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ON MAXIMALITY PRINCIPLES RELATED TO EKELAND'S THEOREM

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The large area of applications of the variational principle of I. Ekeland [5, 6] has determined in the last years a great interest in the generalizations of this principle. These are usually formulated as maximality principles. In the following we refer to the relations between such maximality principles.

The full statement of the theorem of I. Ekeland is the following.

Theorem 1 [5, 6] *Let* (X, d) *be a complete metric space and* $f : X \to \mathbb{R} \cup \{+\infty\}$ *a lower semicontinuous (l.s.c.) lower bounded function nonidentically equal to* $+\infty$ *(i.e. a proper function). Then for each* $\varepsilon > 0$ *and* $x_1 \in X$ *such that*

$$f(x_1) < \inf\{f(x) : x \in X\} + \varepsilon,$$

and for each $\lambda > 0$ *there exists* $x_0 \in X$ *such that*

$$f(x_0) \le f(x_1)$$
$$d(x_0, x_1) \le \lambda \text{ and}$$
$$f(x) > f(x_0) - \frac{1}{\lambda} d(x_0, x) \text{ for each } x \text{ in } X \setminus \{x_0\}.$$

In the paper [3] there are given several equivalent variants of the Ekeland's principle. Denoting dom $f = \{x \in X : f(x) < +\infty\}$, one of the simplest is the following one (known as Brondsted's Lemma).

Theorem 2 [4] *Let (X, d) be a complete metric space and $f : X \to \mathbb{R} \cup \{+\infty\}$ a l.s.c. lower bounded proper function. Then the relation defined by*

$$x \lesssim y \text{ iff } x = y \text{ or } x, y \in \operatorname{dom} f, \quad d(x, y) \leq f(x) - f(y)$$

is on order relation and for each x_0 in dom f *there is a maximal element $x^* \in X$ such that $x_0 \lesssim x^*$.*

By order relation we mean a relation which is reflexive, transitive and antisymmetric.

It is clear that for every relation on X one can define a multivalued mapping $F : X \to 2^X$ such that $y \in F(x)$ iff $x \lesssim y$. The relation $\lesssim$ is reflexive iff $x \in F(x)$ for each $x \in X$, antisymmetric iff ($y \in F(x)$, $x \in F(y)$ implies $x = y$) and transitive iff ($y \in F(x)$ implies $F(y) \subseteq F(x)$ for each x, y in X). The transitivity condition has the equivalent form $F^2(x) \subseteq F(x)$ for each x in X, where $F^2(x) = \{t \in X :$ there exists $y \in F(x)$ such that $t \in F(y)\}$.

In the papers [3,8] one gives the following theorem, which is also restated in the terms of the multivalued mapping F.

Theorem 3 [3] *Let (X, d) be a metric space and $\lesssim$ an order relation on X such that*

(1) *for each monotone sequence $\{x_n\}_{n \in \mathbb{N}}$ (i.e. $x_n \lesssim x_m$, $\forall n \leq m$) we have $d(x_n, x_{n+1}) \xrightarrow{n} 0$*

(2) *the set $(x, \lesssim) = \{y \in X : x \lesssim y)$ is closed for each x in X.*

Then for each x_0 in X there is a maximal element $\overline{x}$ in X such that $x_0 \lesssim \overline{x}$.

A generalization of Theorem 2 was given in the paper [9] in the following way.

Theorem 4 [9] *Let (X, d) be a quasimetric space (that is the function $d : X^2 \to \mathbb{R}_+$ satisfies all the requirements of a metric except sufficiency) and $\lesssim$ a reflexive and transitive relation on X which satisfies the conditions*

(1') for each monotone sequence $\{x_n\}_{n\in\mathbb{N}}$, inf $d(x_n, x_{n+1}) = 0$ and

(3) for each monotone sequence $\{x_n\}_{n\in\mathbb{N}}$ which is a Cauchy one there exists z in X such that $x_n \lesssim z$ for each n in $\mathbb{N}$.

