Bilateral approximations for some Aitken-Steffensen-Hermite type methods of order three

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Abstract

We study the local convergence of some Aitken-Steffensen-Hermite type methods of order three. We obtain that under some reasonable conditions on the monotony and convexity of the nonlinear function, the iterations offer bilateral approximations for the solution, which can be efficiently used as a posteriori estimations.

MSC 2000: 65H05.

Keywords: nonlinear equations, Aitken-Steffensen type methods, Hermite inverse interpolatory polynomials, divided differences.

1 Introduction

As it is well known, the Steffensen, Aitken and Aitken-Steffensen methods are interpolatory methods, with controlled nodes. More precisely, they can be obtained from the first degree Lagrange inverse interpolatory polynomial, for which the two interpolation nodes are controlled by one or two auxiliary functions suitably chosen [1]–[6], [9]–[12], [17].

Some generalizations of these methods have been obtained in [12], [15], [16], by controlling the interpolation nodes in the Lagrange, respectively Hermite inverse interpolatory polynomials of degree 2; the resulted methods have convergence order 3.

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Moreover, in [13], [15], [16], were obtained conditions under which the Steffensen, Aitken or Aitken-Steffensen type methods lead to sequences which approximate bilaterally the solution.

In this paper we propose two iterative methods of Aitken-Steffensen type, which are obtained from the inverse interpolatory Hermite polynomial of degree 2, and which have the q-convergence order 3. The interpolation nodes are obtained using two auxiliary functions.

In the first section we present the two methods and we show that they have the convergence order 3. In Section 2 we obtain convergence results depending on the monotony of the function (increasing/decreasing) and on its convexity (convex/concave). In Section 3 we show how the auxiliary functions may be constructed such that they verify the convergence results obtained in Section 2, while the last section contains some numerical example which illustrate the results.

Consider the equation

$$f(x) = 0, \tag{1}$$

where $f : [a, b] \to \mathbb{R}$, $a, b \in \mathbb{R}$, $a < b$, and the additional two equations, equivalent to the above one:

$$\begin{aligned} x - p(x) &= 0 \\ x - q(x) &= 0 \end{aligned} \tag{2}$$

where $p, q : [a, b] \to [a, b]$. Denote $h : [a, b] \to [a, b]$ given by

$$h(x) = q(p(x)). \tag{3}$$

Here we shall study the method obtained when the nodes are controlled by p and h.

Denote $F = f([a, b])$ and let $s, k \in \mathbb{N}$ and $n = s + k$. We shall make the following hypotheses on f :

α) $f \in C^n([a, b])$

β) $f'(x) \neq 0, \ \forall x \in [a, b]$.

It can be easily seen that α) and β) imply that f is continuous on $[a, b]$ and is invertible, therefore there exists the inverse $f^{-1} : F \to [a, b]$ and the following result holds.

Theorem 1.1 [10], [12], [18] *If f obeys α) and β) then $f^{-1} \in C^n(F)$ and for all $x \in [a,b]$ and $j \in \mathbb{N}$, $1 \le j \le n$ we have*

$$(4) \quad \left[f^{-1}(y)\right]^{(j)} = \\ = \sum \frac{(2j-i_1-2)(-1)^{j+i_1-1}}{i_2!i_3!...i_j![f'(x)]^{2j-1}} \cdot \left(\frac{f'(x)}{1!}\right)^{i_1} \left(\frac{f''(x)}{2!}\right)^{i_2} ... \left(\frac{f^{(j)}(x)}{j!}\right)^{i_j}$$

where $y = f(x)$, and the above sum extends over all nonnegative integer solutions of the system

$$i_2 + 2i_3 + ... + (j-1)i_j = j-1;$$
$$i_1 + i_2 + ... + i_j = j-1.$$

We recall below some particular cases of (4), which will be subsequently used, i.e., $j = 1, 2, 3$:

$$\left[f^{-1}(y)\right]' = \frac{1}{f'(x)};$$
$$\left[f^{-1}(y)\right]'' = -\frac{f''(x)}{\left[f'(x)\right]^3};$$
$$(5) \quad \left[f^{-1}(y)\right]''' = \frac{3\left[f''(x)\right]^2 - f'(x)f'''(x)}{\left[f'(x)\right]^5},$$

where $y = f(x)$.

