An estimation of a generalized divided difference in uniformly convex spaces

MIRA-CRISTIANA ANISIU (CLUJ-NAPOCA)

VALERIU ANISIU (CLUJ-NAPOCA)

Abstract

The rest in some approximation formulae can be expressed in terms of a generalized divided difference on three knots. We provide an estimation of such a divided difference for functions defined on a uniformly convex space.

KEY WORDS: uniformly convex space; Fréchet derivative; generalized divided difference

MSC 2000: 46B20, 46A55, 46T20

1 Introduction

The convexity properties of the functions were used by Tiberiu Popoviciu to give estimations of the rest in some approximation formulae. A synthesis of this type of results can be found in the book [4].

Several theorems of representation of linear functionals were proved by Raşa [5], [6]. To mention two of them, let E denote a locally convex Hausdorff real space and X a compact convex metrizable subset of E; for $f \in C(X)$, $x, y \in X$ and $a \in [0,1]$, we denote

$$(x, a, y; f) = (1-a) f(x) + a f(y) - f((1-a)x + ay). \qquad (1)$$

We remark that, since X is a metrizable space, there exist strictly convex functions in $C(X)$; we denote by φ such a function.

Theorem 1 *Let $L : C(X) \to \mathbb{R}$ be a linear functional such that $L(g) > 0$ for each strictly convex function $g \in C(X)$. Then for every $f \in C(X)$ there exists $x, y \in X$, $x \neq y$ and $a \in (0,1)$ such that*

$$L(f) = L(\varphi) \frac{(x, a, y; f)}{(x, a, y; \varphi)}.$$

We consider now $C(X)$ endowed with the uniform norm.

Theorem 2 *Let $L : C(X) \to \mathbb{R}$ be a continuous and linear functional such that $L(g) \geq 0$ for each convex function $g \in C(X)$. Then for every $f \in C(X)$ there exists $x, y \in X$, $x \neq y$ and $a \in (0,1)$ such that*

$$L(f) = L(\varphi) \frac{(x, a, y; f)}{(x, a, y; \varphi)}.$$

For the special case $E = \mathbb{R}$, $X = [0,1]$ and $\varphi(t) = t^2$, $t \in [0,1]$, Ivan and Raşa [3] showed that

$$(x, a, y; \varphi) = (1-a)\, x^2 + ay^2 - ((1-a)\, x + ay)^2 = a\,(1-a)\,(x-y)^2\,,$$

for all $x, y, a \in [0,1]$. In this case it follows that

$$\frac{(x, a, y; f)}{(x, a, y; \varphi)} = [x, (1-a)\, x + ay, y]\,,$$

where the last expression is the classical divided difference of the real function f on the knots x, $(1-a)\, x + ay$ and y. In the general case,

$$[x, a, y; f, \varphi] := \frac{(x, a, y; f)}{(x, a, y; \varphi)} \tag{2}$$

with $(x, a, y; f)$ given by (1) was then named *generalized divided difference on three knots.*

2 Main results

We give an estimate of the generalized divided difference (2) in the case of a real uniformly convex space.

Let $(E, \|\cdot\|)$ be a real smooth uniformly convex space and X a compact subset of E. Consider the (strictly convex) function $\varphi_r \in C(X)$ given by

$$\varphi_r(x) = \|x\|^r, \ x \in X,$$

where $1 < r \leq 2$.

We need upper and lower bounds for the expression $(x, a, y; f)$. An upper bound for $|(x, a, y; f)|$ was found in [3], for f twice Fréchet differentiable on an open set Y and $\|f''(y)\| \leq M$ for each $y \in Y$, namely

$$|(x, a, y; f)| \leq \frac{M}{2} a(1-a) \|x-y\|^2 . \tag{3}$$

It was proved for Hilbert case, but it can be shown that (3) holds in our setting too.

If f is a convex function, $(x, a, y; f)$ is ≥ 0 and is related with the modulus of uniform strict convexity. We recall some definitions from [7], [2]. The *modulus of uniform strict convexity* at x (named *gage of uniform convexity* in [7]) is

$$\mu_f(x,t) = \inf_{\substack{y \in \mathrm{dom}(f) \\ \|x-y\|=t \\ \lambda \in (0,1)}} \frac{(x, \lambda, y; f)}{\lambda(1-\lambda)}, \ t \geq 0.$$

A related function is

$$\overline{\mu}_f(x,t) = \inf_{\substack{y \in \mathrm{dom}(f) \\ \|x-y\|=t}} \left(f(x) + f(y) - 2f\left(\frac{x+y}{2}\right) \right).$$

One has [2]:

$$\frac{1}{2}\mu \leq \overline{\mu} \leq \mu. \tag{4}$$

The function f is said to be *uniformly convex* at x if $\mu_f(x,t) > 0$ for each $t > 0$. The *modulus of total convexity* of f at x is defined by

$$\nu_f(x,t) = \inf_{\substack{y \in \mathrm{dom}(f) \\ \|x-y\|=t}} \left(f(y) - f(x) - d^+ f(x, y-x) \right) \tag{5}$$

where $d^+ f(x,h)$ denotes the directional derivative (it exists for f convex).