Then for each x_0 in X there exists a d-maximal element $\overline{x}$ in X (i.e. $\overline{x} \lesssim x$ implies $d(\overline{x}, x) = 0$) such that $x_0 \lesssim \overline{x}$.

Remark 1 *It is clear that any relation which satisfies the conditions in Theorem 3 satisfies also those in Theorem 4. Indeed, let (X, d) be a complete metric space and $\lesssim$ an order relation which verifies (1) and (2). It remains to verify that $\lesssim$ satisfies (3). Let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in X; the completeness of X implies that $\{x_n\}_{n\in\mathbb{N}}$ converges to an element z in X. But $x_m \in (x_n, \lesssim)$ for each $m > n$ and $(x_n, \lesssim)$ is closed, hence $z \in (x_n, \lesssim)$ for each n in $\mathbb{N}$. It follows $x_n \lesssim z$ for each n in $\mathbb{N}$ and (3) is satisfied.*

Remark 2 *In a metric space (X, d), the antisymmetry of $\lesssim$ follows from the transitivity and the condition (1), respectively (1').*

Let $x, y \in X$ such that $x \lesssim y$ and $y \lesssim x$. It follows that the sequence $\{x_n\}_{n\in\mathbb{N}}$, $x_{2n-1} = x, x_{2n} = y$ is monotone. Using (1) or (1') it follows $d(x, y) = 0$, hence $x = y$ and $\lesssim$ is antisymmetric.

In the following we give a generalization of the above theorems.

Theorem 5 *Let (X, d) be a quasimetric space and $\lesssim$ a transitive relation such that (3) holds and*

(4) *for each* x *in* X *and* $\varepsilon > 0$, *there is* $y = y_{\varepsilon,x}$ *such that* $x \leq y$ *and* $d(y) \leq \varepsilon$, *where* $d(y) = \sup\{d(z, y) : y \lesssim z\}$.

Then for each x_0 *in* X *there is a d-maximal element* $\overline{x}$ *in* X *such that* $x_0 \lesssim \overline{x}$.

Remark 3 *The condition* (4) *implies the fact that* $(x, \lesssim)$ *is nonvoid for each* x *in* X; *it appears in* [9] *as condition* (1) *in the proof of Theorem 1* [9].

Proof of Theorem 5. Let $x_0 \in X$ be given. Using (4), we obtain inductively a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that

for x_0 and $\varepsilon = 1/2$, there is $x_1 \in X$, $x_0 \lesssim x_1$ and $d(x_1) \leq 1/2$;

for x_{n-1} already obtained and $\varepsilon = 1/2^n$, there exists $x_n \in X$, $x_{n-1} \lesssim x_n$ and $d(x_n) \leq 1/2^n$.

The sequence $\{x_n\}_{n\in\mathbb{N}}$ is monotone and it has the property that $d(x_n) \leq 1/2^n$ for each n in $\mathbb{N}$. But $x_n \lesssim x_{n+1}$ and, applying the transitivity, $x_n \lesssim x_{n+p}$; it follows that $d(x_{n+1}, x_{n+p}) \leq 2d(x_n) \leq 1/2^{n-1}$, hence $\{x_n\}$ is Cauchy sequence. The condition (3) implies the existence of $\overline{x}$ in X such that $x_n \lesssim \overline{x}$ for each n in $\mathbb{N}$. It is obvious that $x_0 \lesssim \overline{x}$.

