REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Tome 25, N^{os} 1–2, 1996, pp. 33–41

A NOTE ON INEXACT SECANT METHODS

EMIL CĂTINAŞ
(Cluj-Napoca)

1. INTRODUCTION

Given a nonlinear function $F : \mathbb{R}^n \to \mathbb{R}^n$, in order to approximate a solution of the equation

$$F(x) = 0,$$

the classical Newton method is usually applied:

$$x_{k+1} = x_k - F'(x_k)^{-1} F(x_k), \quad k = 0, 1, \dots, \ x_0 \in \mathbb{R}^n \text{ given},$$

or, equivalently,

$$F'(x_k)\, s_k = -F(x_k), \quad \text{where } s_k = x_{k+1} - x_k.$$

However, in many cases it turns out that the above linear systems are not solved exactly, so, in order to study the convergence and the convergence order of the method, an error term must be taken into account.

In the paper [3] there are considered the inexact Newton methods:

$$F'(x_k)\, s_k = -F(x_k) + r_k,$$

Local convergence of these methods is studied and also necessary and sufficient conditions on the magnitude of r_k are imposed for a certain convergence order to be achieved.

In the present paper we shall show how these results can be easily extended to the chord method, which has a theoretically higher efficiency index than the Newton method. Unfortunately the chord method is not very safe for all practical cases since, for example, a too fast convergence of one component in the sequence (x_k) may lead to overflow errors or division by zero.

Definition 1.1. [4] *We call the first order divided difference of F at the distinct points $x, y \in \mathbb{R}^n$, denoted by $[x, y; F]$, a linear operator belonging to $\mathcal{L}(\mathbb{R}^n)$ and satisfying*

1) $[x,y;F](y-x)=F(y)-F(x)$;
2) *If F is Fréchet differentiable at $z\in\mathbb{R}^n$ then*

$$\lim_{x,y\to z}[x,y;F]=F'(z).$$

It is known [1] that there are several ways to choose the linearly independent first order divided differences for given F,x,y, depending on the dimension $n\in\mathbb{N}$.

For example, if $x=(x_i)_{i=\overline{1,n}}, y=(y_i)_{i=\overline{1,n}}\in\mathbb{R}^n$, with $x_i\neq y_i$, $i=\overline{1,n}$ then we can take

$$[x,y;F]_{i,j}=\frac{F_i(y_1,\ldots,y_j,x_{j+1},\ldots,x_n)-F_i(y_1,\ldots,y_{j-1},x_j,\ldots,x_n)}{y_j-x_j}.$$

The chord method is given by the iteration

$$x_{k+1}=x_k-[x_{k-1},x_k;F]^{-1}F(x_k),\quad k=1,2,\ldots,\quad x_0,x_1\in\mathbb{R}^n\ \text{given},$$

and it has the r-convergence order $\frac{1+\sqrt{5}}{2}$. The inexact chord method studied is

$$[x_{k-1},x_k;F]s_k=-F(x_k)+r_k,\quad k=1,2,\ldots,\ x_0,x_1\in\mathbb{R}^n\ \text{given},\tag{1.1}$$

the residual r_k satisfying

$$\frac{\|r_k\|}{\|F(x_k)\|}\le\eta_k,$$

where $\|\cdot\|$ is a given norm in $\mathbb{R}^n$.

We shall suppose hereafter that the function F obeys the following conditions:

C1) There exists $x^*\in\mathbb{R}^n$ such that $F(x^*)=0$;
C2) F is continuously Fréchet differentiable at x^*;
C3) $F'(x^*)$ is nonsingular.

2. LOCAL CONVERGENCE OF INEXACT CHORD METHODS

As in the case of inexact Newton methods, when the forcing sequence (η_k) is uniformly less than one, an attraction theorem can be stated, i.e. for any sufficiently good initially guesses x_0 and x_1, the sequence (x_k) converges to x^*.

