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On the pointwise convergence of a family of functionals on $\mathcal{C}(I)$

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Abstract. Given a continuous function $u : [0, \infty) \to \mathbb{R}$, a family of functionals $\varphi_\alpha : \mathcal{C}(I) \to \mathbb{R}$, $\alpha > 0$, is defined by $\varphi_\alpha(f) = \frac{1}{\alpha} \int\limits_0^\alpha u(t) f(t/\alpha)\, \mathrm{d}t$. It is proved that the necessary and sufficient conditions for the family φ_α, $\alpha > 0$ to satisfy $\lim\limits_{\alpha\to\infty} \varphi_\alpha(f) = \left(\lim\limits_{\alpha\to\infty} \frac{1}{\alpha} \int\limits_0^\alpha u(t)\mathrm{d}t\right) \cdot \int\limits_0^1 f$ are:

I. $\exists \lim\limits_{\alpha\to\infty} \frac{1}{\alpha} \int\limits_0^\alpha u(t)\mathrm{d}t$; II. $\sup\limits_{\alpha>0} \frac{1}{\alpha} \int\limits_0^\alpha |u(t)|\, \mathrm{d}t < \infty$.

If $f \in \mathcal{C}^1(I)$, condition I alone implies the existence of $\lim\limits_{\alpha\to\infty} \varphi_\alpha(f)$.

A sequence of functionals $(\varphi_n)_{n\in\mathbb{N}}$ is attached to a numerical sequence $(a_n)_{n\in\mathbb{N}}$ which is Cesàro-convergent to a, namely

$$\varphi_n(f) = \frac{1}{n} \sum_{k=1}^{n} a_k f(k/n), \; f \text{ Riemann integrable.}$$

Additional conditions are imposed on the sequence $(a_n)_{n\in\mathbb{N}}$ in order to prove that

$$\lim_{n\to\infty} \varphi_n(f) = a \cdot \int_0^1 f.$$

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1 Introduction

For the interval $I=[0,1]$ and a Banach space $F\neq\{0\}$, let us denote by $\mathcal{B}(I,F)$ the Banach space of bounded functions $f:I\to F$ endowed with the sup norm. The subspace of $\mathcal{B}(I,F)$ of regular functions (which admit side limits at each $t\in I$) will be denoted by $\mathcal{R}(I,F)$; the Banach space of continuous functions $\mathcal{C}(I,F)$ is a subspace of $\mathcal{R}(I,F)$.

Given a sequence of real numbers $(a_n)_{n\in\mathbb{N}}$, a sequence of operators $\varphi_n:\mathcal{R}(I,F)\to F,\ n\in\mathbb{N}$,

$$\varphi_n(f)=\frac{1}{n}\sum_{k=1}^{n}a_k f\left(\frac{k}{n}\right) \tag{1.1}$$

can be generated. It was proved in [1] that:

A. The operators φ_n are linear and continuous.

B. If the numeric sequence satisfies the conditions:

B_1. $(a_n)_{n\in\mathbb{N}}$ is Cesàro-convergent to a ($\lim\limits_{n\to\infty}\frac{a_1+\ldots+a_n}{n}=a$);

B_2. the sequence $\left(\frac{|a_1|+\ldots+|a_n|}{n}\right)_{n\in\mathbb{N}}$ is bounded,

then the sequence $(\varphi_n(f))_{n\in\mathbb{N}}$ is convergent and

$$\lim_{n\to\infty}\varphi_n(f)=a\cdot\int_0^1 f. \tag{1.2}$$

C. If $\lim\limits_{n\to\infty}\varphi_n(f)$ exists for every $f\in\mathcal{C}(I,F)\subseteq\mathcal{R}(I,F)$, the conditions B_1 and B_2 from above are also necessary.

D. If $f\in\mathcal{C}^1(I,F)$ (i.e. f is continuous with a continuous derivative), the result in B holds even if condition B_2 is omitted.

The aim of this paper is to provide continuous variants of these results for families of functionals (for the sake of simplicity we consider $F=\mathbb{R}$).

