Families of Similar Orbits in the Inverse Problem of Dynamics

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Abstract

The family of orbits given in advance in the inverse problem of dynamics can be described in implicit or parametric form. It is proved that the similar curves expressed in parametric form can be rewritten implicitly, the corresponding first order partial differential equation satisfied by the potential being integrable by quadratures. An example from astrophysics (figure-eight curves) is worked out to illustrate the theoretical results.

1 Introduction

The aim of the two-dimensional inverse problem of Dynamics is to find the potentials V (or force fields) which can give rise to a prescribed monoparametric family of planar trajectories traced by a unit mass material point. The family of curves is given as

$$f(x,y) = c, \tag{1}$$

the parameter c varying along the family. An important tool for the inverse problem is Szebehely's linear first order partial differential equation

$$f_x V_x + f_y V_y = 2W(E - V), \tag{2}$$

where $E = E(c)$ represents the energy dependence and $W = W(x,y)$ is a function related to the curvature of the family, given by

$$W = \frac{f_{xx} f_y^2 - 2 f_{xy} f_x f_y + f_{yy} f_x^2}{f_x^2 + f_y^2}.$$

Szebehely (1974) showed that the potential V, under whose action a unit mass material point describes the curves in family (1) with the energy dependence given in advance, verifies equation (2). This equation has been simplified by Bozis (1983) using a function γ, representing the slope of the curves of the family orthogonal to (1). The simplified equation is

$$V_x + \gamma V_y + \frac{2\Gamma}{1+\gamma^2}(E - V) = 0, \tag{3}$$

where

$$\gamma = \frac{f_y}{f_x},\ \Gamma = \gamma\gamma_x - \gamma_y. \tag{4}$$

Grigoriadou et al (1999) proved that equation (2), respectively (3), can be integrated by quadratures if the two equations

$$\frac{dy}{dx} = \gamma(x,y) \tag{5}$$

and

$$\frac{dy}{dx} = -\frac{1}{\gamma(x,y)} \tag{6}$$

are solvable. They also listed some cases in which both equations (5) and (6) are solvable and offered some criteria to assure solvability in other cases.

For example, for

$$\gamma(x,y)=\sigma_1(x)\,\sigma_2(y)\,,$$

or γ homogeneous of degree zero in x and y,

$$\gamma(x,y)=\sigma\left(\frac{y}{x}\right),$$

equations (5) and (6) are solvable and the solution of (3) can be obtained by quadratures.

2 Families given in parametric form

Recently the case of families of orbits given in parametric form was considered. The motivation is that some curves (as, for example, Lissajoux ones) can thus be described in a simpler way; some families (appearing in astrophysics) are more naturally modelled parametrically.

The family of curves is given by the parametric equations

(7) $$x=F(\lambda,b)\qquad y=G(\lambda,b)$$

where the parameter b varies from member to member of the family, as c did for (1), while λ varies along each specific curve for a fixed b. The first order equation satisfied by the potential in this case was derived by Bozis and Borghero (1998), while the expression of a general force field (not necessarily conservative) was determined by Anisiu and Pal (1999).

Families of ellipses in parametric form were considered by Bozis and Caranicolas (1997) and other examples (Bozis and Borghero, 1998) arose in connection with isotach orbits (orbits along which the kinetic energy $T=E-V$ is constant), namely the family of homocentric circles

$$x=b\cos\lambda\quad y=b\sin\lambda,$$

or logarithmic spirals

$$x=be^{k\lambda}\cos\lambda\quad y=be^{k\lambda}\sin\lambda.$$

Other parametric families appeared in astrophysics, figure-eight curves

$$x=\cos\lambda,\ y=b\sin 2\lambda$$

detected by N - body simulations in barred galaxies being studied by Caranicolas (1998).

These special families of curves have in common the property that, under natural conditions, they can be transformed in the "classical" form (1) of families in the inverse problem. In the following it is shown that this is true for the class of similar curves.

3 Families of similar curves

The curves in a family given in parametric form are called *self-similar* if the family is given by

(8) $$x=b\phi(\lambda)\quad y=b\psi(\lambda),$$

they are called *x-similar* if the family is given by

(9) $$x=b\phi(\lambda)\quad y=\psi(\lambda),$$

and *y-similar* if

(10) $$x=\phi(\lambda)\quad y=b\psi(\lambda),$$

with $b \in I_0 \subset \mathbb{R}^+$ or $\mathbb{R}^-$, $\lambda \in I \subset \mathbb{R}$ and $\phi, \psi \in C(I)$.