But $x_n \lesssim \overline{x}$ for each n in $\mathbb{N}$, hence $d(\overline{x}, x_n) \leq 1/2^n$. We show that $\overline{x}$ is d-maximal. Let $x \in X$, $\overline{x} \lesssim x$. It follows that $x_n \lesssim x$ for each n in $\mathbb{N}$, so $d(x, x_n) \leq d(x_n) \leq 1/2^n$. We obtain $d(x, \overline{x}) \leq d(x, x_n) + d(x_n, \overline{x}) \leq 1/2^{n-1}$, hence $d(x, \overline{x}) = 0$ and the theorem is proved. ■

Theorem 4, and, following Remark 1, Theorem 3 too, is a corollary of Theorem 5. Indeed, it remains to prove that, the conditions in Theorem 4 being satisfied, the condition (4) holds. Suppose that there exists x in X and $\varepsilon > 0$ such that for each y in X, $x \lesssim y$ we have $d(y) > \varepsilon$. The relation $\lesssim$ being reflexive, $(x, \lesssim)$ is nonvoid for each x in X. Let $x_1 \in X$ such that $x \lesssim x_1$; but $d(x_1) > \varepsilon$, and we obtain $x_2 \in X$, $x_1 \lesssim x_2$ such that $d(x_1, x_2) \geq \varepsilon - \varepsilon/2$. We have $x \lesssim x_2$, so $d(x_2) > \varepsilon$ and there exists $x_3 \in X$, $x_2 \lesssim x_3$, $d(x_2, x_3) \geq \varepsilon - \varepsilon/_{2^2}$. Inductively, for

$x_n \in X$, $x \lesssim x_n$ with $d(x_n) > \varepsilon$ we obtain $x_{n+1} \in X$, $x_n \lesssim x_{n+1}$ and $d(x_n, x_{n+1}) \geq \varepsilon - \varepsilon/_{2^n}$. We have $\inf(x_n, x_{n+1}) \geq \inf\{\varepsilon - \varepsilon/_{2^n}\} = \varepsilon/2$, contradicting (1') in Theorem 4. It follows that (4) is satisfied and using Theorem 5 the conclusion of Theorem 4 holds.

In the following example, Theorem 5 applies, but Theorem 4 doesn't.

Example 1 *Let* $X = [0,1] \cup [2,3]$ *with the usual metric on* $\mathbb{R}$ *and the reflexive and transitive relation given by*

$$x \lesssim y \text{ if } x \leq y \text{ or } (x = 2,\ y = 1),$$

$\leq$ *being the natural order relation on* $\mathbb{R}$.

Condition (1) *fails to be satisfied, because the sequence* $x_1 = 1$, $x_2 = 2$, $x_3 = 1, \ldots$ *is monotone and* $d(x_n, x_{n+1}) \nrightarrow 0$. *The relation* $\lesssim$ *is not antisymmetric.*

It is worth mentioning that the theorem of Brézis-Browder [1] and Ekeland are simple consequences of Theorem 4, as it was shown in [9].

In the following we give a characterization of the completeness of a metric space in the terms of maximal elements for a relation, respectively strict fixed points for a mutivalued mapping.

In the paper [7], S. Park gives seven characterizations of metric completeness related to the conditions in theorems of Caristi-Ekeland type [2,5]. We propose one more characterization to be added to the mentioned list.

Let (X, d) be a metric space. Among the equivalent statements given in the theorem in [7], there are the following two (denoted there by (iii), respectively (iv)).

(a) For every sequence $\{F_n\}_{n\in\mathbb{N}}$ of nonempty closed subsets of X such that $F_{n+1} \subseteq F_n$, $n \in \mathbb{N}$, and the sequence $\{diam F_n\}_{n\in\mathbb{N}}$ converges to 0, we have $\bigcap\limits_{n=1}^{\infty} F_n \neq \emptyset$.

(b) Every lower semicontinuous function $h : X \to (-\infty, +\infty)$ which is bounded from below has a d-point p in X, that is $h(p) - h(x) < d(p, x)$ for every point x in X, $x \neq p$.

We consider now the following statement.