LEMMA 2.1. *If the conditions $C1)$–$C3)$ hold, then for any $\gamma>0$ there exists $\varepsilon>0$ such that if $x,y\in B(x^*,\varepsilon)=\{z\in\mathbb{R}^n \mid \|z-x^*\|\le\varepsilon\}$ then $[x,y;F]$ is nonsingular and*

1) $\big\|[x,y;F]-F'(x^*)\big\|\le\gamma$;

2) $\left\| [x,y;F]^{-1} - F'(x^*)^{-1} \right\| \le \gamma.$

LEMMA 2.2. *If F is differentiable at x^*, then for any $\gamma > 0$ there exists $\varepsilon > 0$ such that*

$$\|F(y) - F(x^*) - F'(x^*)(y - x^*)\| \le \gamma \|y - x^*\|,$$

if $\|y - x^*\| \le \varepsilon$.

The lemmas are immediate consequences of Definition 1.1, respectively of the definition of $F'(x^*)$.

THEOREM 2.3. *Suppose that $\eta_k \le \overline{\eta} < q < 1$ for all $k \in \mathbb{N}$ and let $\mu = \max\left\{ \|F'(x^*)\|, \|F'(x^*)^{-1}\| \right\}$. Then there exists $\varepsilon \ge 0$ such that if $x_0, x_1 \in B(x^*, \varepsilon)$ are chosen to satisfy $\|x_1 - x^*\| \le \frac{q}{\mu^2} \|x_0 - x^*\|$ then the sequence (x_k) given by* (1.1) *converges to x^*, and, moreover, the following inequalities hold:*

$$\|x_{k+1} - x^*\|_* \le q \|x_k - x^*\|_*, \quad k \ge 0, \tag{2.1}$$

where $\|y\|_* = \|F'(x^*) y\|$.

Proof. From the definition of μ we get

$$\tfrac{1}{\mu} \|y\| \le \|y\|_* \le \mu \|y\|, \quad \text{for all } y \in \mathbb{R}^n. \tag{2.2}$$

Since $\overline{\eta} \le q$, there exists $\gamma > 0$ sufficiently small that

$$(1 + \gamma\mu)\left(\overline{\eta}(1 + \gamma\mu) + 2\gamma\mu\right) \le q.$$

Now, by Lemmas 2.1 and 2.2, choose $\varepsilon \ge 0$ sufficiently small that

$$\left\| [x,y;F] - F'(x^*) \right\| \le \gamma \tag{2.3}$$

$$\left\| [x,y;F]^{-1} - F'(x^*)^{-1} \right\| \le \gamma \tag{2.4}$$

$$\left\| F(y) - F(x^*) - F'(x^*)(y - x^*) \right\| \le \gamma \|y - x^*\|, \tag{2.5}$$

for $\|x - x^*\| \le \mu^2 \varepsilon$ and $\|y - x^*\| \le \mu^2 \varepsilon$.

We prove (2.1) by induction.

For $k = 0$, by (2.2) we get

$$\tfrac{1}{\mu} \|x_1 - x^*\|_* \le \|x_1 - x^*\| \le \tfrac{q}{\mu^2} \|x_0 - x^*\| \le \tfrac{q}{\mu} \|x_0 - x^*\|_*,$$

i.e. (2.1) hold.

Supposing now that it holds for $i = 0, 1, \dots, k-1$, it follows that

$$\|x_k - x^*\| \le \mu \|x_k - x^*\|_* \le \mu q^k \|x_0 - x^*\|_* \le \mu^2 q^k \|x_0 - x^*\| < \mu^2 \varepsilon,$$

so that (2.3)–(2.5) hold for $x = x_{k-1}$ and $y = x_k$.

By Lemma 2.1 we have now from (1.1) that x_{k+1} exists. Moreover, since

$$\begin{aligned} F'(x^*)(x_{k+1} - x^*) = &\Big(I + F'(x^*)\left([x_{k-1}, x_k; F]^{-1} - F'(x^*)^{-1}\right)\Big) \cdot \\ &\cdot \Big(r_k + \left([x_{k-1}, x_k; F] - F'(x^*)\right)(x_k - x^*) \\ &\quad - \left(F(x_k) - F(x^*) - F'(x^*)(x_k - x^*)\right)\Big), \end{aligned}$$