A family of curves can be described in the implicit way $f(x,y) = c$ (1) or in parametric form $x = F(\lambda, b)$, $y = G(\lambda, b)$ (7), the parameter b varying along the family and λ varying on each curve for a fixed b. The parametric description of the family can be regarded as a transformation from a domain in the xy Cartesian plane to a domain in the λb plane. As mentioned by Anisiu and Pal (1999), this transformation is one-to-one and with a C^1 inverse (at least locally) if it is of C^1 class and has the Jacobian

$$J = F_\lambda G_b - F_b G_\lambda$$

different from zero.

Proposition 1 *For $\phi, \psi \in C^1(I)$ the Jacobians of the transformations (8), (9) and (10) are respectively*

$$\begin{aligned} J_1 &= b\,(\phi'(\lambda)\,\psi(\lambda) - \phi(\lambda)\psi'(\lambda)), \\ J_2 &= -\phi(\lambda)\psi'(\lambda), \\ J_3 &= \phi'(\lambda)\psi(\lambda). \end{aligned}$$

For these specific transformations we can give the expression of the inverse. This will allow us to find the corresponding families of curves in the xy plane which originate from families of similar curves.

4 The case of self-similar curves

Let the functions ϕ and ψ in (8) be given so that the Jacobian J_1 is different from 0 and $\phi(\lambda) \neq 0$ for each $\lambda \in I$; it follows that $\frac{\psi}{\phi}$ is one-to-one. A similar result holds for $\psi(\lambda) \neq 0$ for each $\lambda \in I$, $\frac{\phi}{\psi}$ being one-to-one. Then from (8) we obtain

$$\frac{y}{x} = \frac{\psi}{\phi}(\lambda).$$

So we get

$$\lambda = \left(\frac{\psi}{\phi}\right)^{-1}\left(\frac{y}{x}\right), \tag{11}$$

and then

$$b = \sqrt{\frac{x^2 + y^2}{\phi^2(\lambda) + \psi^2(\lambda)}}, \tag{12}$$

if $I \subset \mathbb{R}^+$ (for $I \subset \mathbb{R}^-$, $b = -\sqrt{\frac{x^2+y^2}{\phi^2(\lambda)+\psi^2(\lambda)}}$), λ being given by (11). The case of self-similar curves gives rise to the family of curves (12) described in the classical way for the inverse problem of dynamics. The function f which appears in this case can be written as

$$f_1(x,y) = xF_1\left(\frac{y}{x}\right), \quad x > 0 \tag{13}$$

(respectively $f_1(x,y) = -xF_1\left(\frac{y}{x}\right)$ for $x < 0$), where

$$F_1(u) = \sqrt{\frac{1 + u^2}{\phi^2\left(\left(\frac{\psi}{\phi}\right)^{-1}(u)\right) + \psi^2\left(\left(\frac{\psi}{\phi}\right)^{-1}(u)\right)}}. \tag{14}$$

If the functions are of C^1 class and we calculate $\gamma = \frac{f_y}{f_x}$ corresponding to f, we obtain

$$\gamma_1 = \frac{F_1'\left(\frac{y}{x}\right)}{F_1\left(\frac{y}{x}\right) - F_1'\left(\frac{y}{x}\right)\frac{y}{x}}. \tag{15}$$

The Jacobian can be written as

$$J_1 = -b\phi^2(\lambda)\left(\frac{\psi}{\phi}\right)'(\lambda).$$

We have proved

Theorem 2 *Let $\phi, \psi \in C^1(I)$ be given so that $\phi(\lambda) \neq 0$ and $\left(\frac{\psi}{\phi}\right)'(\lambda) \neq 0$ for each $\lambda \in I$. The family (8) can be described in the form (1) with f given by (13)-(14), and γ given by the function in (15) which is homogeneous of degree zero in x and y .*

5 The case of x-similar curves

Let the functions ϕ and ψ in (9) be given so that $\phi(\lambda) \neq 0$ and $\psi'(\lambda) \neq 0$ for each $\lambda \in I$; it follows that ψ is one-to-one. From the second equality in (9) we obtain $\lambda = \psi^{-1}(y)$, and from the first one

$$b = \frac{x}{\phi \circ \psi^{-1}(y)}.$$

The function f which can describe the family of orbits in this case is of the form

$$f_2(x, y) = \frac{x}{F_2(y)}, \tag{16}$$

with

$$F_2(y) = \phi \circ \psi^{-1}(y). \tag{17}$$

For γ we obtain the value

$$\gamma_2 = -\frac{xF_2'(y)}{F_2(y)}. \tag{18}$$

In this case we have

Theorem 3 *Let $\phi, \psi \in C^1(I)$ be given so that $\phi(\lambda) \neq 0$ and $\psi'(\lambda) \neq 0$ for each $\lambda \in I$. The family (9) can be written in the form (1) with f given by (16)-(17), γ being given by the expression with separable variables in (18).*

