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ON SOME STEFFENSEN-TYPE ITERATIVE METHODS FOR A CLASS OF NONLINEAR EQUATIONS

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1. INTRODUCTION

Consider a Banach space X and the equation

$$F(x) + G(x) = 0, \tag{1}$$

where $F, G : X \to X$ are nonlinear operators, F being Fréchet differentiable and G being continuously but nondifferentiable. This is the case when we study an equation $H(x) = 0$, with $H : X \to X$ a nondifferentiable operator to which we cannot apply Newton's method. H is then split into two parts: a differentiable part and a nondifferentiable one.

Various methods have been proposed for solving these kind of problems.

In [8, 9, 10] are considered the Newton-like methods:

$$x_{n+1} = x_n - F'(x_n)^{-1}\big(F(x_n) + G(x_n)\big), \quad n = 0, 1, \ldots, x_0 \in X, \tag{2}$$

and, more generally,

$$x_{n+1} = x_n - A(x_n)^{-1}\big(F(x_n) + G(x_n)\big), \quad n = 0, 1, \ldots, x_0 \in X, \tag{3}$$

where A is a linear operator approximating F'.

In [1] is studied the secant-type method

$$x_{n+1} = x_n - [x_{n-1}, x_n; F]^{-1}\big(F(x_n) + G(x_n)\big), \quad n = 1, 2, \ldots, x_0, x_1 \in X, \tag{4}$$

$[x, y; F]$ denoting the first order divided difference of F at the points x, y. The convergence order of these sequences is linear (as it can be also seen in the numerical example).

In [3] is considered a combination of Newton's method and the secant method:

$$x_{n+1} = x_n - \big(F'(x_n) + [x_{n-1}, x_n; G]\big)^{-1}\big(F(x_n) + G(x_n)\big), \quad n = 1, 2, \ldots, x_0, x_1 \in X, \tag{5}$$

having the convergence order $\frac{1+\sqrt{5}}{2} \approx 1.618$, i.e., the convergence order of the secant method.

In the present paper we propose a method based on Steffensen's method and Newton's method, having quadratic convergence:

(6)
$$x_{n+1} = x_n - \big(F'(x_n) + [x_n, \varphi(x_n); G]\big)^{-1}\big(F(x_n) + G(x_n)\big), \quad n = 0, 1, \ldots, x_0 \in X,$$

where $\varphi : X \to X$,

$$\varphi(x) = x - \lambda\big(F(x) + G(x)\big),$$

λ being a fixed positive number.

2. THE CONVERGENCE OF THE METHOD

We shall use, as in [4, 5] the known definitions for the divided differences of an operator:

Definition 1. *An operator* $[x_0, y_0; G]$ *belonging to the space* $\mathcal{L}(X, X)$ *(the Banach space of the linear and bounded operators from* X *to* X*) is called the first order divided difference of the operator* $G : X \to X$ *at the points* x_0, $y_0 \in X$ *if the following properties hold:*

a) $[x_0, y_0; G](y_0 - x_0) = G(y_0) - G(x_0)$, *for* $x_0 \neq y_0$;

b) if G *is Fréchet differentiable at* x_0, *then*

$$[x_0, x_0; G] = G'(x_0).$$

Definition 2. *An operator belonging to the space* $\mathcal{L}(X, \mathcal{L}(X, X))$, *denoted by* $[x_0, y_0, z_0; G]$, *is called the second-order divided difference of the operator* $G : X \to X$ *at the points* $x_0, y_0, z_0 \in X$ *if the following properties hold:*

a) $[x_0, y_0, z_0; G](z_0 - x_0) = [y_0, z_0; G] - [x_0, y_0; G]$, *for the distinct points* $x_0, y_0, z_0 \in X$;

b) if G *is two times Fréchet differentiable at* $x_0 \in X$, *then*

$$[x_0, x_0, x_0; G] = \tfrac{1}{2} G''(x_0).$$

We shall denote by $B_r(x_0) = \{x \in X \mid \|x - x_0\| < r\}$ the open ball having the center at $x_0 \in X$ and the radius $r > 0$.