and

$$F(x_k) = F'(x^*)(x_k - x^*) + \left(F(x_k) - F(x^*) - F'(x^*)(x_k - x^*)\right),$$

taking norms, using (2.3)–(2.5) and the choice of γ we obtain

$$\begin{aligned} \|x_{k+1} - x^*\|_* \le &\Big(1 + \|F'(x^*)\| \left\|[x_{k-1}, x_k; F]^{-1} - F'(x^*)^{-1}\right\|\Big) \cdot \\ &\cdot \Big(\|r_k\| + \left\|[x_{k-1}, x_k; F] - F'(x^*)\right\| \cdot \|x_k - x^*\| \\ &\quad + \left\|F(x_k) - F(x^*) - F'(x^*)(x_k - x^*)\right\|\Big) \\ \le &(1 + \gamma\mu)\left(\eta_k \|F(x_k)\| + \gamma \|x_k - x^*\| + \gamma \|x_k - x^*\|\right) \\ \le &(1 + \gamma\mu)\left(\eta_k \left(\|x_k - x^*\|_* + \gamma \|x_k - x^*\|\right) + 2\gamma \|x_k - x^*\|\right) \\ \le &(1 + \gamma\mu)\left(\overline{\eta}(1 + \gamma\mu) + 2\gamma\mu\right) \|x_k - x^*\|_* \le q \|x_k - x^*\|, \end{aligned}$$

which proves the linear convergence of (x_k). □

3. RATE OF CONVERGENCE OF INEXACT CHORD METHODS

In this section the convergence order of inexact chord methods is characterized in terms of the rate of convergence of the relative residuals and of residuals.

Definition 3.1. [3] *Let (x_k) be a sequence which converges to x^*. Then*

1. *the sequence (x_k) converges to x^* q-superlinearly if*

$$\|x_{k+1} - x^*\| = o(\|x_k - x^*\|), \quad \text{as } k \to \infty;$$

2. *the sequence (x_k) converges to x^* with r-order at least q $(q > 1)$ if*

$$\limsup_{k \to \infty} \|x_k - x^*\|^{1/q^k} < 1.$$

Lemma 3.2. [3] *Let* $\alpha = \max\left\{ \|F'(x^*)\| + \frac{1}{2\beta}, 2\beta\right\}$, *where* $\beta = \|F'(x^*)^{-1}\|$. *Then the following inequalities hold:*

$$\tfrac{1}{\alpha}\|y - x^*\| \le \|F(y)\| \le \alpha\|y - x^*\|,$$

for $\|y - x^*\|$ *sufficiently small.*

Theorem 3.3. *Assume that the sequence* (x_k) *given by* (1.1) *converges to* x^*. *Then* $x_k \to x^*$ *q-superlinearly if and only if*

$$\|r_k\| = o(\|F(x_k)\|), \quad as \quad k \to \infty.$$

Proof. Assume that $x_k \to x^*$ superlinearly. Since

$$\begin{aligned} r_k = &\big(F(x_k) - F(x^*) - F'(x^*)(x_k - x^*)\big) - \big([x_{k-1}, x_k; F] - F'(x^*)\big)(x_k - x^*) \\ &+ \Big(F'(x^*) + \big([x_{k-1}, x_k; F] - F'(x^*)\big)\Big)(x_{k+1} - x^*), \end{aligned}$$

taking norms,

$$\begin{aligned} &\|r_k\| \le \\ &\le \big\|F(x_k) - F(x^*) - F'(x^*)(x_k - x^*)\big\| + \big\|[x_{k-1}, x_k; F] - F'(x^*)\big\|\,\|x_k - x^*\| \\ &\quad + \big(\|F'(x^*)\| + \big\|[x_{k-1}, x_k; F] - F'(x^*)\big\|\big)\|x_{k+1} - x^*\| \\ &= o(\|x_k - x^*\|) + o(1)\|x_k - x^*\| + \big(\|F'(x^*)\| + o(1)\big)o(\|x_k - x^*\|) \\ &= o(\|x_k - x^*\|) = o(\|F(x_k)\|), \quad \text{as } k \to \infty, \end{aligned}$$

by Lemmas 2.1, 2.2 and 3.2.