6 The case of y-similar curves

This case is analogous to the case of x-similar curves. Considering ϕ and ψ so that $\psi(\lambda) \neq 0$ and $\phi'(\lambda) \neq 0$ for each $\lambda \in I$ (hence ϕ one-to-one), from the first equality (10) we obtain $\lambda = \phi^{-1}(x)$, and from the second one

$$b = \frac{y}{\psi \circ \phi^{-1}(x)}.$$

The function f is then given by

$$f_3(x, y) = \frac{y}{F_3(x)}, \tag{19}$$

where

$$F_3(x) = \psi \circ \phi^{-1}(x), \tag{20}$$

γ being

$$\gamma_3 = -\frac{F_3(x)}{yF_3'(x)}. \tag{21}$$

We can state

Theorem 4 *Let $\phi, \psi \in C^1(I)$ be given so that $\psi(\lambda) \neq 0$ and $\phi'(\lambda) \neq 0$ for each $\lambda \in I$. The family (10) can be written in the implicit form (1) with f given by (19)-(20), γ being a function with separable variables expressed in (21).*

7 Conclusions and examples

As it was mentioned in the introduction, Grigoriadou et al (1999) proved that the basic partial differential equation (3) of the inverse problem can be integrated by quadratures in the case that the function γ is homogeneous of order 0 in x and y, or with separable variables. The results in the above theorems allow us to rewrite any family of similar curves (satisfying some natural conditions) in the classical form, and, more than that, to obtain the potential satisfying equation (3) by quadratures. So, for these types of families it is useful to deal with the classical description instead of the parametric one.

Let us consider the family of figure-eight curves

$$x = \cos\lambda,\ \ y = b\sin 2\lambda$$

studied by Caranicolas (1997), for which an approximate potential was found (containing the first terms of a power series in the small parameter b). Applying theorem 4 for this y-similar family of curves with $b \in (0,\infty)$, $\lambda \in \left(0, \frac{\pi}{2}\right)$ one obtains $f_3(x,y) = \frac{y}{x\sqrt{1-x^2}}$, $x \in (0,1)$, $y \in (0,\infty)$, which gives rise to the family

(22)
$$\frac{y^2}{x^2(1-x^2)} = c\ ;$$

the functions γ and Γ in (4) are in this case

$$\gamma = -\frac{x(x^2-1)}{y(2x^2-1)},\quad \Gamma = -\frac{x^3(x^2-1)(2x^2-3)}{y^2(2x^2-1)^3}.$$

The first equation of the subsidiary system of ordinary differential equations for Szebehely's partial differential equation (3) is

$$\frac{dy}{dx} = -\frac{x(x^2-1)}{y(2x^2-1)},$$

which gives the integral

(23)
$$y^2 + \frac{1}{2}x^2 - \frac{1}{4}\ln\left|\frac{2x^2-1}{2}\right| = c_1.$$

The second equation is

$$\frac{dV}{dx} = u(x,c_1)V - u(x,c_1)\bar{E}(x,c_1),$$

where $u(x,c_1)$ and $\bar{E}(x,c_1)$ are obtained from $\frac{2\Gamma(x,y)}{1+\gamma^2(x,y)}$, respectively $E(f(x,y))$, by substituting $y^2 = c_1 - \frac{1}{2}x^2 + \frac{1}{4}\ln\left|\frac{2x^2-1}{2}\right|$. We obtain

$$V = \exp\left(\int^x u(s,c_1)ds\right)\left(c_2 - \int^x u(s,c_1)\bar{E}(s,c_1)\exp\left(-\int^s u(t,c_1)dt\right)ds\right).$$

The calculations show that the function u has the form

$$u(x,c_1) = \frac{v'(x,c_1)}{v(x,c_1)},$$

where v' denotes the derivative with respect to x of the function

$$v(x,c_1) = c_1 - \frac{1}{2}x^2 + \frac{1}{4}\ln\left|\frac{2x^2-1}{2}\right| + \frac{x^2(x^2-1)^2}{(2x^2-1)^2}.$$

The expression of V can be written

$$V = v(x,c_1)\left(c_2 - \int^x \frac{v'(s,c_1)}{v^2(s,c_1)}\bar{E}(s,c_1)ds\right).$$

After an integration by parts we get

$$V = \bar{E}(x, c_1) + v(x, c_1)\left(c_2 + I(x, c_1)\right),$$

where

$$I(x, c_1) = -\int^x E'\left(\frac{c_1 - \frac{1}{2}s^2 + \frac{1}{4}\ln\left|\frac{2s^2-1}{2}\right|}{s^2(1-s^2)}\right)\frac{2(2s^2-1)}{s^3(s^2-1)^2}ds \tag{24}$$

and E' denotes the derivative of the function E with respect to its argument.