Let $u:[0,\infty)\to\mathbb{R}$ be a continuous function. We define a family of functionals associated to u, namely $\varphi_\alpha : \mathcal{C}(I) \to \mathbb{R}$, $\alpha > 0$, given by

$$\varphi_\alpha(f) = \frac{1}{\alpha} \int_0^\alpha u(t) f\left(\frac{t}{\alpha}\right) \mathrm{d}t, \ f \in \mathcal{C}(I). \tag{1.3}$$

Proposition 1.1 *For each $\alpha > 0$, the functional φ_α is linear and continuous, and its norm is given by*

$$\|\varphi_\alpha\| = \frac{1}{\alpha} \int_0^\alpha |u(t)| \,\mathrm{d}t. \tag{1.4}$$

Proof. This result is classical; see for a simple proof [3]. It also holds if $\mathcal{C}(I)$ is replaced with the Banach space $\mathcal{R}(I)$ of regular functions. ∎

2 Main results for families of functionals

As mentioned in B and C in the introduction, in the discrete case the conditions B_1 and B_2 are necessary and sufficient for the sequence (1.1) to converge, the limit being given by (1.2). We can prove a similar result for the continuous case.

Theorem 2.1 *Let there be given $f \in \mathcal{C}(I)$ and $u \in \mathcal{C}([0,\infty))$. If the function u satisfies the conditions:*

I. $\exists \lim\limits_{\alpha\to\infty} \frac{1}{\alpha} \int_0^\alpha u(t)\mathrm{d}t$;

II. $\sup\limits_{\alpha>0} \frac{1}{\alpha} \int_0^\alpha |u(t)| \,\mathrm{d}t < \infty$,

there exists the limit $\lim\limits_{\alpha\to\infty} \varphi_\alpha(f)$ and

$$\lim_{\alpha\to\infty} \varphi_\alpha(f) = \left(\lim_{\alpha\to\infty} \frac{1}{\alpha} \int_0^\alpha u(t)\mathrm{d}t\right) \cdot \int_0^1 f. \tag{2.1}$$

The conditions I and II are also necessary in order to have (2.1) for each $f \in \mathcal{C}(I)$.

Proof. Let us suppose that conditions I and II hold. We can prove (2.1) even for the more general case of a regular function f. To this end it suffices to prove (2.1) for $f \in E$ with $E \subseteq \mathcal{R}(I)$ and $\overline{\text{sp}}E = \mathcal{R}(I)$ (condition II allows then to obtain (2.1) for any $f \in \mathcal{R}(I)$). As a set E we choose the set of characteristic functions $\chi_{[a,b]}$, $[a,b] \subseteq I$.

For $f = \chi_{[a,b]}$ we have

$$\begin{aligned}\varphi_\alpha(f) &= \int_0^1 u(\alpha t) f(t)\mathrm{d}t = \int_a^b u(\alpha t)\mathrm{d}t = \tfrac{1}{\alpha}\int_{\alpha a}^{\alpha b} u(t)\mathrm{d}t \\ &= b\tfrac{1}{\alpha b}\int_0^{\alpha b} u(t)\mathrm{d}t - a\tfrac{1}{\alpha a}\int_0^{\alpha a} u(t)\mathrm{d}t.\end{aligned}$$

It follows

$$\begin{aligned}\lim_{\alpha\to\infty}\varphi_\alpha(f) &= (b-a)\lim_{\alpha\to\infty}\tfrac{1}{\alpha}\int_0^{\alpha} u(t)\mathrm{d}t \\ &= \left(\lim_{\alpha\to\infty}\tfrac{1}{\alpha}\int_0^{\alpha} u(t)\mathrm{d}t\right)\cdot\int_0^1 f.\end{aligned}$$

Let us suppose now that (2.1) takes place for each $f \in \mathcal{C}(I)$. For $f(x) = 1$, $\forall x \in I$ we obtain condition I. To prove II, we apply the Banach-Steinhaus principle for sequences $\alpha_n \to \infty$, because it cannot be used directly for generalized sequences. (In fact, if X is an infinite dimensional Banach space, then its dual X^* is sequentially closed and dense in $\left(X^{\#}, \text{weak}^*\right)$, see [6, p. 138].) It follows that $\sup_{\alpha>0}\|\varphi_\alpha\| < \infty$, and using the expression of $\|\varphi_\alpha\|$ given in (1.4) we obtain II. ■