Conversely, assume that $\|r_k\| = o(\|F(x_k)\|)$. As in the proof of Theorem 2.3,

$$\begin{aligned} &\|x_{k+1} - x^*\| \le \\ &\le \big(\|F'(x^*)^{-1}\| + \big\|[x_{k-1}, x_k; F]^{-1} - F(x^*)^{-1}\big\|\big)\cdot \\ &\quad \cdot\Big(\|r_k\| + \big\|[x_{k-1}, x_k; F] - F'(x^*)\big\|\,\|x_k - x^*\| \\ &\qquad + \big\|F(x_k) - F(x^*) - F'(x^*)(x_k - x^*)\big\|\Big) \\ &= \big(\|F'(x^*)^{-1}\| + o(1)\big)\big(o(\|F(x_k)\|) + o(1)\|x_k - x^*\| + o(\|x_k - x^*\|)\big) \\ &= o(\|F(x_k)\|) + o(\|x_k - x^*\|) = o(\|x_k - x^*\|), \quad \text{as } k \to \infty, \end{aligned}$$

again by Lemmas 2.1, 2.2 and 3.2. □

Definition 3.4. [4] *Given the distinct points $x, y, z \in \mathbb{R}^n$, the second order divided difference of F at x, y, and z is a linear operator $[x, y, z; F] \in \mathcal{L}(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n))$ such that*

1. $[x, y, z; F](z - x) = [y, z; F] - [x, y; F]$;
2. *if F is twice differentiable at $u \in \mathbb{R}^n$ then*

$$\lim_{x,y,z\to u}[x, y, z; F] = \tfrac{1}{2}F''(u).$$

Lemma 3.5. *If there exist $\varepsilon > 0$ and $K \geq 0$ such that for all $x, y \in B(x^*, \varepsilon)$ we have*

$$\big\|[x^*, x, y;\ F]\big\| \leq K$$

then

$$\big\|[x, y; F] - F'(x^*)\big\| \leq K\big(\,\|x - x^*\| + \|y - x^*\|\,\big), \quad \forall x, y \in B(x^*, \varepsilon).$$

Proof. By Definition 3.4 we have

$$\begin{aligned}\big\|[x, y; F] - F'(x^*)\big\| &= \big\|[x, y; F] - [x^*, x^*; F]\big\| \\ &= \|[x, y; F] - [x^*, x; F] + [x^*, x; F] - [x^*, x^*; F]\| \\ &= \big\|[x^*, x, y; F](y - x^*) + [x^*, x^*, x; F](x - x^*)\,\big\| \\ &\leq K\big(\|y - x^*\| + \|x - x^*\|\big).\end{aligned}$$

□

Remark 3.6. The condition imposed in the above Lemma holds, for example, when F is twice continuously differentiable at x^*. □

Definition 3.7. [6] *The mapping F' is Hölder continuous with exponent p $(0 < p \leq 1)$ at x^* if there exists $L \geq 0$ such that*

$$\big\|F'(y) - F'(x^*)\big\| \leq L\,\|y - x^*\|^p$$

for $\|y - x^*\|$ sufficiently small.

Lemma 3.8. [6] *If F' is Hölder continuous with exponent p at x^*, then there exists $L' \geq 0$ such that*

$$\big\|F(y) - F(x^*) - F'(x^*)(y - x^*)\big\| \leq L'\,\|y - x^*\|^{1+p},$$

for $\|y - x^*\|$ sufficiently small.

Theorem 3.9. *Assume that the sequence* (x_k) *given by* (1.1) *converges to* x^*, F' *is Hölder continuous with exponent* p_0 *at* x^*, *where* $1+p_0=\frac{1+\sqrt{5}}{2}$.

If $x_k \to x^*$ *with r-order at least* $1+p_0$ *then* $r_k \to 0$ *with r-order at least* $1+p_0$.

If $r_k \to 0$ *with r-order at least* $1+p_0, F$ *satisfies the condition of Lemma* 3.5 *and there exists* $k_1 \in \mathbb{N}$ *sufficiently large such that* $\|x_{k_1}-x^*\| \leq c\gamma$ *and* $\|x_{k_1+1}-x^*\| \leq c\gamma^{1+p_0}$, *with* c, γ *given below by the r-convergence of* r_k, *then* $x_k \to x^*$ *with r-order at least* $1+p_0$.