Concerning the convergence of the iterative process (6) we shall prove the following theorem:

Theorem 3. *If there exists the element* $x_0 \in X$, *and the positive real numbers* K, l, ε, M, r *such that:*

i) *G is continuous on* $B_r(x_0)$;

ii) *F is Fréchet differentiable on* $B_r(x_0)$, *with the Fréchet derivative satisfying the Lipschitz condition*

$$\left\|F'(x) - F'(y)\right\| \le l \left\|x - y\right\|, \quad \forall x, y \in B_r(x_0);$$

iii) *The second-order divided difference of G is uniformly bounded on* $B_r(x_0)$*:*

$$\left\|[x, y, z; G]\right\| \le K, \quad \forall x, y, z \in B_r(x_0);$$

iv) *The operators* $F'(x) + [x, \varphi(x); G]$ *are invertible, with the inverses uniformly bounded:* $\forall x \in B_r(x_0)$ *with* $\varphi(x) \in B_r(x_0)$ *there exists* $\left(F'(x) + [x, \varphi(x); G]\right)^{-1}$ *and*

$$\left\|\left(F'(x) + [x, \varphi(x); G]\right)^{-1}\right\| \le M;$$

v) λ *is chosen such that* $\lambda \le M$;

vi) $q := M^2 \varepsilon\left(\frac{l}{2} + 2K\right) < 1$ *and the radius is given by*

$$r := \frac{1}{M\left(\frac{l}{2}+2K\right)} \sum_{k=0}^{\infty} q^{2^k},$$

then

j) *The sequence* $(x_n)_{n\ge0}$ *given by* (6) *is well defined and* $(x_n)_{n\ge0} \subset B_r(x_0)$;

jj) $(x_n)_{n\ge0}$ *converges to some* $x^* \in \overline{B_r(x_0)}$, *which is a solution of equation* (1)*;*

jjj) *The following estimation holds:*

$$\|x^* - x_n\| \le \frac{q^{2^n}}{M\left(\frac{l}{2} + 2K\right)\left(1 - q^{2^n}\right)}.$$

Proof. From the hypothesis *i)* concerning F it is known [6] that we get

$$\left\|F(y) - F(x) - F'(x)(y - x)\right\| \le \frac{l}{2} \left\|y - x\right\|^2. \tag{7}$$

From the definitions of the divided differences we obtain

$$G(y) - G(x) - [x, \varphi(x); G](y - x) = [x, \varphi(x), y; G](y - \varphi(x))(y - x). \tag{8}$$

Indeed,

$$\begin{aligned}
&[x,\varphi(x),y;G](y-\varphi(x))(y-x)=\\
&=[\varphi(x),y;G](y-\varphi(x))-[x,\varphi(x);G](y-\varphi(x))\\
&=G(y)-G(\varphi(x))+[x,\varphi(x);G](\varphi(x)-x)-[x,\varphi(x);G](y-x)\\
&=G(y)-G(\varphi(x))+G(\varphi(x))-G(x)-[x,\varphi(x);G](y-x)\\
&=G(y)-G(x)-[x,\varphi(x);G](y-x).
\end{aligned}$$

We shall prove by induction that

$$\begin{aligned}
&x_k,\varphi(x_k)\in B_r(x_0),\quad k\in\mathbb{N}\\
&\|F(x_k)+G(x_k)\|\le M^{-2}\left(\tfrac{l}{2}+2K\right)^{-1}q^{2^k},\quad k\in\mathbb{N}.
\end{aligned}\tag{9}$$

From the above inequality it can be easily deduced by (6) that $\exists x_{k+1}$ and

$$\begin{aligned}
\|x_{k+1}-x_k\|&=\left\|(F'(x_k)+[x_k,\varphi(x_k);G])^{-1}(F(x_k)+G(x_k))\right\|\\
&\le M^{-1}\left(\tfrac{l}{2}+2K\right)^{-1}q^{2^k}.
\end{aligned}\tag{10}$$

For $k=0$ we have:

$$\begin{aligned}
&x_0\in B_r(x_0);\\
&\|x_0-\varphi(x_0)\|=\|x_0-x_0+\lambda(F(x_0)+G(x_0))\|\le\lambda\varepsilon\le M\varepsilon<r,
\end{aligned}$$

which imply that

$$\begin{aligned}
&\varphi(x_0)\in B_r(x_0)\\
&\|F\,(x_0)+G(x_0)\|\le\varepsilon=M^{-2}\left(\tfrac{l}{2}+2K\right)^{-1}q^{2^0}.
\end{aligned}$$

Suppose now that (9) is true for $k=\overline{1,n-1}$. By (10) it follows that $\exists x_n$, and we have that $x_n\in B_r(x_0)$. Indeed,

$$\|x_n-x_0\|\le\|x_1-x_0\|+\cdots+\|x_n-x_{n-1}\|\le M^{-1}\left(\tfrac{l}{2}+2K\right)^{-1}\sum_{k=0}^{n-1}q^{2^k}<r.$$

