be the mapping given by $\left(\bigcup\limits_{i\in I} F_i\right)(x) = \bigcup\limits_{i\in I} F_i(x)$. Similarly, $\bigcap\limits_{i\in I} F_i : X \multimap Y$ will be given by $\left(\bigcap\limits_{i\in I} F_i\right)(x) = \bigcap\limits_{i\in I} F_i(x)$. These mappings are called the union (respectively the intersection) of the family F_i, $i \in I$.

Theorem 5.1 *The union $F = \bigcup\limits_{i\in I} F_i$ of a family of l.s.c. mappings is a l.s.c. mapping.*

Proof. Let $G \subset Y$ be an open set. We have $F^{-}(G) = \{x \in X | \bigcup\limits_{i\in I} F_i(x) \cap G \neq \emptyset\} = \bigcup\limits_{i\in I} F_i^{-}(G)$, so $F^{-}(G)$ is an open set and F is l.s.c. ■

Theorem 5.2 *The intersection $F = \bigcap\limits_{i\in I} F_i$ of a family of u.s.co. mappings is a u.s.co. mapping.*

Proof. Let $i_0 \in I$; using Th. 3.6 we obtain that $F_0 = \bigcap\{F_i | i \in I \setminus \{i_0\}\}$ is a closed mapping. By Th. 3.8 it follows that $F = F_0 \cap F_{i_0}$ is u.s.co. as the intersection of a closed mapping with a u.s.co. one. ■

Remark 5.1 *In Th. 3.6 we proved that the intersection of a family of closed mappings is closed.*

Theorem 5.3 *The intersection $F = \bigcap\limits_{i=1}^{n} F_i$ of a finite family of l.s.c. mappings $F_i : X \multimap Y$, $i = \overline{1,n}$ is generally not a l.s.c. mapping.*

Proof. Let $G \subset Y$ be an open set. We have $F^{-}(G) = \{x \in X | \bigcap\limits_{i=1}^{n} F_i(x) \cap G \neq \emptyset\} \subseteq \bigcap\limits_{i=1}^{n} F_i^{-}(G)$, so it is not necessarily an open set and F may be not l.s.c. ■

Theorem 5.4 *The union $F = \bigcup\limits_{i=1}^{n} F_i$ of a finite family of u.s.co. mappings $F_i : X \multimap Y$, $i = \overline{1,n}$ is also a u.s.co. mapping.*

Proof. For any $x \in X$, the set $F(x) = \bigcup_{i=1}^{n} F_i(x)$ is compact as a finite union of compact sets. For the open set $G \subset Y$, we have $\{x \in X | \bigcup_{i=1}^{n} F_i(x) \subset G\} = \bigcap_{i=1}^{n} \{x \in X | F_i(x) \subset G\}$, so it is an open set and F is u.s.co. ■

Theorem 5.5 *The union $F = \bigcup_{i=1}^{n} F_i$ of a finite family of closed mappings $F_i : X \multimap Y$, $i = \overline{1,n}$ is also a closed mapping.*

Proof. Let $x_0 \in X$ and $y_0 \in Y \setminus F(x_0)$. We have $y_0 \notin F(x_0)$, hence $y_0 \notin F_i(x_0)$, $i = \overline{1,n}$. The mapping F_i, $i = \overline{1,n}$ being closed, there are $U_i \in \mathcal{V}(y_0)$ and $V_i \in \mathcal{V}(x_0)$ with $F_i(V_i) \cap U_i = \emptyset$, $i = \overline{1,n}$. Let $U = \bigcap_{i=1}^{n} U_i$ and $V = \bigcap_{i=1}^{n} V_i$.

Because $V \subset V_i$, $i = \overline{1,n}$ we obtain by Th. 2.1, Ch. II that $F_i(V) \subset F_i(V_i)$; we have then $F(V) \subset \bigcup_{i=1}^{n} F_i(V_i)$. It follows $F(V) \cap U \subset \bigcup_{i=1}^{n} (F_i(V_i)) = \bigcup_{i=1}^{n} (F_i(V_i) \cap U) = \emptyset$, and F is closed. ■

Remark 5.2 *The proofs of Th. 5.3, 5.4 and 5.5 show that the properties of these theorems are not generally true for infinite families of mappings.*

Theorem 5.6 *The cartesian product $F = \prod_{i=1}^{n} F_i$ of a finite family of l.s.c. mappings $F_i : X \multimap Y_i$ is a l.s.c. mapping $F : X \multimap \prod_{i=1}^{n} Y_i$ given by $F(x) = (F_1(x), ..., F_n(x))$ for any $x \in X$.*

Proof. Let $G \subset Y$ an open set; G will be a union of elementary open sets $E^k = \prod_{i=1}^{n} G_i^k$, G_i^k being open sets in Y_i, $i = \overline{1,n}$.

We have $F^-\left(E^k\right) = \{x \in X | \prod_{i=1}^{n} F_i(x) \cap \prod_{i=1}^{n} G_i^k \neq \emptyset\} = \bigcap_{i=1}^{n} F_i^-\left(G_i^k\right)$. We obtain that $F^-\left(E^k\right)$ is an open set, so $F^-(G) = \bigcup_{k \in K} F^-\left(E^k\right)$ is open; it follows that F is l.s.c. ■

Theorem 5.7 *The cartesian product $F = \prod_{i=1}^{n} F_i$ of a family of u.s.co. mappings $F_i : X \multimap Y_i$ is a u.s.co. mapping $F : X \multimap \prod_{i=1}^{n} Y_i$.*

Proof. We give the proof for $n = 2$, so $F : X \multimap Y_1 \times Y_2$. By Th. 7.13, Ch. I., $F_i(x)$ and $F_2(x)$ being compact sets, it follows that $F(x) = F_1(x) \times F_2(x)$ is a compact set for any $x \in X$.

Let $G \subset Y_1 \times Y_2$ and $a \in \{x \in X | \, F(x) \subset G\}$ (if this set is void, it is obviously open). The set $F(a)$, being compact and closed in G, can be covered with a finite number of elementary open sets included in G, let then be $E^1, E^2, ..., E^p$.

For $(y_1, y_2) \in F(a)$, let $E(y_i) = \cup\{E^k \big| \, E^k \cap (\{y_1\} \times F_2(a)) \neq \emptyset\}$ and $E(y_2) = \cup\{| \, E^k \big| \, E^k \cap (F_1(a) \times \{y_2\}) \neq \emptyset\}$. By the projection of $E(y_1)$ on Y_2 we obtain the open set $\pi_2 E(y_1)$; when y_1 varies we obtain a finite number of such sets. The situation is similarly for $\pi_1 E(y_2)$. Let $G_1 = \bigcap_{y_2 \in F_2(a)} \pi_1 E(y_2)$ and $G_2 = \bigcap_{y_1 \in F_1(a)} \pi_2 E(y_1)$.

The set $E = G_1 \times G_2$ is an elementary open set in Y and $\{x \in X | \, F_1(x) \times F_2(x) \subset E\} = \{x \in X | \, F_1(x) \subset G_1\} \cap \{x \in X | \, F_2(x) \subset G_2\}$, so it is an open set in X.

From $F(a) \subset E \subset G$, we have $a \in \{x \in X | \, F_1(x) \times F_2(x) \subset E\} \subset \{x \in X | \, F(x) \subset G\}$, and the set $\{x \in X | \, F(x) \subset G\}$ being a neighbourhood for any of its points is open. It follows that F is u.s.co. ■

Theorem 5.8 *The cartesian product* $F = \prod_{i=1}^{n} F_i$ *of a family of closed mappings* $F_i : X \multimap Y_i$ *is a closed mapping* $F : X \multimap \prod_{i=1}^{n} Y_i$.

Proof. Let $x_0 \in Y$ and $y_0 \notin F(x_0)$, $y_0 = (y_1, ..., y_n)$; there is then an index $i \in \{1, ..., n\}$ such that $y_i \notin F_i(x_0)$, let $i = 1$. Because F_1 is a closed set, there is $V \in \mathcal{V}(x_0)$ and $U_1 \in \mathcal{V}(y_1)$ with $F_1(V) \cap U_1 = \emptyset$. The set $U = U_1 \times Y \times ... \times Y \in \mathcal{V}(y_0)$ and $F(V) \cap U = \emptyset$ (because of $F_1(V) \cap U = \emptyset$), hence F is closed. ■

The next theorems show that the composed mapping has the same properties as the initial ones.

Theorem 5.9 *Let* $F_1 : X \multimap Y$ *and* $F_2 : Y \multimap Z$ *l.s.c mappings. Then the composed mapping* $F = F_2 \circ F_1 : X \multimap Z$ *is a l.s.c. mapping.*

Proof. Let $G \subset Z$ be an open set and $F^-(G) = \{x \in X | \, F_2 \circ F_1(X) \cap G \neq \emptyset\} = \{x \in X | \, F_2(F_1(x)) \cap G \neq \emptyset\} = \{x \in X | \, F_1(x) \cap F_2^-(G) \neq \emptyset\} = F_1^-\left(F_2^-(G)\right)$. Because F_1 and F_2 are l.s.c. mappings, $F^-(G)$ is an open set and F is l.s.c. ■

Theorem 5.10 *Let* $F_1 : X \multimap Y$ *and* $F_2 : Y \multimap Z$ *u.s.co. mappings. Then the composed mapping* $F = F_2 \circ F_1 : X \multimap Z$ *is a u.s.co. mapping.*

Proof. For $x \in X$, $F(x) = F_2(F_1(x))$ is a compact set, because $F_1(x)$ is compact and for the u.s.co. mapping F_2 one can apply Th. 2.9.