Remark 2.1 *If the function u is periodic with $u(t+T) = u(t)$ for each $t > 0$, then*

$$\lim_{\alpha\to\infty}\frac{1}{\alpha}\int_0^{\alpha} u(t)\mathrm{d}t = \frac{1}{T}\int_0^{T} u(t)\mathrm{d}t$$

and condition II automatically holds because of the boundedness of u. In *this special case we obtain*

$$\lim_{\alpha\to\infty}\varphi_\alpha(f)=\frac{1}{T}\int_0^T u(t)\mathrm{d}t\cdot\int_0^1 f, \tag{2.2}$$

which is a result due to L. Fejér, see [4, p. 114].

Remark 2.2 *Condition II does not follow from condition I. Indeed, let* $u\in\mathcal{C}([0,\infty))$ *be given by*

$$u(x)=\begin{cases}2\sqrt{n+1}\,(x-2n)\,,\ x\in[2n,2n+1/2)\\ -2\sqrt{n+1}\,(x-2n-1)\,,\ x\in[2n+1/2,2n+3/2)\\ 2\sqrt{n+1}\,(x-2n-2)\,,\ x\in[2n+3/2,2n+2)\end{cases}$$

$(n\in\mathbb{N})$*. For this function we have*

$$\sup_{\alpha>0}\frac{1}{\alpha}\int_0^\alpha|u(t)|\,\mathrm{d}t\ge\sup_{n\in\mathbb{N}^*}\frac{1}{2n}\int_0^{2n}|u(t)|\,\mathrm{d}t\ =\sup_{n\in\mathbb{N}^*}\frac{1}{2n}\sum_{k=1}^n\sqrt{k}=\infty.$$

For $2n\le\alpha<2n+2$ *it follows* $\frac{1}{\alpha}\int_0^\alpha u(t)\mathrm{d}t=\frac{1}{\alpha}\int_{2n}^\alpha u(t)\mathrm{d}t\le\frac{\sqrt{n+1}}{2\alpha}\le\frac{\sqrt{n+1}}{4n}$ *and* $\lim_{\alpha\to\infty}\frac{1}{\alpha}\int_0^\alpha u(t)\mathrm{d}t=0$.

If we consider only functions in $\mathcal{C}^1(I)$, we obtain the corresponding property as in D mentioned in the introduction for sequences of operators.

Theorem 2.2 *Let there be given a function* $f\in\mathcal{C}^1(I)$ *and* $u\in\mathcal{C}([0,\infty))$*. If the function* u *satisfies condition I from Theorem 2.1* $\left(\exists\lim_{\alpha\to\infty}\frac{1}{\alpha}\int_0^\alpha u(t)\mathrm{d}t=a\right)$, *then there exists* $\lim_{\alpha\to\infty}\varphi_\alpha(f)$ *and its value is given by (2.1).*

Proof. Let U be an antiderivative of u with $U(0) = 0$ (i.e. $U(x) = \int_0^x u(t)\mathrm{d}t$). We have for $f \in \mathcal{C}^1(I)$

$$\begin{aligned}\varphi_\alpha(f) &= \frac{1}{\alpha}\int_0^\alpha U'(t) f\left(\frac{t}{\alpha}\right)\mathrm{d}t \\ &= \frac{1}{\alpha}U(\alpha)f(1) - \frac{1}{\alpha^2}\int_0^\alpha U(t)f'\left(\frac{t}{\alpha}\right)\mathrm{d}t \\ &= \frac{1}{\alpha}U(\alpha)f(1) - \frac{1}{\alpha}\int_0^1 U(\alpha t)f'(t)\,\mathrm{d}t.\end{aligned}$$

But $\frac{1}{\alpha}U(\alpha t) = \frac{1}{\alpha}\int_0^{\alpha t} u(s)\mathrm{d}s$ and we get $\lim\limits_{\alpha\to\infty}\frac{1}{\alpha}U(\alpha t) = ta$. Using the theorem of dominated convergence we get

$$\lim_{\alpha\to\infty}\varphi_\alpha(f) = af(1) - a\int_0^1 tf'(t)\,\mathrm{d}t = a\cdot\int_0^1 f.$$

■

3 A new result for sequences of operators

In connection with the result mentioned in B in the introduction for the sequence of operators $\varphi_n : \mathcal{R}(I,F) \to F,\ n \in \mathbb{N}$, given by (1.1), an open question was formulated in [1]: Is the conclusion in B true if $F = \mathbb{R}$ for f Riemann integrable (instead of regular)?