Then, using (6), (7), (8) and (9),

$$\begin{aligned}
\|F\,(x_n)+G(x_n)\|&\le\left\|F(x_n)-F(x_{n-1})-F'(x_{n-1})(x_n-x_{n-1})\right\|\\
&\quad+\|G(x_n)-G(x_{n-1})-[x_{n-1},\varphi(x_{n-1});G](x_n-x_{n-1})\|\\
&\le\tfrac{l}{2}\|x_n-x_{n-1}\|^2+K\|x_n-x_{n-1}\|\cdot\|x_n-\varphi(x_{n-1})\|\le
\end{aligned}$$

$$\leq \tfrac{l}{2} M^{-2} \left(\tfrac{l}{2}+2K\right)^{-2} q^{2^n}$$
$$+ K M^{-1} \left(\tfrac{l}{2}+2K\right)^{-1} q^{2^{n-1}} \Big(M^{-1} \left(\tfrac{l}{2}+2K\right)^{-1} q^{2^{n-1}} + \lambda M^2 \left(\tfrac{l}{2}+2K\right)^{-1} q^{2^{n-1}} \Big)$$
$$\leq M^{-2} \left(\tfrac{l}{2}+2K\right)^{-1} q^{2^n}.$$

It remains to show that $\varphi(x_n) \in B_r(x_0)$:

$$\begin{aligned} \|x_0 - \varphi(x_n)\| &\leq \|x_0 - x_n\| + \lambda \|F(x_n) + G(x_n)\| \\ &\leq M^{-1} \left(\tfrac{l}{2}+2K\right)^{-1} \sum_{k=0}^{n-1} q^{2^k} + M^{-1} \left(\tfrac{l}{2}+2K\right)^{-1} q^{2^n} < r. \end{aligned}$$

The induction (9) is proved.

Now we prove that the sequence $(x_n)_{n\geq 0}$ is a Cauchy sequence, hence it converges to some element $x^* \in \overline{B_r(x_0)}$:

$$\begin{aligned} \|x_{n+p} - x_n\| &\leq \|x_{n+p} - x_{n+p-1}\| + \dots + \|x_{n+1} - x_n\| \\ &\leq M^{-1} \left(\tfrac{l}{2}+2K\right)^{-1} \sum_{k=n}^{n+p-1} q^{2^k} \\ &= M^{-1} \left(\tfrac{l}{2}+2K\right)^{-1} q^{2^n} \sum_{k=n}^{n+p-1} q^{2^k - 2^n} \\ &\leq M^{-1} \left(\tfrac{l}{2}+2K\right)^{-1} q^{2^n} (1 + q^{2 \cdot 2^n - 2^n} + q^{4 \cdot 2^n - 2^n} + \dots) \\ &\leq M^{-1} \left(\tfrac{l}{2}+2K\right)^{-1} \frac{q^{2^n}}{1 - q^{2^n}}. \end{aligned}$$

Passing to limit for $n \to \infty$ in relation (6) and taking into account the hypotheses concerning F and G, we get that x^* is a solution of (1).

The estimation jjj) is obtained from the above relation, for $p \to \infty$. □

3. NUMERICAL EXAMPLES

Given the system

$$\begin{cases} 3x^2 y + y^2 - 1 + |x-1| = 0 \\ x^4 + xy^3 - 1 + |y| = 0 \end{cases}$$

we shall consider $X = (\mathbb{R}^2, \|\cdot\|_\infty)$ and $F, G : X \to X, F = (f_1, f_2), G = (g_1, g_2)$, with $f_1(x,y) = 3x^2 y + y^2 - 1,\ f_2(x,y) = x^4 + xy^3 - 1,\ g_1(x,y) = |x-1|$, $g_2(x,y) = |y|$.

We shall take $[x,y;G]\in\mathbb{M}_2(\mathbb{R})$ given by

$$[x,y;G](i,1)=\tfrac{g_i(y^1,y^2)-g(x^1,y^2)}{y^1-x^1},\qquad [x,y;G](i,2)=\tfrac{g_i(x^1,y^2)-g_i(x^1,x^2)}{y^2-x^2},$$

$i=1,2$.

Using the method (2) with $x_0=(1,0)$, we obtain

n	x_n^1	x_n^2	$\\|x_n-x_{n-1}\\|$
0	$1.000\,000\,000\,000\,00\cdot 10^{+0}$	$0.000\,000\,000\,000\,00\cdot 10^{+0}$	
1	$1.000\,000\,000\,000\,00\cdot 10^{+0}$	$3.333\,333\,333\,333\,33\cdot 10^{-1}$	$3.33\cdot 10^{-01}$
2	$9.065\,502\,183\,406\,11\cdot 10^{-1}$	$3.540\,029\,112\,081\,51\cdot 10^{-1}$	$9.344\cdot 10^{-02}$
3	$8.853\,284\,006\,634\,12\cdot 10^{-1}$	$3.380\,272\,763\,613\,32\cdot 10^{-1}$	$2.122\cdot 10^{-02}$
4	$8.913\,295\,568\,328\,00\cdot 10^{-1}$	$3.266\,139\,765\,935\,66\cdot 10^{-1}$	$1.141\cdot 10^{-02}$
⋮	⋮	⋮	⋮
39	$8.946\,553\,733\,346\,87\cdot 10^{-1}$	$3.278\,265\,217\,462\,98\cdot 10^{-1}$	$5.149\cdot 10^{-19}$