For $G \subset Z$ open set, we have $\{x \in X | F_2 \circ F_1(x) \subset G\} = \{x \in X | F_1(x) \subset M\} = N$, where $M = \{x \in X | F_2(x) \subset G\}$ is an open set, so N is also open. It follows that F is a u.s.co. mapping. ∎

The Theorem 2.9 shows that u.s.co. mappings preserve the compactness; this is not generally true for the l.s.c. and u.s.c. mapping (Ex. 4.3). The next theorem is related to a class of mappings preserving the connectedness.

Theorem 5.11 *Let $F : X \multimap Y$ be a l.s.c. (u.s.c.) mapping. If F is pointwise connected, then $F(C)$ is connected for any connected set C.*

Proof. Let F be l.s.c. Suppose that $F(C)$ is disconnected, so there are two non-void connected sets A_1 and A_2 with $F(C) = A_1 \cup A_2$, $A_1 \cap \overline{A}_2 = \emptyset$, $\overline{A}_1 \cap A_2 = \emptyset$. Let $B_i = \{x \in C | F(x) \subset A_i\}$, $i = \overline{1,2}$.

We have obviously $B_1 \cup B_2 \subset C$. For $x \in C$, we have $F(x) \subset A_1$ or $F(x) \subset A_2$, because $F(x)$ is connected. It follows that $C = B_1 \cup B_2$. We have $B_i \neq \emptyset$, $i = \overline{1,2}$, for $A_i \neq \emptyset$, $i = \overline{1,2}$.

Let $x \in B_1$ and $y \in F(x)$. Suppose that $F(x) \subset A_1$; then $y \in A_1$ and form $A_1 \cap \overline{A}_2 = \emptyset$ we obtain an open neighbourhood $U \in \mathcal{V}(y)$ with $U \cap A_2 = \emptyset$. F being a l.s.c. mapping, there is $V \in \mathcal{V}(x)$ an open neighbourhood with $F(z) \cap A_1 \neq 0$, $\forall z \in V$. For any $z \in C \cap V$ we have $F(z) \cap A_1 \neq \emptyset$ and $F(z) \subset A_1$. It follows that $V \cap B_2 = \emptyset$, so $B_1 \cap \overline{B}_2 = \emptyset$. Similarly one shows $\overline{B}_1 \cap B_2 = \emptyset$, contradiction with the connectedness of C. It follows that $F(C)$ must be connected.

Let now F be u.s.c and $F(C)$ disconnected. For A_1 and A_2 defined above we put $B_i = \{x \in C | F(x) \subset A_i\}$, $i = \overline{1,2}$. We have obviously $B_1 \cup B_2 \subset C$. For $x \in C$, we have $F(x) \subset A_1$, or $F(x) \subset A_2$, so $C = B_1 \cup B_2$; $A_i \neq \emptyset$ implies $B_i \neq \emptyset$, $i = \overline{1,2}$.

Let $x \in B_1$; then $F(x) \subset A_1 \subset \complement_Y \overline{A}_2$. Because $\complement_Y \overline{A}_2$ is open and $F(x) \subset \complement_Y \overline{A}_2$ we have that there is $V \in \mathcal{V}(x)$ with $F(z) \subset \complement_Y \overline{A}_2$ for any $z \in V$. Let $z \in V \cap B_2$; then $F(z) \subset \complement_Y \overline{A}_2$ and $F(z) \subset A_2 \subset \overline{A}_2$, which is a contradiction. It follows that $V \cap B_2 = \emptyset$, and $B_1 \cap \overline{B}_2 = \emptyset$. Similarly, $\overline{B}_1 \cap B_2 = \emptyset$, contradiction with the connectedness of C. It follows that $F(C)$ is a connected set. ∎

At the end of this section we give the maximum theorem whose preliminary results shows the connection between the semicontinuity of functions and of certain mappings.

Definition 5.1 *Let $(X, \mathcal{T})$ be a topological space and $\mathbb{R}$ endowed with the usual topology. A function $f : X \to \mathbb{R}$ is called* lower (upper) semicontinuous *at x_0 if for any $\varepsilon > 0$ there is a neighbourhood $V \in \mathcal{V}(x_0)$ such that $f(x_0) - \varepsilon < f(x)$ $(f(x) < f(x_0) + \varepsilon)$ for any $x \in V$.*

These notions for functions were given in 1899 by R. Baire.

Theorem 5.12 *If $f : X \times Y \multimap \mathbb{R}$ is a lower semicontinuous function and $F : X \multimap Y$ is a l.s.c. mapping with $F(x) \neq \emptyset$, $\forall x \in X$, then the real valued function $M : X \to \mathbb{R}$, $M(x) = \sup\{f(x,y) | \, y \in F(x)\}$ is l.s.c.*

Proof. Let $x_0 \in X$ and $\varepsilon > 0$; the definition of $M(x_0)$ implies the existence of $y_0 \in F(x_0)$ such that $f(x_0, y_0) > M(x_0) - \frac{\varepsilon}{2}$. The function f being l.s.c. there is $V \in \mathcal{V}(x_0)$ and $U \in \mathcal{V}(y_0)$ such that for any $(x,y) \in V \times U$, $f(x,y) > f(x_0, y_0) - \frac{\varepsilon}{2} > M(x_0) - \varepsilon$.

Because F is a l.s.c. mapping, there is $V' \in \mathcal{V}(x_0)$ with $F(x) \cap U \neq \emptyset$, $\forall x \in V'$. For $x \in V \cap V'$ we have $M(x) > M(x_0) - \varepsilon$, and M is lower semicontinuous. ■

Theorem 5.13 *If $f : X \times Y \to \mathbb{R}$ is an upper semicontinuous function and $F : X \multimap Y$ is a u.s.co. mapping with $F(x) \neq \emptyset$, $\forall x \in X$, then the real valued function $M(x) = \sup\{f(x,y) | \, y \in F(x)\}$ is upper semicontinuous.*

Proof. Let $x_0 \in X$ and $\varepsilon > 0$; f being upper semicontinuous, for any $y \in F(x_0)$ there is $V_y \in \mathcal{V}(x_0)$ and $U_y \in \mathcal{V}(y)$ such that for $(x,z) \in V_y \times U_y$ we have $f(x,z) < f(x_0, y) + \varepsilon$.

$F(x_0)$ being a compact set, it can be covered by a finite number of neighbourhoods $U_{y_1}, ..., U_{y_n}$. For $V = \bigcap\limits_{i=1}^{n} V_{y_i}$ and $U = \bigcup\limits_{i=1}^{n} U_{y_i}$ we have

$x \in V$, $y \in U \Rightarrow f(x,y) \leq \max\limits_{i=\overline{1,n}} f(x_0, y_i) + \varepsilon \leq M(x_0) + \varepsilon$.

For U we obtain $V' \in \mathcal{V}(x_0)$ such that $x \in V'$ implies $F(x) \subset U$, so $x \in V' \cap V$ and $M(x) = \max\limits_{y \in F(x)} f(x,y) \leq M(x_0) + \varepsilon$. ■

Theorem 5.14 *(the maximum theorem) Let $f : Y \to \mathbb{R}$ a continuous function and $F : X \multimap Y$ a l.s.c. and u.s.co. mapping with $F(x) \neq \emptyset$, $\forall x \in X$. Then the function $M : X \to \mathbb{R}$, $M(x) = \max\{f(y) | \, y \in F(x)\}$ is continuous and $\Phi(x) = \{y \in F(x) | \, f(y) = M(x)\}$ is a u.s.co. mapping $\Phi : X \multimap Y$.*

Proof. By Th. 5.12 and Th. 5.13 above it follows that M is a continuous function. The mapping $G : X \multimap Y$, $G(x) = \{y \in Y | \, M(x) - f(y) \leq 0\}$ is closed (Ex.4.6). It follows that $\Phi = F \cap G$ is u.s.co., by Th. 3.8. ■

CHAPTER IV

CONTINUITY OF THE POINT-TO-SET MAPPINGS

Definition 1 *The mapping $F : X \multimap Y$ is called* continuous at $x_0 \in X$ *if it is l.s.c. and u.s.c. at x_0.*
Definition 2 *The mapping $F : X \multimap Y$ is called* continuous *if it is continuous at any point $x \in X$.*

The values of F being sets, the point-to-set mapping F may be considered as a function having the values in a family of subsets of Y. It is then possible to define a topology on families of subsets of a given set and to study the point-to-set mappings as functions. It was in 1905 when D. Pompeiu [25] defined a metric on the set of non-void bounded closed subsets of the complex plane. The one who introduced a similar metric for arbitrary metric spaces was F. Hausdorff; because his book [11] published in 1914 was well-known, the metric is usually called the Hausdorff metric.