We shall prove that this result holds if the sequence $(a_n)_{n\in\mathbb{N}}$ is bounded from above or below, or if $\left(\frac{|a_1|+\cdots+|a_n|}{n}\right)_{n\in\mathbb{N}}$ is convergent.

Theorem 3.1 *Let there be given a Riemann integrable function $f : I \to \mathbb{R}$ and a sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers satisfying the conditions:*

1. $\lim\limits_{n\to\infty}\frac{a_1+\ldots+a_n}{n} = a$;
2. *the sequence* $\left(\frac{|a_1|+\ldots+|a_n|}{n}\right)_{n\in\mathbb{N}}$ *is bounded;*

3. the sequence $(a_n)_{n\in\mathbb{N}}$ is bounded from above or from below, or $\left(\frac{|a_1|+\ldots+|a_n|}{n}\right)_{n\in\mathbb{N}}$ is convergent.

Then the sequence $(\varphi_n(f))_{n\in\mathbb{N}}$ given by (1.1) is convergent to $a\cdot\int_0^1 f$.

Proof. (i) Let us consider a sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \geq 0$, which satisfies condition 1 (hence also condition 2). Given $\varepsilon > 0$, for the Riemann integrable function f there exist the continuous functions $u, v : I \to \mathbb{R}$ such that

$$u \leq f \leq v \text{ and } \int (v-u) < \varepsilon. \tag{3.1}$$

Then the functionals φ_n given by (1.1) will satisfy

$$\varphi_n(u) \leq \varphi_n(f) \leq \varphi_n(v). \tag{3.2}$$

From the result in [1] mentioned at B in the introduction we have $\lim\limits_{n\to\infty} \varphi_n(v) = a\cdot\int_0^1 v$, hence there exists $n_1 \in \mathbb{N}$ so that for any $n \geq n_1$, $\varphi_n(v) < a\cdot\int_0^1 v + \varepsilon$. Condition (3.1) implies that $\int_0^1 v < \int_0^1 f + \varepsilon$, hence

$$\varphi_n(v) < a\cdot\int_0^1 f + \varepsilon(a+1). \tag{3.3}$$

Similarly, there exists $n_2 \in \mathbb{N}$ so that for any $n \geq n_2$,

$$a\cdot\int_0^1 f - \varepsilon(a+1) < \varphi_n(u). \tag{3.4}$$

From (3.2), (3.3) and (3.4) we obtain

$$a \cdot \int_0^1 f - \varepsilon(a+1) < \varphi_n(f) < a \cdot \int_0^1 f + \varepsilon(a+1),$$

hence $\left|\varphi_n(f) - a \cdot \int_0^1 f\right| \le \varepsilon(a+1)$ for $n \ge \max\{n_1, n_2\}$ and the conclusion holds.

(ii) Let us consider $(a_n)_{n\in\mathbb{N}}$ which satisfies the conditions 1 and 2 and is bounded from below, i. e. $a_n \ge -\alpha$. The sequence $(b_n)_{n\in\mathbb{N}}$, $b_n = a_n + \alpha$ is Cesàro-convergent to $a+\alpha$ and has $b_n \ge 0$; applying the result proved in (i) it follows that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (a_n + \alpha) f\left(\frac{k}{n}\right) = (a+\alpha) \cdot \int_0^1 f,$$

hence $\lim_{n\to\infty} \varphi_n(f) = a \cdot \int_0^1 f$.

(iii) If the sequence $(a_n)_{n\in\mathbb{N}}$ satisfies 1 and 2, and is bounded from above $(a_n \le \alpha)$, the sequence $(c_n)_{n\in\mathbb{N}}$, $c_n = \alpha - a_n$ is Cesàro-convergent to $\alpha - a$ and has $c_n \ge 0$. Applying again the result proved in (i) we obtain the conclusion.