One has

$$\nu_f \geq \mu_f, \tag{6}$$

but a reversed inequality holds only when f is Fréchet differentiable, namely in this case there exists a positive constant α such that

$$\overline{\mu}_f(x,t) \geq \alpha \nu_f(x, \frac{t}{2}). \tag{7}$$

We are interested now in the case $f(x) = \varphi_r(x) = \|x\|^r$ for $r > 1$.

Lemma 3 *If E is a uniformly convex space for which the norm is smooth, then for each $R > 0$ there exists a positive constant K such that*

$$\mu_{\varphi_r}(z,t) \geq K t^r$$

for each $z \in E$, $\|z\| \leq R$.

Proof. Denote by δ_E the modulus of uniform convexity of the space. Using theorem 1 in [1], there exists a positive constant $K_1 > 0$ such that

$$\nu_{\varphi_r}(z,t) \geq r K_1 t^r \int_0^1 \tau^{r-1} \delta_E \left(\frac{\tau t}{2\left(\|z\| + \tau t\right)} \right) d\tau. \tag{8}$$

Using the well known fact that $t \mapsto \delta_E(t)/t$ is increasing, one obtains

$$\delta_E \left(\frac{\tau t}{2\left(\|z\| + \tau t\right)} \right) \geq \delta_E \left(\frac{\tau t}{2\left(R + \tau t\right)} \right) > 0,$$

so the integral in (8) is $\geq K_2$. It follows that $\nu_{\varphi_r}(z,t) \geq t^r r K_1 K_2$. Because φ_r is Fréchet differentiable, one can use (7) and get

$$\overline{\mu}_{\varphi_r}(z,t) \geq \alpha \left(\frac{t}{2} \right)^r r K_1 K_2.$$

Using (4) we have

$$\mu_{\varphi_r}(z,t) \geq 2\alpha \left(\frac{t}{2} \right)^r r K_1 K_2 = K t^r.$$

■

Theorem 4 *Let Y be an open set, $X \subset Y \subset E$ and $f : Y \to \mathbb{R}$ a function with continuous second order Fréchet derivative, such that $\|f''(y)\| \leq M$ for all $y \in Y$. Then there exists a constant K depending only on X such that*

$$|[x, a, y; f, \varphi_r]| \leq KM \tag{9}$$

for all $x, y \in X$, $x \neq y$ and $a \in (0,1)$.

Proof. As we have mentioned above,

$$|(x,a,y;f)| \leq \frac{M}{2}a(1-a)\left\|x-y\right\|^2.$$

Using lemma 3 with R such that $X \subseteq B(0,R)$, one has

$$(x,a,y;\varphi_r) \geq a(1-a)\mu_{\varphi_r}(x,\|x-y\|) \geq$$
$$a(1-a)K_1\left\|x-y\right\|^r,$$

and then

$$|[x,a,y;f,\varphi_r]| \leq \frac{M}{2K_1}\left\|x-y\right\|^{2-r} \leq KM.$$

■

In the special case when E is a Hilbert space we have

$$(x,a,y;\varphi_2) = a(1-a)\left\|x-y\right\|^2,$$

and the following result obtained in [3] holds.

Corollary 5 *Let E be a Hilbert space and $r=2$. In the conditions of theorem 4, the inequality (9) is satisfied with $K=1/2$.*

References

[1] D. Butnariu, A. N. Iusem, E. Resmerita, Total convexity for powers of the norm in uniformly convex Banach spaces, *J. Convex Analysis* **7** (2000), 319-334

[2] D. Butnariu, A. N. Iusem, C. Zălinescu, On uniform convexity, total convexity and convergence of the proximinal point and outer Bregman projection algorithms in Banach spaces, *J. Convex Analysis* **10** (2003), 35-61

[3] M. Ivan, I. Raşa, The rest in some approximation formulae, *Séminaire de la Théorie de la Meilleure Approximation, Convexité et Optimization*, Cluj-Napoca, Srima 2002, 87-92

[4] E. Popoviciu, *Teoreme de medie din analiza matematică şi legătura lor cu teoria interpolării.* Editura Dacia, Cluj, 1972

[5] I. Raşa, Sur les fonctionelles de la forme simple au sens de T. Popoviciu, *Anal. Numér. Théor. Approx.* **9** (1980), 261-268

[6] I. Raşa, *Funcţionale de formă simplă în sensul lui Tiberiu Popoviciu* (PhD Thesis), Cluj-Napoca, 1982

[7] C. Zălinescu, *Convex Anaysis in General Vector Spaces* , World Scientific, 2002

Mira-Cristiana ANISIU
T. Popoviciu Institute of Numerical Analysis
P. O. Box 68
400110 Cluj-Napoca, Romania
(E-mail: mira@math.ubbcluj.ro)

Valeriu ANISIU
Faculty of Mathematics and Computer Science
1, Kogălniceanu st
400084 Cluj-Napoca, Romania
(E-mail: anisiu@math.ubbcluj.ro)