Proof. Let L be the Hölder constant and let α and L' be the constants given in Lemma 3.2 and Lemma 3.8 respectively. Pick $\varepsilon>0$ sufficiently small that for all $x, y \in B(x^*,\varepsilon)$, there exists $[x,y;F]^{-1}$ and

$$\begin{aligned}\|F(y)\| &\leq \alpha\|y-x^*\| \\ \left\|[x,y;F]^{-1}\right\| &\leq \alpha \\ \left\|F'(y)-F'(x^*)\right\| &\leq L\|y-x^*\|^{p_0} \\ \|F(y)-F(x^*)-F(x^*)(y-x^*)\| &\leq L'\|y-x^*\|^{1+p_0}.\end{aligned}$$

Such an ε exists by Lemma 3.2, the continuity and the Hölder continuity of F' at x^* and Lemma 3.8.

Assume that $x_k \to x^*$ with r-order at least $1+p_0$. Then there exist constants $0 \leq \gamma < 1$ and $k_0 \geq 0$ such that

$$\|x_k - x^*\| \leq \gamma^{(1+p_0)^k}, \quad \text{for } k \geq k_0. \tag{3.1}$$

Now choose $k_1 \geq k_0$ sufficiently large that

$$\|x_k - x^*\| \leq \varepsilon, \quad \text{for } k \geq k_1.$$

From the definition of r_k,

$$\begin{aligned}\|r_k\| &\leq \|[x_{k-1},x_k;F]\| \cdot \|x_{k+1}-x_k\| + \|F(x_k)\| \\ &\leq \alpha(\|x_{k+1}-x^*\| + \|x_k-x^*\|) + \alpha\|x_k-x^*\|\end{aligned}$$

by (3.1),

$$\|r_k\| \leq \alpha\big(\gamma^{(1+p_0)^{k+1}} + \gamma^{(1+p_0)^k}\big) + \alpha\gamma^{(1+p_0)^k} = \big(\alpha\gamma^{p_0(1+p_0)^k}+2\alpha\big)\gamma^{(1+p_0)^k},$$

so that

$$\|r_k\| \leq 3\alpha\gamma^{(1+p_0)^k}, \quad \text{for } k \geq k_1.$$

The result now follows immediately from the definition of r-order of convergence.

Conversely, assume that $\|r_k\| \to 0$ with r-order at least $1+p_0 = \frac{1+\sqrt{5}}{2}$.
Then there exist constants $0 \leq \gamma < 1$ and $k_0 \geq 0$ such that

$$\|r_k\| \leq \gamma^{(1+p_0)^k}, \quad \text{for } k \geq k_0.$$

Suppose also that $\|[x^*, x, y; F]\| \leq K$ for $x, y \in B(x^*, \varepsilon)$.

Let

$$c = \left(2\alpha\left(2K + L'\right)\right)^{-1}, \quad \text{if} \quad \left(2\alpha\left(2K + L'\right)\right)^{-1} > 1$$

$$c = \left(2\alpha\left(2K + L'\right)\right)^{-1/p_0}, \quad \text{if} \quad \left(2\alpha\left(2K + L'\right)\right)^{-1} \leq 1,$$

and

$$\|x_k - x^*\| \leq \min\{\varepsilon, c\gamma\}, \quad \text{for } k \geq k_1, \text{ while } \|x_{k+1} - x^*\| \leq c\gamma^{1+p_0}$$

and admit that $k_1 \geq k_0$ is sufficiently large that

$$\tfrac{\alpha}{c}\gamma^{(1+p_0)^k[1-(1+p_0)^{1-k_1}]} \leq \tfrac{1}{2}, \quad \text{for } k \geq k_1,$$

From the definition of r-convergence if suffices to prove that

$$\|x_k - x^*\| \leq c\gamma^{(1+p_0)^{k-k_1}}, \quad \text{for } k \geq k_1.$$

The proof is by induction. When $k = k_1$ it follows from the assumptionon k_1 that

$$\|x_{k_1} - x^*\| \leq c\gamma \text{ and } \|x_{k_1+1} - x^*\| \leq c\gamma^{1+p_0},$$