(iv) Let us consider the sequence $(a_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} \frac{a_1+\ldots+a_n}{n} = a$ and $\lim_{n\to\infty} \frac{|a_1|+\ldots+|a_n|}{n} = a^*$. We can write

$$a_n = \frac{a_n + |a_n|}{2} - \frac{|a_n| - a_n}{2}.$$

The sequence $\left(\frac{a_n+|a_n|}{2}\right)_{n\in\mathbb{N}}$ is Cesàro-convergent to $\frac{a+a^*}{2}$ and has non-negative terms, hence it follows from (i) that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(\frac{a_n + |a_n|}{2}\right) f\left(\frac{k}{n}\right) = \frac{a+a^*}{2} \cdot \int_0^1 f.$$

Similarly, for $\left(\frac{|a_n|-a_n}{2}\right)_{n\in\mathbb{N}}$ we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{|a_n|-a_n}{2}\right)f\left(\frac{k}{n}\right)=\frac{a^*-a}{2}\cdot\int_0^1 f.$$

It follows

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}a_nf\left(\frac{k}{n}\right)=\frac{a+a^*}{2}\cdot\int_0^1 f-\frac{a^*-a}{2}\cdot\int_0^1 f=a\cdot\int_0^1 f.$$

■

Remark 3.1 *The conditions in 3 of Theorem 3.1 are not consequences of 1 and 2, as the following examples show.*

Example 3.1 *A Cesàro-convergent sequence* $(a_n)_{n\in\mathbb{N}}$ *which is not bounded from above or from below, for which* $\left(\frac{|a_1|+...+|a_n|}{n}\right)_{n\in\mathbb{N}}$ *is bounded, is given by*

$$a_n=\begin{cases} k,\ for\, n=2^k,\ k\ even\\ -k,\ for\, n=2^k,\ k\ odd\\ 0,\ otherwise.\end{cases}$$

Example 3.2 *A Cesàro-convergent sequence* $(a_n)_{n\in\mathbb{N}}$, *for which* $\left(\frac{|a_1|+...+|a_n|}{n}\right)_{n\in\mathbb{N}}$ *is bounded without being convergent, can be obtained from a bounded sequence* $d_n\geq 0$ *for which* $\frac{d_1+...+d_n}{n}$ *does not converge, as*

$$a_n=\begin{cases} d_{n/2},\ for\, n\ even\\ -d_{(n+1)/2},\ for\, n\ odd.\end{cases}$$

Such a sequence $(d_n)_{n\in\mathbb{N}}$ *is, for example,*

$$d_n=\begin{cases} 1,\ for\, n\in\left[2^k,2^{k+1}\right),\ k\ even\\ \\ 0,\ for\, n\in\left[2^k,2^{k+1}\right),\ k\ odd.\end{cases}$$

Indeed, we have

$$\lim_{k\to\infty,\, k\ even} \frac{d_1 + ... + d_{2^{k+1}-1}}{2^{k+1}-1} = \lim_{k\to\infty} \frac{2^0 + 2^2 + ... + 2^k}{2^{k+1}-1} = \frac{2}{3},$$

and

$$\lim_{k\to\infty,\, k\ odd} \frac{d_1 + ... + d_{2^{k+1}-1}}{2^{k+1}-1} = \lim_{k\to\infty} \frac{2^0 + 2^2 + ... + 2^{k-1}}{2^{k+1}-1} = \frac{1}{3}.$$

Remark 3.2 *We mention, in connection with Theorem 3.1, the following result from [2] (see also [5]):*

For $k > 0$, if $f, g : [0,\infty) \to \mathbb{R}$ satisfy:

− $f \in \mathcal{C}^1([0,\infty))$ is decreasing with $f(\infty) = 0$ and there exists $\lim\limits_{\alpha\to\infty} \int\limits_0^\alpha f(x + k\alpha)\mathrm{d}x = a$;

− $g \in \mathcal{C}([0,\infty))$ is bounded and $\exists \lim\limits_{\alpha\to\infty} \frac{1}{\alpha}\int\limits_0^\alpha g = b$,

then $\lim\limits_{\alpha\to\infty} \int\limits_0^\alpha f(x + k\alpha)g(x)\mathrm{d}x = ab$.

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