Later, in 1921, L. Vietoris defined the topology which has now his name, demanding no more than the initial space be a metric one [33]. There are also other topologies necessary to characterize the semi-continuity of point-to-set mappings [10, 23].

1 The Hausdorff metric

Definition 1.1 *Let (X, d) a metric space and 2^X the family of the non-void bounded closed subsets of X. For $A, B \in 2^X$ we take, according to the definitions given by Pompeiu and Hausdorff*

$$\rho(x, B) = \inf\{ d(x, y) | \, y \in B\}$$
$$\rho(y, A) = \inf\{ d(x, y) | \, x \in A\}.$$

We denote $\rho(A, B) = \sup\{ \rho(x, B) | \, x \in A\}$ and $\rho(B, A) = \sup\{ (y, A) | \, y \in B\}$.

It is called the Hausdorff distance *of the sets A and B the number $D(A, B) = \max\{\rho(A, B), \rho(B, A)\}$.*

Remark 1.1 *Defining $D(A, B) = \inf\{ d(x, y) | \, x \in A, \ y \in B\}$ we do not obtain a metric, because D does not satisfy the triangular inequality. Indeed, for the subsets of $\mathbb{R}$ $A = \{0\}$, $B = [1, 2]$ and $C = \{3\}$ we have $D(A, B) = 1, D(B, C) = 1$ and $D(A, C) = 3$.*

Theorem 1.1 *The function $D : 2^X \times 2^X \to \mathbb{R}$ is a metric.*

Proof. We have $D(A,B) \geq 0$ for any $A, B \in 2^X$ because the infimum and supremum are taken for non-negative numbers.

1^0 Let $D(A,B,) = 0$. Then $\sup\limits_{x\in A} \rho(x,B) = 0$ and $\sup\limits_{y\in B} \rho(y,A) = 0$ so $A \subset \overline{B} = B$ and $B \subset \overline{A} = A$. It follows then $A = B$.

2^0 $D(A,B) = D(B,A)$ follows immediately from the definition.

3^0 $D(A,C) \leq D(A,B) + D(B,C)$.

d being a metric, we have $d(x,z) \leq d(x,y) + d(y,z)\,, \forall x,y,z \in X$. Using the properties of the infimum and supremum, we obtain

$$
\begin{aligned}
&\inf\{\, d(x,z) |\, z \in C\} \leq d(x,y) + \inf_{z\in C} d(y,z) \\
&\rho(x,C) \leq d(x,y) + \rho(y,C) \leq d(x,y) + D(B,C) \\
&\rho(x,C) \leq \inf\{\, d(x,y) |\, y \in B\} + D(B,C) = \rho(x,B) + D(B,C) \leq \\
&D(A,B) + D(B,C) \\
&\sup\{\, \rho(x,C) |\, x \in A\} \leq D(A,B) + D(B,C) \\
&\rho(A,C) \leq D(A,B) + D(B,C)\,.
\end{aligned}
$$

We obtain similarly $\rho(z,A) \leq D(C,B) + D(B,A)$ and $\sup\{\, \rho(z,A) |\, z \in C\} \leq D(A,B) + D(B,C)$.

It follows $\rho(C,A) \leq D(A,B) + D(B,C)$, so $D(A,C) \leq D(A,B) + D(B,C)$.

We have proved that $\left(2^X, D\right)$ is a metric space. ∎

Theorem 1.2 *For $A, B \in 2^X$ we have the equivalence*

$$
D(A,B) \leq \varepsilon \Leftrightarrow \begin{cases} A \subset V_\varepsilon(B) \\ B \subset V_\varepsilon(A) \end{cases}
$$

where $V_\varepsilon(A) = \{x \in X |\, \rho(x,A) \leq \varepsilon\}$.

Proof. $D(A,B) \leq \varepsilon \Leftrightarrow \rho(A,B) \leq \varepsilon$ and $\rho(B,A) \leq \varepsilon \Leftrightarrow$

$\Leftrightarrow \forall x \in A,\ \forall y \in B : \rho(x,B) \leq \varepsilon$ and $\rho(y,A) \leq \varepsilon \Longleftarrow \forall y \in B,\ \forall x \in A$,

$x \in V_\varepsilon(B)$ and $y \in V_\varepsilon(A) \Leftrightarrow A \subset V_\varepsilon(B)$ and $B \subset V_\varepsilon(A)$. ∎

Theorem 1.3 *If B is a compact set, then $A \subset V_\varepsilon(B)$ iff $V(x) \cap B \neq \emptyset$ for any $x \in A$.*

Proof. $A \subset V_\varepsilon(B) \Leftrightarrow \forall x \in A,\ \rho(x,B) \leq \varepsilon \Leftrightarrow \forall x \in A$,

$\inf\{\, d(x,y) |\, y \in B\} \leq \varepsilon \Leftrightarrow \forall x \in A,\ \exists y_0 \in B : d(x,y_0) \leq \varepsilon$ (B being a compact set) $\Leftrightarrow \forall x \in A,\ \exists y_0 \in V_\varepsilon(x) \cap B \Leftrightarrow \forall x \in A,\ V_\varepsilon(x) \cap B \neq \emptyset$. ∎

Because a point-to-set mapping may be considered as a function with values in 2^Y, it is important to study the properties inherited by $\left(2^X, D\right)$ from (X,d). To prove the inheritance of the compactness we give at first two lemmas.

Lemma 1.1 *If (X,d) is totally bounded, $\left(2^X, D\right)$ is totally bounded too.*

Proof. Let $\varepsilon > 0$. We prove that there is an ε-net in 2^X. Putting $\varepsilon' = \frac{\varepsilon}{2}$, we obtain in X an ε'-net $A = \{a_1, ..., a_n\}$. It follows that for any $x \in X$ there is $a_i \in A$ with $d(x, a_i) < \varepsilon'$. Let $H_1, ..., H_k$ be the system of all the non-void subsets of A. This system forms an ε-net in 2^X.

Indeed, let $M \in 2^X$; let i be the index of the subset $H_i = \{x \in A | d(x, M) < \varepsilon'\}$. Because $d(x, M) < \varepsilon'$, for any $x \in H_i$, we have that $\rho(H_i, M) \leq \varepsilon'$. For $y \in M$ arbitrarily chosen, there is $a_j \in A$ with $d(y, a_j) < \varepsilon'$; from the construction of H_i, we have $a_j \in H_i$. It follows that $d(y, H_i) \leq \varepsilon'$ for any $y \in M$, hence $\rho(M, H_i) \leq \varepsilon'$. The fact that $\rho(H_i, M) \leq \varepsilon'$ and $\rho(M, H_i) \leq \varepsilon'$ implies $D(H_i, M) \leq \varepsilon' < \varepsilon$, so $\{H_i\}_{i=\overline{1,k}}$ is an ε-net in 2^X; the space 2^X is then totally bounded. ■

Lemma 1.2 *If the space (X,d) is complete, $\left(2^X, D\right)$ is also complete.*

Proof. Let $(A_n)_{n\in\mathbb{N}}$ be a fundamental sequence in 2^X and $A = \{x \in X | \forall V \in \mathcal{V}(x), \forall n_0 \in \mathbb{N}, \exists n > n_0$ such that $V \cap A_n \neq \emptyset\}$ (A is the superior limit of the sequence of sets in the sense of Kuratowski).

We shall prove that A is the limit of the sequence $(A_n)_{n\in\mathbb{N}}$ in the space $\left(2^X, D\right)$.

Let $\varepsilon > 0$ and $\varepsilon' = \varepsilon/3$. The sequence $(A_n)_{n\in\mathbb{N}}$ being fundamental, there is $n_\varepsilon \in \mathbb{N}$ such that $D(A_n, A_{n_\varepsilon}) < \varepsilon', \forall n > n_\varepsilon$. We prove at first that $D(A, A_n) \leq 2\varepsilon'$.

a) Let $x \in A$; for n defined above, there is a number $n > n_\varepsilon$ such that $B(x, \varepsilon') \cap A_n \neq \emptyset$; hence we obtain $a_n \in A_n$ with $d(x, a_n) < \varepsilon'$. It follows that $\rho(x, A_{n_\varepsilon}) = d(x, A_{n_\varepsilon}) \leq d(x, a_n) + d(a_n, A_{n_\varepsilon}) < \varepsilon' + D(A_n, A_{n_\varepsilon}) < 2\varepsilon'$, so $\rho(x, A_{n_\varepsilon}) < 2\varepsilon'$ for any $x \in A$.

b) Let $x \in A_{n_\varepsilon}$. We denote $n_k = n_{\varepsilon'/2^k}$, considering $n_k > n_{k-1}$.