The general solution of the partial differential equation (3) will be given by $c_2 = A(c_1)$ with A an arbitrary function of c_1 from (23). For the family of curves (22) traced with a preassigned energy $E(f)$, the potentials creating it are given by

$$V(x, y) = E(f(x, y)) + \left(y^2 + \frac{x^2(x^2-1)^2}{(2x^2-1)^2}\right)\left(A(c_1) + I(x, c_1)\right), \tag{25}$$

where I is given by (24) and $c_1 = y^2 + \frac{1}{2}x^2 - \frac{1}{4}\ln\left|\frac{2x^2-1}{2}\right|$.

It is known that during the motion of a material point of unit mass along an orbit of the family the inequality

$$E(f(x, y)) - V(x, y) \geq 0, \tag{26}$$

must be observed, so real motion will take place only in the region where

$$A(c_1) + I(x, c_1) \leq 0, \tag{27}$$

c_1 being given by (23).

If we are interested in finding the potentials producing families of orbits with constant energy (i.e. the energy has the same value e for all the curves in the family), we obtain

$$V(x, y) = e + \left(y^2 + \frac{x^2(x^2-1)^2}{(2x^2-1)^2}\right) A\left(y^2 + \frac{1}{2}x^2 - \frac{1}{4}\ln\left|\frac{2x^2-1}{2}\right|\right),$$

in this case I being identically null.

If we choose $E(f) = f + 1$, it follows $E'(f) = 1$ and $I(x, c_1)$ will depend only on x,

$$I(x, c_1) = \frac{1}{x^2(x^2-1)}.$$

The value of the potential will be obtained from (25)

$$V(x, y) = \frac{x^2(4x^2-3)}{(2x^2-1)^2} + \left(y^2 + \frac{x^2(x^2-1)^2}{(2x^2-1)^2}\right) A\left(y^2 + \frac{1}{2}x^2 - \frac{1}{4}\ln\left|\frac{2x^2-1}{2}\right|\right).$$

In this case, considering the special value of the arbitrary function A, $A(z) = 4$ we obtain the potential

$$V_p(x, y) = 4y^2 + x^2$$

which is a two-dimensional harmonic oscillator potential with the ratio of frequencies 1:2. This represents a very simple potential producing figure-eight orbits.

We have proved that the case of orbits (22) given in parametric form can be successfully treated by using the classical implicit description of the family. The most general form of potentials producing this family is given by (25) and, choosing the dependence of the energy on the family, we can obtain specific potentials. The inequality (27) allows us to programme motion in some regions of the space chosen in advance, having at our disposal the arbitrary function A. The possibility of programming motion was studied in detail by Anisiu and Bozis (2000) for the case of families of curves $f(x, y) = y - h(x)$; for that type of families the integration by quadratures of Szebehely's equation can also be accomplished.

References

[1] Anisiu, M.-C., Pal, A., Two-dimensional inverse problem of dynamics for families in parametric form, *Inverse Problems* **15** (1999), 135-140

[2] Anisiu, M.-C., Bozis, G., Programmed motion for a class of families of planar orbits, *Inverse Problems* **16** (2000), 19-32

[3] Bozis, G., Inverse problem with two parametric families of planar orbits, *Celest. Mech.* **31** (1983), 129-142

[4] Bozis, G., Caranicolas, N. D., The Earth and the Universe, Special Volume of the Aristotle University of Thessaloniki (ed. Asteriadis et al), 1997

[5] Bozis, G., Borghero, F., A new formulation of the two-dimensional inverse problem of dynamics, *Inverse Problems* **14** (1998), 41-51

[6] Caranicolas, N. D., Potentials for the central parts of a barred galaxy, *Astronomy and Astrophysics* **332** (1998), 88-92

[7] Grigoriadou, S., Bozis, G., Elmabsout, B., Solvable cases of Szebehely's equation, *Celest. Mech.***74** (1999), 211-221

[8] Szebehely, V., On the determination of the potential by satellite observations, *Proc. Int. Meeting on Earth's Rotations by Satellite Observations* Ed. E. Proverbio (Bologna: Pitagora Editrice), 1974, 31-35