Using the method (5) with $x_0=(1,1)$, $x_1=(2,2)$ we get:

n	x_n^1	x_n^2	$\\|x_n-x_{n-1}\\|$
0	$2.000\,000\,000\,000\,00\cdot 10^{+0}$	$2.000\,000\,000\,000\,00\cdot 10^{+0}$	
1	$1.000\,000\,000\,000\,00\cdot 10^{+0}$	$1.000\,000\,000\,000\,00\cdot 10^{+0}$	$1.000\cdot 10^{+00}$
2	$3.333\,333\,333\,333\,33\cdot 10^{-1}$	$1.333\,333\,333\,333\,33\cdot 10^{+0}$	$6.666\cdot 10^{+01}$
3	$9.620\,253\,164\,556\,96\cdot 10^{-1}$	$3.544\,303\,797\,468\,35\cdot 10^{-1}$	$9.789\cdot 10^{-01}$
4	$9.006\,962\,171\,562\,64\cdot 10^{-1}$	$3.304\,659\,355\,979\,86\cdot 10^{-1}$	$6.132\cdot 10^{-02}$
5	$8.947\,064\,096\,934\,25\cdot 10^{-1}$	$3.278\,552\,521\,887\,66\cdot 10^{-1}$	$5.989\cdot 10^{-03}$
6	$8.946\,553\,768\,094\,08\cdot 10^{-1}$	$3.278\,265\,245\,651\,25\cdot 10^{-1}$	$5.103\cdot 10^{-05}$
7	$8.946\,553\,733\,346\,87\cdot 10^{-1}$	$3.278\,265\,217\,462\,98\cdot 10^{-1}$	$3.474\cdot 10^{-09}$
8	$8.946\,553\,733\,346\,87\cdot 10^{-1}$	$3.278\,265\,217\,462\,98\cdot 10^{-1}$	$2.003\cdot 10^{-17}$
9	$8.946\,553\,733\,346\,87\cdot 10^{-1}$	$3.278\,265\,217\,462\,98\cdot 10^{-1}$	$2.710\cdot 10^{-20}$

Using method (6) with $\lambda=0.5$ and $x_0=(1,1)$ we get:

n	x_n^1	x_n^2	$\\|x_n-x_{n-1}\\|$
0	$1.000\,000\,000\,000\,00\cdot 10^{+0}$	$0.000\,000\,000\,000\,00\cdot 10^{+00}$	
1	$1.400\,000\,000\,000\,00\cdot 10^{+0}$	$0.000\,000\,000\,000\,00\cdot 10^{+00}$	$1.000\cdot 10^{+00}$
2	$1.154\,212\,949\,626\,24\cdot 10^{+0}$	$1.438\,413\,350\,975\,79\cdot 10^{-01}$	$2.457\cdot 10^{-01}$
3	$1.010\,571\,500\,463\,24\cdot 10^{+1}$	$2.691\,698\,935\,508\,61\cdot 10^{-01}$	$1.436\cdot 10^{-01}$
4	$8.990\,733\,928\,764\,52\cdot 10^{-1}$	$3.762\,673\,831\,093\,11\cdot 10^{-01}$	$1.114\cdot 10^{-01}$
5	$8.950\,225\,056\,578\,07\cdot 10^{-1}$	$3.288\,153\,820\,340\,89\cdot 10^{-01}$	$4.745\cdot 10^{-02}$
6	$8.946\,555\,041\,441\,07\cdot 10^{-1}$	$3.278\,274\,887\,465\,46\cdot 10^{-01}$	$9.878\cdot 10^{-04}$
7	$8.946\,553\,733\,347\,87\cdot 10^{-1}$	$3.278\,265\,217\,468\,06\cdot 10^{-01}$	$9.669\cdot 10^{-07}$
8	$8.946\,553\,733\,346\,87\cdot 10^{-1}$	$3.278\,265\,217\,462\,98\cdot 10^{-01}$	$5.086\cdot 10^{-13}$
9	$8.946\,553\,733\,346\,87\cdot 10^{-1}$	$3.278\,265\,217\,462\,98\cdot 10^{-01}$	$2.710\cdot 10^{-20}$

It seems that the best results are not obtained here for λ taken too small, because the divided differences cannot be computed in this case for $\|x_n - x_{n-1}\| \leq 1.0 \cdot 10^{-16}$.

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