We obtain x_{n_0}, x_{n_1},... in the following way:

$$x_{n_0} = x \in A_{n_0} = A_n.$$

For $x_{n_k} \in A_{n_k}$, we take $x_{n_{k+1}} \in A_{n_{k+1}}$, $d\left(x_{n_{k+1}}, x_{n_k}\right) < \frac{\varepsilon'}{2^k}$ (this is possible, $(A_n)_{n\in\mathbb{N}}$ being a fundamental sequence).

We have $d(x_{n_m}, x_{n_k}) < \frac{\varepsilon'}{2^{k-1}}$, $\forall m > k$, $k \geq 0$. It follows that $(x_{n_k})_{k\in\mathbb{N}}$ is a fundamental sequence in the complete space (X,d), so there is $y = \lim\limits_{k\to\infty} x_{n_k} \in A$.

For $k = 0$, we have $d(x_{n_m}, x) < 2\varepsilon'$, $\forall m \in \mathbb{N}$. When $m \to \infty$, we obtain $d(y, x) \leq 2\varepsilon'$. It follows that $\rho(x, A) = d(x, A) \leq 2\varepsilon'$, $\forall x \in A_{n_\varepsilon}$.

The results from a) and b) imply that $D(A, A_{n_\varepsilon}) \leq 2\varepsilon'$.

For $n > n_\varepsilon$ we have $D(A, A_n) \leq D(A, A_{n_\varepsilon}) + D(A_{n_\varepsilon}, A_n) < 2\varepsilon' + \varepsilon' = \varepsilon$ so $D(A, A_n) < \varepsilon$ and the sequence $(A_n)_{n\in\mathbb{N}}$ converges to A in $\left(2^X, D\right)$.
■

Using Lemma 1.1, Lemma 1.2 and Th. 8.17, Ch. I we obtain.

Theorem 1.4 *If the metric space* (X, d) *is compact, then* $\left(2^X, D\right)$ *is also compact.*

2 The Vietoris finite topology

In this section are exposed some topologies on systems of subsets of a topological space $(X, \mathcal{T})$. The chosen system is the family of the non-void closed sets, which will be denoted by 2^X. One could topologize similarly the system $\mathcal{A}(X)$ of the non-void subsets of X, but the obtained topological space would not have suitable separation properties. So, in this section 2^X denotes the non-void closed subsets of X.

An important topology on 2^X is the Vietoris finite topoligy. To introduce this topology, we give the following definition.

Definition 2.1 *Given the system* $(A_i)_{i=\overline{1,n}}$ *of non-void subsets of* X*, we define the set* $\langle A_1, ..., A_n \rangle = \{ M \in 2^X \big| \, M \subset \bigcup\limits_{i=1}^{n} A_i, \; M \cap A_i \neq \emptyset, \; i = \overline{1, n} \}$.

Theorem 2.1 *The sets* $\langle A_1, ..., A_n \rangle$*, with* A_i*,* $i = \overline{1,n}$ *non-void open subsets of the topological space* X *form a basis for a topology on* 2^X.

Proof. We prove the theorem using Th. 2.2, Ch. I.

Let $M \in 2^X$, we have $M \in \langle X \rangle$, so $2^X \subset \langle X \rangle$. It follows that $2^X = \langle X \rangle$.

Let now $U = \langle A_1, ..., A_n \rangle$ and $V = \langle B_1, ..., B_m \rangle$. We denote $A = \bigcup\limits_{i=1}^{n} A_i$ and $B = \bigcup\limits_{j=1}^{m} B_j$. We have $U \cap V = \{ M \in 2^X \big| \, M \subset A \cap B,$ $M \cap A_i \neq \emptyset, \; i = \overline{1,n}, \; M \cap B_j \neq \emptyset, \; j = \overline{1,m} \} =$
$\{ M \in 2^X \big| \, M \subset \bigcup\limits_{i=1}^{n} \bigcup\limits_{j=1}^{m} \left[(A_i \cap B) \cup (A \cap B_j) \right], \; M \cap (A \cap B_j \neq \emptyset), \; j = \overline{1,m}, \; M \cap (A_i \cap B) \neq \emptyset, \; i = \overline{1,n} \} =$
$\langle A_1 \cap B, ..., A_n \cap B, A \cap B_1, ..., A \cap B_m \rangle$. ■

Definition 2.2 *The topology generated by* $(\langle A_1, ..., A_n \rangle)$ *is called the* Vietoris finite topology.

Before studying the properties of this topology, we introduce two other topologies on 2^X, which will serve to characterize the semicontinuity of mappings.

Theorem 2.2 *The sets* $\{A \in 2^X \,|\, A \subset U\}$, *where* U *is an open set in* X *form a basis for a topology on* 2^X.

The proof is analogous to that of Th. 2.1.

Definition 2.3 *The topology generated by the basis considered in Th. 2.2 is called the* upper semi-finite topology.

Theorem 2.3 *The sets* $\{A \in 2^X \,|\, A \cap U \neq \emptyset\}$, *where* U *is an open set in* X *form a subbasis for a topology on* 2^X.

The theorem is proved showing that the system of the finite intersections of the considered sets determines a basis, following the method from Th. 2.1.

Definition 2.4 *The topology generated by the subbasis considered in Th. 2.3 is called the* lower semi-finite topology.

In the following we consider some important properties of 2^X endowed with the Vietoris finite topology.

The next two theorems give a method to construct closed sets in 2^X.

Theorem 2.4 *If* $A \in 2^X$, *the set* $\{E \in 2^X \,|\, E \subset A\}$ *is closed in* 2^X.

Proof. If $A \in 2^X$, it is a closed set and $\complement_X A$ is open; but $\complement_{2^X}\{E \in 2^X \,|\, E \subset A\} = \{E' \in 2^X \,|\, E' \cap \complement_X A \neq \emptyset\} = \langle X, \complement_X A \rangle$, so it is open. It follows that the set mentioned in the theorem is closed. ■

Theorem 2.5 *If* $A \in 2^X$, *the set* $\{E \in 2^X \,|\, E \cap A \neq \emptyset\}$ *is closed in* 2^X.

Proof. If A is closed, $\complement_X A$ is open; $\complement_{2^X}\{E \in 2^X \,|\, E \cap A \neq \emptyset\} = \{E' \in 2^X \,|\, E' \subset \complement_X A\} = \langle \complement_X A \rangle$, hence the given set is closed. ■

We give in the following some separation properties for 2^X.

Theorem 2.6 *The space* 2^X *is* T_0.

Proof. Let $A, B \in 2^X$, $A \neq B$; there is then $x \in B \setminus A$ (or $x \in A \setminus B$). We have $B \in \langle X, \complement_X A \rangle$ and $A \notin \langle X, \complement_X A \rangle$; because $\langle X, \complement_X A \rangle$ is open in 2^X, it follows that 2^X is T_0. ■

Theorem 2.7 *If* X *is a* T_1*-space,* 2^X *is* T_1 *too.*

Proof. Let A, $B \in 2^X$, $A \neq B$ and $x \in B \setminus A$. X being T_1, $\{x\}$ is a closed set (Th. 3.1, Ch. I), so $\complement_X\{x\}$ is open in X. The next sets, which are open in 2^X, $\langle X, \complement_X A \rangle$ and $\langle \complement_X\{x\} \rangle$ fulfil the conditions:

$B \in \langle X, \complement_X A \rangle$, $A \notin \langle X, \complement_X A \rangle$ and $A \in \langle \complement_X\{x\} \rangle$, $B \notin \langle \complement_X\{x\} \rangle$.

It follows that 2^X is a T_1-space. ■

Remark 2.1 *The fact that 2^X is T_1 does not imply that X is T_1. Indeed, for $X = \{x, y\}$ endowed with the indiscrete topology (which is not T_1), $2^X = \langle X \rangle$ is a T_1-space.*

The next theorem gives a fundamental system of neighbourhoods of $\{x\} \in 2^X$, X being a T_1-space.

Theorem 2.8 *If $(U_\alpha)_{\alpha \in A}$ is a fundamental system of neighbourhoods of $x \in X$, then $(\langle U_\alpha \rangle)_{\alpha \in A}$ is a fundamental system of neighbourhoods of $\{x\}$ in 2^X, X being considered T_1.*

Proof. Let $V \in \mathcal{V}(\{x\})$ in 2^X; there is then an open set U such that $\{x\} \in U \subset V$. The set U is a union of elements of the basis $\left\langle A_1, ..., A_{n_j} \right\rangle$, $j \in J$, so there is an index $j \in J$ such that $x \in A_i$, $i = \overline{1, n_j}$. We obtain $x \in \bigcap\limits_{i=1}^{n_j} A_i$, so $\bigcap\limits_{i=1}^{n_j} A_i \in \mathcal{V}(x)$. Because $(U_\alpha)_{\alpha \in A}$ is a fundamental system of neighbourhoods of x, there is $\alpha \in A$ such that $x \in U_\alpha \subset \bigcap\limits_{i=1}^{n_j} A_i$.