We have

$$\begin{aligned}
&\|x_{k+1} - x^*\| \leq \\
&\leq \|[x_{k-1}, x_k; F]^{-1}\| \big(\|r_k\| + \|[x_{k-1}, x_k; F] - F'(x^*)\| \cdot \|x_k - x^*\| \\
&+ \|F(x_k) - F(x^*) - F'(x^*)(x_k - x^*)\| \big) \\
&\leq \alpha \big(\|r_k\| + K(\|x_k - x^*\| + \|x_{k-1} - x^*\|)\|x_k - x^*\| + L' \|x_k - x^*\|^{1+p_0} \big) \\
&\leq \alpha\gamma^{(1+p_0)^k} + \alpha K \|x_k - x^*\|^{1+p_0} + \alpha K \|x_{k-1} - x^*\| \|x_k - x^*\| + \alpha L' \|x_k - x^*\|^{1+p_0} \\
&= \alpha\gamma^{(1+p_0)^k} + \alpha(K + L')\|x_k - x^*\|^{1+p_0} + \alpha K \|x_{k-1} - x^*\| \|x_k - x^*\| \\
&\leq \alpha\gamma^{(1+p_0)^k} + \alpha(K + L')c^{1+p_0}\gamma^{(1+p_0)^{k+1-k_1}} + \alpha K c\gamma^{(1+p_0)^{k-k_1}} \cdot c\gamma^{(1+p_0)^{k-1-k_1}} \\
&= \alpha\gamma^{(1+p_0)^k} + \alpha(K + L')c^{1+p_0}\gamma^{(1+p_0)^{k+1-k_1}} + \alpha K c^2\gamma^{(1+p_0)^{k+1-k_1}}.
\end{aligned}$$

As in the proof of Theorem 2.3,

$$\begin{aligned}
&\|x_{k+1} - x^*\| \leq \\
&\leq \alpha\gamma^{(1+p_0)^k} + c\gamma^{(1+p_0)^k}\left(\alpha(K+L')c^{p_0} + \alpha K c\right) \\
&= c\gamma^{(1+p_0)^{k+1-k_1}}\left(\tfrac{\alpha}{c}\gamma^{(1+p_0)^{k+1-k_1}[(1+p_0)^{k_1-1}-1]} + \alpha(K+L')c^{p_0} + \alpha K c\right) \\
&\leq c\gamma^{(1+p_0)^{k+1-k_1}}\left(\tfrac{1}{2} + \tfrac{1}{2}\right) \\
&= c\gamma^{(1+p_0)^{k+1-k_1}},
\end{aligned}$$

since $1+p_0$ satisfies $t^2 = t+1$. We get that $x_k \to x^*$ with r-order of convergence $1+p_0$. □

REFERENCES

[1] M. Balázs, G. Goldner, *Remarks on divided differences and method of chords*, Revista de Analiză Numerică şi Teoria Aproximaţiei, **3** (1974) no. 1, pp. 19–30 (in Romanian).

[2] J.E. Dennis Jr., J.J. Moré, *Quasi-Newton methods, motivation and theory*, SIAM Rev., **19** (1977), pp. 46–89.

[3] R.S. Dembo, S.C. Eisenstat, T. Steihaug, *Inexact Newton methods*, SIAM J. Numer. Anal., **19** (1982) 2, pp. 400–408.

[4] G. Goldner, M. Balázs, *On the chord method and on a modification of it for solving nonlinear operator equations*, Stud. Cerc. Mat., **20** (1968) 7, pp. 981-990 (in Romanian).

[5] I. Muntean, *Functional Analysis*, "Babeş-Bolyai" University, Cluj-Napoca, 1993 (in Romanian).

[6] J.M. Ortega, W.C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, New York, 1970.

Received March 7, 1996.

Romanian Academy
"T. Popoviciu" Institute of Numerical Analysis
P.O. Box 68 Cluj-Napoca 1
RO-3400
Romania