Every multivalued mapping $F : X \to 2^X$ such that there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$, $x_{n+1} \in F(x_n)$, $n \in \mathbb{N}$ and

1) $F(z) = \overline{F(z)}$ for each z in $\overline{\{x_n\}_{n\in\mathbb{N}}}$

2) $F(z) \neq \emptyset$ for each z in $\overline{\{x_n\}}_{n\in\mathbb{N}}$

3) $z_2 \in F(z_1) \Rightarrow F(z_2) \subseteq F(z_1)$ for each z_1, z_2 in $\overline{\{x_n\}}_{n\in\mathbb{N}}$

4) diam $F(x_n) \xrightarrow{n} 0$,

has a strict fixed point x^* (i.e. $F(x^*) = \{x^*\}$) and $x^* = \lim_n x_n$.

Theorem 6 *The following implications hold:*

$$(a) \Rightarrow (*) \Rightarrow (b).$$

Proof. $(a) \Rightarrow (*)$. Let $\{x_n\}_{n\in\mathbb{N}}$ be as in the hypothesis of $(*)$. Then the condition **3)** implies $F(x_{n+1}) \subseteq F(x_n)$, $n \in \mathbb{N}$ and using **1)**, **2)** and **4)** we can apply **(a)**, hence $\bigcap_n F(x_n) = \{x^*\}$. From x_{n+1}, $x^* \in F(x_n)$ and b) we obtain $x^* = \lim_n x_n$. But $x^* \in F(x_n)$ and using **3)** we get $F(x^*) \subseteq F(x_n)$, $n \in \mathbb{N}$, so $F(x^*) \subseteq \{x^*\}$. It follows $F(x^*) = \{x^*\}$, because $F(x^*\} \neq \emptyset$.

$(*) \Rightarrow (b)$. For the given function h, we define $F : X \to 2^X$ by $F(x) = \{y \in X : d(x, y) \leq h(x) - h(y)\}$. We have $x \in F(x)$ for x in X, hence $F(x) \neq \emptyset$; the condition **1)** holds on X because h is lower semicontinuous. To verify **3)**, let $z_1, z_2 \in X$, $z_2 \in F(z_1)$ and $z \in F(z_2)$; it means

$d(z_1, z_2) \leq h(z_1) - h(z_2)$ and

$d(z_2, z) \leq h(z_2) - h(z)$,

then $d(z_1, z) \leq h(z_1) - h(z)$, i.e. $\in F(z_1)$ and $F(z_2) \subseteq F(z_1)$.

Let $x_0 \in X$ be given; we obtain inductively a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_{n+1} \in F(x_n), h(x_{n+1}) < \inf h(F(x_n)) + 1/n$. It follows that for x in $F(x_{n+1}) \subseteq F(x_n)$ we have

$$d(x, x_{n+1}) \leq h(x_{n+1}) - h(x) \leq h(x_{n+1}) - \inf h(F(x_n)) < 1/n,$$

hence diam $F(x_n) \overset{n}{\to} 0$. Applying $(*)$ we obtain $p = \lim_n x_n, F(p) = \{p\}$. Then for each x in $X \backslash \{p\}$ we have $x \notin F(p)$, hence $d(p, x) > h(p) - h(x)$. ∎

It follows that condition $(*)$ can be included in the list in [7] as (iii′), being equivalent to the completeness of the space X.

We also mention that $(*)$ can be expressed in the terms of a relation in the following way.

For each relation defined on the metric space X there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \lesssim x_{n+1}$, $n \in \mathbb{N}$ and

1) $(x_n, \lesssim) = \{y \in X : x_n \lesssim y\}$ is a closed set for each n in $\mathbb{N}$

2) $(z, \lesssim) \neq \emptyset$ for each $z \in \overline{\{x_n\}}_{n\in\mathbb{N}}$

3) $\lesssim$ is transitive on the set $\overline{\{x_n\}}_{n\in\mathbb{N}}$

4) diam $(x_n, \lesssim) \overset{n}{\to} 0$

there exists a maximal element x^* (i.e. $x^* \lesssim x$ implies $x^* = x$).

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