We prove that $\langle U_\alpha \rangle \subset V$; indeed, for $M \in \langle U_\alpha \rangle$ we have $M \subset U_\alpha \subset \bigcap\limits_{i=1}^{n_j} A_i$, so $M \cap A_i \neq \emptyset$, $i = \overline{1, n_j}$ and $M \subset \bigcup\limits_{i=1}^{n_j} A_i$. It follows that $M \in \left\langle A_1, ..., A_{n_j} \right\rangle \subset U \subset V$ and the system $(\langle U_\alpha \rangle)_{\alpha \in A}$ is a fundamental system of neighbourhoods for $\{x\}$. ∎

Until the end of this section we consider that the topological space X is T_1.

Theorem 2.9 *The topological space X is T_3 iff 2^X is a Hausdorff space.*

Proof. 1. Let X be a T_3 space and $A, B \in 2^X$, $A \neq B$, $x \in B \setminus A$ (or $x \in A \setminus B$). For x and A there are two open sets U and V such that $A \subset U$, $x \in V$ and $U \cap V = \emptyset$. We have then $A \in \langle U \rangle$, $B \in \langle X, V \rangle$ and $\langle U \rangle \cap \langle X, V \rangle = \emptyset$, so 2^X is a Hausdorff space.

2. We suppose that X is not T_3, so there is $M \in 2^X$ and $x \notin M$ such that x and M cannot be separated with open sets in X. We consider the sets M and $M \cup \{x\}$, which are elements in 2^X. Let G and G' arbitrary open sets in 2^X with $M \in G$ and $M \cup \{x\} \in G'$. We have $G = \cup \langle A_1, ..., A_n \rangle$ and $G' = \cup \langle B_1, ..., B_m \rangle$. It follows that $M \subset \bigcup\limits_{i=1}^{n} A_i = A$ and $x \in B_j$, $j = \overline{1, m}$. X being not T_3, we have $A \cap B_j \neq \emptyset$, $j = \overline{1, m}$. We have $G \cap G' = \cup (\langle A_1, ..., A_n \rangle \cap \langle B_1, ..., B_m \rangle) = \cup \langle A_1 \cap B, ..., A_n \cap B, A \cap B_1, ..., A \cap B_m \rangle \neq \emptyset$, so 2^X is not Hausdorff.

It follows that X is T_3. ∎

The next theorem concerns the compactness of the spaces X and 2^X, generalizing Th. 1.4 where X and 2^X are metric spaces.

Theorem 2.10 *The topological space X is compact iff 2^X is compact.*

Proof. 1. Let X be a compact space. For the open sets A and B, with $B \subset A$ we consider $K(A, B) = \{C \in 2^X \,|\, C \subset A \text{ or } C \cap B \neq \emptyset\}$. The system $(K(A, B))_{A,B\in\mathcal{T}}$ is a subbasis of the topology on 2^X, because $\langle A_1, ..., A_n\rangle = K\left(\bigcup\limits_{i=1}^{n} A_i, \emptyset\right) \cap \left(\bigcap\limits_{i=1}^{n} K(X, A_i)\right)$. We prove the compactness of 2^X using Th. 7.5, Ch. I.

Let $(K(A_\alpha, B_\alpha))_{\alpha\in A}$ a covering of 2^X. For $x \in X$, we have $\{x\} \in 2^X$ (X is supposed to be T_1) and there is $\alpha \in A$ such that $x \in K(A_\alpha, B_\alpha)$; it follows that $x \in A_\alpha$ or $x \in B_\alpha$ and, because of $B_\alpha \subset A_\alpha$, $x \in A_\alpha$. We have then $X = \bigcup\limits_{\alpha\in A} A_\alpha$; X being a compact space, there is a finite subcovering $(A_i)_{i=\overline{1,n}}$. We show that $(K(A_i, B_i))$, $i = \overline{1,n}$ is a finite covering of 2^X.

Let $M \in 2^X$; if $M \not\subseteq A_i$, $i = \overline{1,n}$, we have $M \cap \complement_X A_i \neq \emptyset$, $i = \overline{1,n}$, hence $\bigcap\limits_{i=1}^{n} \complement_X A_i \neq \emptyset$, contradiction with $X = \bigcup\limits_{i=1}^{n} A_i$. It follows that there is $i \in \{1, ..., n\}$ with $M \subset A_i$, so $M \in K(A_i, B_i)$. We have $2^X = \bigcup\limits_{i=1}^{n} K(A_i, B_i)$ and so 2^X is compact.

2. Let 2^X be compact and $(U_\alpha)_{\alpha\in A}$ an open covering of X. Then $(\langle X, U_\alpha\rangle)_{\alpha\in A}$ is an open covering of 2^X, so it will have a finite subcovering $(\langle X, U_i\rangle)_{i=\overline{1,n}}$. Then $(U_i)_{i=\overline{1,n}}$ is a finite covering of X, hence X is a compact space. ∎

Corollary 2.1 *The space X is compact Hausdorff iff 2^X is compact Hausdorff.*

Proof. 1. Let X be a compact Hausdorff space; by Th. 7.10, Ch. I, X is T_3. Th. 2.9 and 2.10 imply then that 2^X is a compact Hausdorff space.

2. We consider now that 2^X is a compact Hausdorff space. Using Th. 2.9 and 2.10, it follows that X is compact and T_3; X being also T_1, it follows that X is also Hausdorff. ∎

If X satisfies one of the axioms of countability, it does not follow generally that 2^X has the same property. However, the space $K(X) = \{E \in 2^X \,|\, E \text{ compact set}\}$ with the finite Vietoris topology preserves these properties, as the next theorems show.

Theorem 2.11 *The space X satisfies the first countability axiom iff $K(X)$ has the same property.*

Proof. 1. Let X satisfy the first countability axiom; by Remark 2.3, Ch. I any $x \in X$ has a fundamental system of open neighbourhoods $(B_i(x))_{i\in\mathbb{N}}$ with $B_i(x) \subset B_j(x)$, $i > j$. Let $M \in K(X)$; for any $i \in \mathbb{N}$, the family $(B_i(x))_{x\in M}$ is an open covering for M, and we can obtain a finite covering $(B_i(x_{ik}))_{k=\overline{1,m_i}}$. We prove that the countable system $(B_i(x_{ik}))_{k=\overline{1,m_i},i\in\mathbb{N}}$ produces a fundamental system of neighbourhoods for M.

Let $\langle A_1, ..., A_n\rangle$ be a neighbourhood for $M \in K(X)$, so $M \subset \bigcup\limits_{j=1}^{n} A_j$; there will be than an index $i \in \mathbb{N}$ such that $M \subset \langle B_i(x_{i1}), ..., B_i(x_{im_i})\rangle \subset \langle A_1, ..., A_n\rangle$, hence the system $(\langle B_i(x_{i1}), ..., B_i(x_{im_i})\rangle)_{i\in\mathbb{N}}$ is a countable fundamental system of neighbourhoods.

2. Let $K(X)$ satisfy the first countability axiom; then for any $x \in M$, $\{x\}$ has a countable fundamental system of neighbourhoods, let this be $(\langle B_{i1}, ..., B_{im_i}\rangle)_{i\in\mathbb{N}}$. We show that $(B_{ik})_{k=\overline{1,m_i},i\in\mathbb{N}}$ is a fundamental system of neighbourhoods for x, which is also countable.

Let $V \in \mathcal{V}(x)$; we have $\{x\} \subset V$, so there is $\langle B_{i1}, ..., B_{im_i}\rangle \subset \langle V\rangle$ and containing $\{x\}$. This inclusion shows that there is a set $B_{ij} \subset V$, so $(B_{ik})_{k=\overline{1,m_i}}$ has the required property. ■

Theorem 2.12 *The space X satisfies the second countability axiom iff $K(X)$ has the same property.*

Proof. 1. Let X satisfy the second countability axiom and $(B_i)_{i\in\mathbb{N}}$ a countable basis. We show that the system $B = (\langle B_{i1}, ..., B_{im_i}\rangle)$, which is countable, generates a basis in 2^X.

It suffices to prove that the sets $\langle A_1, ..., A_n\rangle$ are unions of some elements in $\mathcal{B}$. Indeed, $\langle A_1, ..., A_n\rangle$ can be written as a union by $M \in K(X)$, $M \in \langle A_1, ..., A_n\rangle$ of elements like $\langle B_{i1}, ..., B_{im_i}\rangle$ chosen such that $M \subset \bigcup_{j=1}^{m_i} B_{ij}$,$M \cap B_{ij} \neq \emptyset$, $j = \overline{1, m_i}$ (it is possible since M is compact).

2. Let $(\langle B_{i1}, ..., B_{im_i}\rangle)_{i\in\mathbb{N}}$ a countable basis in $K(X)$. We show that $(B_{ij})_{j=\overline{1,m_i}}$ is a basis in X. By Th. 2.3, Ch. I, $(B_{ij})_{j=\overline{1,m_i}}$ is a basis iff the class of the sets B_{ij} which contain any $x \in X$ is a fundamental system of neighbourhoods for x.

Let $x \in X$ and $V \in \mathcal{V}(x)$; we have $\{x\} \subset V$, so $\{x\} \in \langle B_{i1}, ..., B_{im_i}\rangle$, $\subset \langle V\rangle$, for $i \in \mathbb{N}$. It follows that there is B_{ij} such that $x \in B_{ij} \subset V$ and X satisfies the second countability axiom. ■

3 Characterizations of the continuity for point-to-set mappings

The first statement of this section refers to the continuity of point-to-set mappings which are defined between metric spaces.

Theorem 3.1 *Let X and Y be metric spaces and $K(Y)$ the family of the non-void compact sets in Y; $F : X \multimap Y$ a point-to-set mapping with compact values, such that $F(x) \neq \emptyset$ for any $x \in X$. Then F is continuous iff the function $f : X \to K(Y)$ given by $f(x) = F(x)$, for any $x \in X$ is continuous, $K(Y)$ being endowed with the Hausdorff metric.*

Proof. 1. Let the point-to-set mapping F be continuous. For an arbitrary $x_0 \in X$ and $\varepsilon > 0$, we have obviously $F(x_0) \subset \operatorname{Int} V_\varepsilon(F(x_0))$; F being u.s.co. at x_0, there is $\eta > 0$ such that $d(x, x_0) < \eta$ implies $F(x) \subset \operatorname{Int} V_\varepsilon(F(x_0)) \subset V_\varepsilon(F(x_0))$. We have for x satisfying $d(x, x_0) < \eta$ that $F(x) \subset V_\varepsilon(F(x_0))$.

$F(x_0)$ is a compact set, so we can choose $y_1, ..., y_n \in F(x_0)$ such that $F(x_0) \subset \bigcup_{i=1}^n \operatorname{Int} V_{\varepsilon/2}(y_i)$. F being l.s.c. at x_0, we obtain η' such that for $x \in X$ with $d(x, x_0) < \eta'$ we have $\operatorname{Int} V_{\varepsilon/2}(y_i) \cap F(x) \neq \emptyset$, $i = \overline{1, n}$. Then $V_\varepsilon(y) \cap F(x) \neq \emptyset$, $\forall y \in F(x_0)$. Applying Th. 1.3 it follows that $F(x_0) \subset V_\varepsilon(F(x))$, for x with $d(x, x_0) < \eta'$.

For any $x \in X$ with $d(x, x_0) \leq \min\{\eta, \eta'\}$ we have $D(F(x), F(x_0)) = D(f(x), f(x_0)) \leq \varepsilon$, hence f is a continuous function.

2. Let now f be a continuous function. We consider $x_0 \in X$ and G an open set in Y with $F(x_0) \subset G$. Let $\varepsilon' = \inf\{ d(F(x_0), x) | \, x \in \complement_Y G\}$ and $\varepsilon < \varepsilon'$; then $V_\varepsilon(F(x_0)) \subset G$. f being continuous, there is $\eta > 0$ such that $d(x, x_0) < \eta$ implies $F(x) \subset V_\varepsilon(F(x_0)) \subset G$, so F is u.s.co.

If G is an open set with $G \cap F(x_0) \neq \emptyset$, there is then $y_0 \in F(x_0) \cap G$. We obtain $\varepsilon > 0$ such that $V_\varepsilon(y_0) \subset G$; f being continuous, it follows that there is $\eta > 0$ such that $d(x, x_0) < \eta$ implies $F(x) \cap V_\varepsilon(y_0) \neq \emptyset$, hence $F(x) \cap G \neq \emptyset$. It follows that F is l.s.c. ■

This theorem and Th. 8.17, Ch. I give

Corollary 3.1 *If X and Y are metric spaces and X is compact, any continuous point-to-set mapping $F : X \multimap Y$ is uniformly continuous, in the sense that for any $\varepsilon > 0$ there is $\eta > 0$ such that $d(x, x') < \eta$ implies $D(F(x), F(x')) < \varepsilon$.*

The next theorems refer to point-to-set mappings defined between arbitrary topological spaces.

Theorem 3.2 *The mapping $F : X \multimap Y$ which is pointwise closed with $F(x) \neq \emptyset$ for any $x \in X$ is continuous iff the function $f : X \to 2^Y$ given by $f(x) = F(x)$ is continuous, 2^Y being endowed with the finite Vietoris topology.*

Proof. 1. Let F be continuous and $G \subset 2^Y$ an open set, hence $G = \bigcup_{i\in I} B_i$, where $B_i = \langle A_{i1}, ..., A_{in_i}\rangle$. Because $f^{-1}(G) = f^{-1}\left(\bigcup_{i\in I} B_i\right) = \bigcup_{i\in I} f^{-1}(B_i)$, it is sufficient to prove that $f^{-}(B_i)$ is open in X.

We have $f^{-1}(B_i) = \{x \in X |\, f(x) \in B_i\} = \bigcup_{A\in B_i} \{x \in X |\, f(x) = A, A \subset \bigcup_{j=1}^{n_i} A_{ij},\ A \cap A_{ij} \neq \emptyset,\ j = \overline{1, n_i}\}$.

If all the sets in the union are void, $f^{-1}(B)$ is open. Let now x be an element of the set corresponding to the index i. We have then $F(x) = f(x) \subset \bigcup_{j=1}^{n_i} A_{ij}$; F being u.s.c., there is $V_0 \in \mathcal{V}(x)$ such that $f(V_0) = F(V_0) \subset \bigcup_{j=1}^{n_i} A_{ij}$.

For $j = \overline{1, n_i}$ we have $F(x) \cap A_{ij} = f(x) \cap A_{ij} \neq \emptyset$; F being l.s.c. there will be $V_j \in \mathcal{V}(x)$ such that $f(x') \cap A_j \neq \emptyset$ for any $x' \in V_j$.

Let $V = \bigcap_{j=1}^{n_i} V_j$; we have obviously $V \in \mathcal{V}(x)$ and for any $x' \in V$ we have $f(x') \subset \bigcup_{j=1}^{n_i} A_j$, $f(x') \cap V_j \neq \emptyset$, $j = \overline{1, n_i}$. It follows that $V \subset f^{-1}(B_i)$ so $f^{-1}(B_i)$ is an open set. We obtain by Th. 6.3, Ch. I, that f is continuous.

2. Let f be continuous. We show that F is u.s.c. and l.s.c. Let $x_0 \in X$ and $U \subset Y$ an open set with $F(x_0) \subset U$. Then $F(x_0) \in \langle U\rangle$; f being continuous, $V = f^{-1}(\langle U\rangle)$ is an open set with $x_0 \in V$. Let $x \in V$, so $f(x) \in \langle U\rangle$; it follows that $F(x) = f(x) \subset U$, hence F is u.s.c.

Let now $x_0 \in X$ and $U \subset Y$ an open set with $F(x_0) \cap U \neq \emptyset$, so $F(x_0) \in \langle U, Y\rangle$. Because f is continuous, $V = f^{-1}(\langle U, Y\rangle)$ is open in X and obviously $x_0 \in V$. Let now $x \in V$; we have $f(x) \in \langle U, Y\rangle$, hence $f(x) \cap U \neq \emptyset$. Because $f(x) = F(x)$ we obtain that F is l.s.c. ■

The next two theorems have analogous proofs.

Theorem 3.3 *The mapping $F : X \multimap Y$ which is pointwise closed with $F(x) \neq \emptyset$ for any $x \in X$ is u.s.c. iff the function $f : X \to 2^Y$ is continuous, 2^Y being endowed with the upper semifinite topology.*

Theorem 3.4 *The mapping $F : X \multimap Y$ which is pointwise closed with $F(x) \neq \emptyset$ for any $x \in X$ is l.s.c. iff the function $f : X \to 2^Y$ is continuous, 2^Y being endowed with the lower semifinite topology.*

If F is pointwise compact and we denote by $K(X)$ (respectively $K(Y)$) the family of the non-void compact subsets of X (respectively Y) endowed with the finite Vietoris topology, we obtain

Theorem 3.5 *If $F : X \multimap Y$ is continuous and pointwise compact, then the function $\widehat{F} : K(X) \to K(Y)$ given by $\widehat{F}(A) = F(A)$ is continuous.*

Proof. The fact that $\widehat{F}(A) \in K(Y)$ is a consequence of Th. 2.9, Ch. III.

We show that $\widehat{F^{-1}}(B)$ is an open set, B being an element of the basis. Let $B = \langle A_1, ..., A_n \rangle$. We have

$$F^{-1}(B) = \{ M \in K(X) | \, F(M) \subset \bigcup_{i=1}^{n} A_i, \widehat{F}(M) \cap A_i \neq \emptyset, \; i = \overline{1,n} \} =$$
$$\{ M \in K(X) | \, F(M) \subset \bigcup_{i=1}^{n} A_i, F(M) \cap A_i \neq \emptyset, \; i = \overline{1,n} \}.$$

Let $L = \{ x \in X | \, F(x) \subset \bigcup_{i=1}^{n} A_i \}$ and $L_i = \{ x \in X | \, F(x) \cap A_i \neq \emptyset \}$, $i = \overline{1,n}$; these sets are open in X, by Th. 2.5, Ch. III and Th. 1.5, Ch. III. We obtain $\widehat{F^{-1}}(B) = \{ M \in K(X) | \, M \subset L, M \cap L_i \neq \emptyset, \; i = \overline{1,n} \} = \langle L \rangle \cap \langle X, L_1, ..., L_n \rangle$, which is an open set in $K(X)$. It follows that the function F is continuous. ■

4 Point-to-set mappings in mathematical programming and optimal control

The mathematical programming is a domain where point-to-set mapping appear frequently. W.I. Zangwill [7, 36] looks to be the first who applied constantly the idea of the point-to-set mappings in mathematical programming.

One finds point-to-set mappings even in the algorithms for linear problems. For example, let us consider the Simplex method and suppose that the point x was generated, so it is a basic feasible solution of the linear system of inequalities. We have now to generate the next point y, which is also a basic solution. This successor is not well-defined, because the variable which enters the basis can be chosen in different ways. The same ambiguity concerning the successor of a point appears in other algorithms too. So we are obliged to consider methods which generate a point y from a certain set. The set of all possible successors for a point x will be well determined.

Point-to-set mappings are of greater importance in the problems of nonlinear programming. We analyze first a general problem of nonlinear programming and we insist then on an example.

The general problem of the nonlinear programming is this: let the continuous function $f : \mathbb{R}^n \to \mathbb{R}$ be minimized on the set of the points $x \in \mathbb{R}^n$ which satisfy the system of inequalities $g_i(x) \leq 0$, $i = \overline{1,m}$. The set $F = \{x \in \mathbb{R}^n | \, g_i(x) \leq 0, \ i = \overline{1,m}\}$ is called the *feasible set* (we suppose $F \neq \emptyset$). A point $x \in F$ which minimizes f is called an optimal point for the considered problem.

The problems of this type are usually solved by algorithmic methods. An algorithm is an iterative procedure, which determines a successor x_{k+1} for a point x_k already obtained. Sometimes it is possible to define a function $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $x_{k+1} = h(x_k)$. This function defines then the iterative procedure. In lots of cases such a function cannot be defined, because for a given x there is not a unique value $h(x)$. In these cases it is necessary to introduce point-to-set mappings. Sometimes the procedure depends on the number k of the iterations already made. There is a large class of algorithms where this dependence does not appear; the procedures are then called antonomous.

We can describe now an iterative autonomous method generated by a point-to-set mapping $H : \mathbb{R}^n \multimap \mathbb{R}^n$. For a given $x_0 \in \mathbb{R}^n$, let us suppose that $x_1, ..., x_k$ have been generated. If $H(x_k) = \emptyset$, the procedure stops. If $H(x_k) \neq \emptyset$, any $y \in H(x_k)$ is a possible value for x_{k+1} and we obtain the successor $x_{k+1} \in H(x_k)$.

Some of the simplest algorithms refer to the finding of the unconstrained minimum for a function $f : \mathbb{R}^n \to \mathbb{R}$. An important class of such algorithms, named the *unconstrained feasible directions algorithms*, is a class of convergent algorithms. Any algorithm in this class has a continuous function $b : \mathbb{R}^n \to \mathbb{R}^n$ which serves as a direction. For a point x_k, the successor x_{k+1} is generated by maximizing f in the direction $b(x_k)$.

If $\bigtriangledown f(x)$ denotes the gradient of f at x, a point x is called a solution if $\bigtriangledown f(x) \cdot b(x) = 0$. We define a point-to-set mapping $H : \mathbb{R}^n \multimap \mathbb{R}^n$ taking $y \in H(x)$ iff y is an optimal solution for the problem $\min\{f(x + tb(x)) | \, t > 0\}$.

The unconstrained feasible directions algorithm is then described as follows. If x_k is a solution, the procedure stops. If not, we consider a successor $x_{k+1} \in H(x_k)$ and we test if it is a solution.

The convergence is assured if the following conditions are fulfilled:

1^0. either the problem has no solutions, or the set $\{x \in \mathbb{R}^n | \, f(x) \leq$

$f(x_0)\}$ is compact for any $x_0 \in \mathbb{R}^n$.

2^0. if x is not a solution, then $y \in H(x)$ implies $f(x) < f(x)$.

An example of unconstrained feasible directions algorithm is the Cauchy method for the case when f is a convex and differentiable function, and $b = \bigtriangledown f$. We consider that the set $\{x|\, f(x) \leq f(x_0)\}$ is compact for any $x_0 \in \mathbb{R}^n$.

We consider the point-to-set mappings $G : \mathbb{R}^n \to \mathbb{R}^n$, $G(x) = \{y \in \mathbb{R}^n|\ y = x + t \bigtriangledown f(x), t \in (0, +\infty)\}$ and $H : \mathbb{R}^n \to \mathbb{R}^n$, $H(x) = \{y \in G(x)|\, f(y) = \min\limits_{z \in G(x)} f(z)\}$. The point-to-set mapping H is the one that describes the algorithms, in the sense that $x_{k+1} \in H(x_k)$.

Theorem 4.1 *The point-to-set mapping* $G : \mathbb{R}^n \multimap \mathbb{R}^n$, $G(x) = \{y \in \mathbb{R}^n|\, y = x + t \bigtriangledown f(x), t \in (0, +\infty)\}$ *is l.s.c. and closed on* $\mathbb{R}^n \setminus \{x \in \mathbb{R}^n|\bigtriangledown f(x) = 0\}$.

Proof. We prove first that G is l.s.c.

Let $x_k \to x_0\,(k \to \infty)$ and $y_0 \in G(x_0)$; we prove that there is a sequence $(y_k)_{k \in \mathbb{N}}$, $y_k \in G(x_k)$, $y_k \to y_0$. Because $y_0 \in G(x_0)$, we have $y_0 = x_0 + t_0 \bigtriangledown f(x_0)$. For any x_k we obtain $G(x_k) = \{y|\, y = x_k + t \bigtriangledown f(x_k)\}$. We choose a sequence $(t_k)_{k \in \mathbb{N}}$, $t_k \to t_0$, $t_k > 0$ and consider $y_k = x_k + t_k \bigtriangledown f(x_k) \in G(x_k)$. It follows that $\lim\limits_{k \to \infty} y_k = \lim\limits_{k \to \infty}(x_k + t_k \bigtriangledown f(x_k)) = x_0 + t_0 \bigtriangledown f(x_0) = y_0$. The sequence $(y_k)_{k \in \mathbb{N}}$ has the required properties.

We prove that G is a closed mapping.

Let $x_k \to x_0\,(k \to \infty)$ and $y_k \in G(x_k)$, $y_k \to y_0$. We prove that $y_0 \in G(x_0)$. The fact that $x_k \to x_0$, $y_k \to y_0$ and $y_k = x_k + t_k \bigtriangledown f(x_k)$ implies $t_k \bigtriangledown f(x_k) \to y_0 - x_0$. We obtain than $t_k \bigtriangledown f(x_0) \to y_0 - x_0$, hence there is a t_0 such that $t_0 \bigtriangledown h(x_0) = y_0 - x_0$ and $t_k \to t_0\,(k \to \infty)$. It follows that $y_0 = x_0 + t_0 \bigtriangledown h(x_0) \in G(x_0)$ and the proof is over. ■

Theorem 4.2 *The point-to-set mapping* $H : \mathbb{R}^n \multimap \mathbb{R}^n$ *given by* $H(x) = \{y \in G(x)|\, f(y) = \min\limits_{z \in G(x)} f(z)\}$ *is a closed mapping on* $\mathbb{R} \setminus \{\, x \in \mathbb{R}^n|$ $\bigtriangledown f(x) = 0\}$.

Proof. We prove that $x_k \to x_0$, $y_k \to y_0$ and $y_k \in H(x_k)$ imply $y_0 \in H(x_0)$, which is equivalent with $y_0 \in G(x_0)$ and $f(y_0) = \min\limits_{y \in G(x_0)} f(y)$.

Because G is a closed mapping, it follows immediately that $y_0 \in G(x_0)$. We suppose now that $f(x_0) \neq \min\limits_{y \in G(x_0)} f(y)$, so there is $z_0 \in G(x_0)$ such that $f(y_0) > f(z_0)$. We have now $x_k \to x_0$, $z_0 \in G(x_0)$; G being l.s.c., there is a sequence $(z_k)_{k \in \mathbb{N}}$, $z_k \in G(x_k)$, $z_k \to z_0$. $f(z_0) =$

$\lim\limits_{k\to\infty} f(z_k) \geq \lim\limits_{k\to\infty} f(y_k) = f(y_0)$, contradiction with $f(y_0) > f(z_0)$. It follows that H is a closed mapping. ■

Point-to-set mappings appear also in the problems concerning the optimal control. We give an example of optimal control [18] and we indicate a continuous point-to-set mapping related to it.

An optimal control problem includes:

1. the process. The process connects the state $x(t)$ and the controller $u(t)$ by means of a differential equations system

(1) $\dot{x} = A(t)\,x + B(t)\,u + v(t)$,

where $A(t)$ is a $n\times n$ matrix, $B(t)$ a $n\times m$ matrix and $v(t)$ a column vector in $\mathbb{R}^n$, all of them being measurable on $\mathbb{R}$; their norms $|A(t)|, |B(t)|$ and $|v(t)|$ are integrable on any compact interval of $\mathbb{R}$.

2. the initial and target states. The initial state x_0 is given and the target state is a fixed set G. Sometimes the target state is a compact set $G(t)$ which varies when $t \in [\tau_0, \tau_1]$.

3. the class of the admissible controllers $\triangle$. This is composed of bounded measurable functions defined on an interval $[t_0, t_1]$ $(t_0 < t_1 < +\infty)$ with values in a non-void set $\Omega \subset \mathbb{R}^m$. An answer or a solution $x(t)$ is an absolutely continuous function $x(t)$ defined on $[t_0, t_1]$ with values in $\mathbb{R}^n$ which satisfies the system (1), with $x(t_0) = x_0$ and $x(t_1) \in G(t_1)$.

4. the cost (objective) functional. It is an accepted criterion for the efficiency of the control u from $\triangle$. If $\triangle$ is the class of the controllers that take x_0 in the final state, the cost is often defined by

(2) $C(u) = \int_{t_0}^{t_1} f(t, x(t), u(t))\,dt$,

where f is a continuous function.

Definition 4.1 *A controller $u^* \in \triangle$ is called* optimal *related to the cost functional C if $C(u^*) \leq C(u)$ for any $u \in \triangle$.*

In this problem we find a point-to-set mapping whose continuity will be proved.

Definition 4.2 *Let a control linear system (1) be given having as initial state x_0 and as controllers the functions $u : [t_0, t_1] \to \Omega$, x being the corresponding answer which satisfies $x(t_0) = x_0$. The* attainability set *$\mathcal{K}((1), \Omega, x_0, t_0, t_1) = \mathcal{K}(t_1)$ is the set of the final points $x(t_1)$ in $\mathbb{R}^n$. We denote also $\mathcal{K}(t_0) = \{x_0\}$.*

We can prove now the next theorem

Theorem 4.3 *Let a linear process governed by (1), with Ω compact convex set, initial state x_0 and controllers $u : [t_0, t_1] \to \Omega$. Then the attainability set $K(t_1)$ is compact, convex and varies continuously with t_1, for $t_1 \geq t_0$.*

Proof. a) $K(t_1)$ is a compact set for any $t_1 \geq t_0$. Let $(x_r(t_1))_{r\in\mathbb{N}}$ a sequence in $K(t_1) \subset \mathbb{R}^n$; let u_r be the corresponding controller for x_r, $r \in \mathbb{N}$. The parameters variation formula gives

$$x_r(t) = \phi(t)\, x_0 + \phi(t) \int\limits_{t_0}^{t} \phi(s)^{-1} \left[B(s)\, u_r(s) + v(s)\right] ds,$$

where ϕ is the matrix of the fundamental solutions of the homogenous system $\dot{x} = A(t)\, x$, with $\phi(t_0) = E$, E denoting the identical matrix $n \times n$.

The set of the controllers $\triangle$ is weakly compact, so we can obtain a subsequence u_{r_i} weakly convergent to a controller $\overline{u} \in \triangle$, i.e.

$$\lim_{is1\infty} \int\limits_{t_0}^{t} \phi(s)^{-1} B(s)\, u_{r_i}(s)\, ds = \int\limits_{t_0}^{t} \phi(s)^{-1} B(s)\, \overline{u}(s)\, ds.$$

Let $\overline{x}$ be the answer to the controller $\overline{u}$; we have for any $t \in [t_0, t_1]$

$$\overline{x}(t) = \phi(t)\, x_0 + \phi(t) \int\limits_{t_0}^{t} \phi(s)^{-1} \left[B(s)\, \overline{u}(s) + v(s)\right] ds = \lim_{i\to\infty} x_{r_i}(t).$$

We proved the existence of a convergent subsequence of $(x_r(t_1))_{r\in\mathbb{N}}$ to $\overline{x}(t_1) \in K(t_1)$, so $\mathcal{K}(t_1)$ is a compact set.

b) $K(t_1)$ is a convex set.

Let $x_0(t_1)$ and $x_1(t_1) \in K(t_1)$ and u_0, u_1 the corresponding controllers. We define for $\lambda \in [0,1]$ the controllers $u_\lambda(t) = (1-\lambda)\, u_0(t) + \lambda u_1(t)$, $t \in [t_0, t_1]$ which are also in $\triangle$. The answer x_λ to u_λ is given by

$$x_\lambda(t) = \phi(t)\, x_0 + \phi(t) \int\limits_{t_0}^{t} \phi(s)^{-1} \left[B(s)\, u_\lambda(s) + v(s)\right] ds.$$

It follows easily that $x_\lambda(t_1) = (1-\lambda)\, x_0(t_1) + \lambda x_1(t_1) \in K(t_1)$, so $K(t)$ is a convex set.

c) $K(t_1)$ varies continuously with $t_1 \geq t_0$.

We prove that for any $\varepsilon > 0$, there is $\delta > 0$ such that $D(K(t_1), K(t_2)) < \varepsilon$ for $|t_1 - t_2| < \delta$, where D is the Hausdorff metric.

Let $t_1 \geq t_0$ and $\widehat{u} \in \triangle$ a controller with the answer $\widehat{x}$ on $t_0 \leq t \leq$

t_1+1. We have then for $t_1, t_2 \in (t_0, t_1+1)$

$$\widehat{x}(t_2) - \widehat{x}(t_1) = \phi(t_2) \int_{t_0}^{t_2} \phi(s)^{-1} [B(s)\widehat{u}(s) + v(s)]\, ds$$

$$- \phi(t_2) \int_{t_0}^{t_1} \phi^{-1}(s) [B(s)\widehat{u}(s) + v(s)]\, dt$$

$$+ [\phi(t_2) - \phi(t_1)] \left(\int_{t_0}^{t_1} \phi(s)^{-1} [B(a)\widehat{u}(s) + v(s)]\, ds + x_0 \right).$$

On $[t_0, t_1+1]$ the continuous functions ϕ and ϕ^{-1} are bounded; let $C_1 > 0$ such that $|\phi(t)| < C_1$, $\left|\phi(t)^{-1}\right| < C_1$. Because of the integrability of $|B(t)|$ and $|v(t)|$, and the boundedness of $|\widehat{u}(t)|$ we have

$$|x_0| + \int_{t_0}^{t_1+1} \left|\phi(s)^{-1}\right| \cdot |B(s)u(s) + v(s)|\, ds < C_2.$$

We have also, for an arbitrary ε, that there is $\delta > 0$ such that

$$\left| \int_{t_1}^{t} \phi(s)^{-1} [B(s)u(s) + v(s)]\, ds \right| < \tfrac{\varepsilon}{2C_1} \text{ and}$$

$$|\phi(t) - \phi(t_1)| \leq \left| \int_{t_1}^{t} A(s)\phi(s)\, ds \right| < \tfrac{\varepsilon}{2C_2} \text{ for } |t - t_1| < \delta.$$

For $|t_2 - t_1| < \delta$ we obtain

$$|\widehat{x}(t_2) - \widehat{x}(t_1)| \leq |\phi(t_2)| \cdot \left| \int_{t_1}^{t_2} \phi(s)^{-1} [B(s)\widehat{u}(s) + v(s)]\, ds \right| +$$

$$+ |\phi(t_2) - \phi(t_1)| \left[\int_{t_0}^{t_1+1} \left|\phi(s)^{-1}\right| \cdot |B(s)\widehat{u}(s) + v(s)|\, ds + |x_0| \right],$$

hence $|\widehat{x}(t_2) - \widehat{x}(t_1)| < C_1 \frac{\varepsilon}{2C_1} + C_2 \frac{\varepsilon}{2C_2} = \varepsilon$.

Let now $\widehat{x}(t_1) \in K(t_1)$ corresponding to the controller $\widehat{u}$ on $[t_0, t_1]$. We define $\widehat{u}$ on $[t_0, t_1+1]$ taking $\widehat{u}(t) = \widehat{u}(t_1)$ for $t \in [t_1, t_1+1]$; let $\widehat{x}$ be the answer. We have then $\widehat{x}(t_2) \in K(t_2)$ and $|\widehat{x}(t_2) - \widehat{x}(t_1)| < \varepsilon$.

Similarly, if $\widetilde{x}(t_2) \in K(t_2)$ corresponds to the controller $\widetilde{u}$ on $[t_0, t_2]$, we extend $\widetilde{u}$ at $[t_0, t_1+1]$ and we obtain $|\widetilde{x}(t_1) - \widetilde{x}(t_2)| < \varepsilon$. We obtain $D(K(t_1), K(t_2)) < \varepsilon$ for $|t_1 - t_2| < \delta$, where $0 < \delta < 1$ depends on ε and t.

We prove analogously that $D\left(K\left(t_0\right), K\left(t_1\right)\right) < \varepsilon$ for $|t_1 - t_0| < \delta$.

The point-to-set mapping $K : [t_0, +\infty] \multimap \mathbb{R}^n$, having non-void compact values and appearing in a natural way in the problem of optimal control, is a continuous mapping. ■

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