Let $x_1, x_2 \in [a,b]$ be two interpolation nodes and $y_1 = f(x_1)$, $y_2 = f(x_2)$. Obviously $f^{-1}(y_1) = x_1$ and $f^{-1}(y_2) = x_2$. Moreover under the hypotheses of Theorem 1.1 we can compute the successive derivatives of f^{-1} at y_1, y_2: $(f^{-1}(y_1))'$, $(f^{-1}(y_1))''$, ..., $(f^{-1}(y_1))^{(s-1)}$ resp. $(f^{-1}(y_2))'$, $(f^{-1}(y_2))''$, ..., $(f^{-1}(y_2))^{(k-1)}$. These values determine the unique Hermite polynomial of degree $n-1$, denoted by $H(y)$, satisfying

$$H^{(i)}(y_1) = \left[f^{-1}(y_1)\right]^{(i)}, \ i = \overline{0, s-1};$$
$$H^{(i)}(y_2) = \left[f^{-1}(y_2)\right]^{(i)}, \ i = 0, \ k-1,$$

We denote this polynomial by $H(y_1, s; y_2, k; f^{-1}\,|y\,)$.

It can be easily seen that under the hypotheses of Theorem 1.1 we have that

$$(6) \quad f^{-1}(y) = H(y_1, s; y_2, k; f^{-1}\,|y\,) + \frac{\left[f^{-1}(\eta)\right]^{(n)}}{n!}(y-y_1)^s(y-y_2)^k,$$

where $\eta \in \mathrm{int}(F)$.

If $x^* \in [a,b]$ is a solution of (1), then $x^* = f^{-1}(0)$ and by (6) we get

$$x^* = H(y_1, s; y_2, k; f^{-1} \,|0) + (-1)^n \frac{[f^{-1}(\eta_0)]^{(n)}}{n!} y_1^s y_2^k, \tag{7}$$

where η_0 is an interior point of the smallest interval containing $0, y_1, y_2$.

If in (7) we neglect the remainder, we obtain for x^* an approximation x_3 given by

$$x_3 = H(y_1, s; y_2, k; f^{-1} \,|0) .$$

If $x_3 \in [a,b]$, then we can take $y_2 = f(x_2)$ and $y_3 = f(x_3)$ as the new interpolation nodes, and continue the process.

In general, if $x_{m-1},\ x_m \in [a,b]$ and $y_{m-1} = f(x_{m-1}),\ y_m = f(x_m)$ then we take

$$x_{m+1} = H(y_{m-1}, s; y_m; k; f^{-1} \,|0),\ m = 2, 3, ... \tag{8}$$

By (7) we have that

$$x^* - x_{m+1} = (-1)^n \frac{[f^{-1}(\eta_m)]^{(n)}}{n!} y_{m-1}^s y_m^k. \tag{9}$$

The above relation shows that if all the elements of the sequence $(x_m)_{m \geq 1}$ generated by (8) remain in $[a,b]$ and converge to x^*, then the r-convergence order ω is given by the positive root of the following equation [7], [8], [12], [14]:

$$t^2 - kt - s = 0,$$

i.e.

$$\omega = \frac{k + \sqrt{k^2 + 4s}}{2}.$$

The convergence order can be higher than ω if the interpolation nodes in (8) are controlled with the aid of p and h.

If, given $x_m \in [a,b]$, we take as interpolation nodes the values $y_m = f(p(x_m))$, $\bar{y}_m = f(h(x_m))$, $m \geq 1$, then we obtain the iterations

$$x_{m+1} = H(y_m, s; \bar{y}_m, k; f^{-1} \,|0),\ m = 1, 2, ... \tag{10}$$

with error

$$x^* - x_{m+1} = (-1)^n \frac{[f^{-1}(\bar{\eta}_m)]^{(n)} y_m^s \bar{y}_m^k}{n!} \tag{11}$$

where $\bar{\eta}_m \in \mathrm{int}(F)$. We call (10) as an Aitken-Steffensen-Hermite type method with two steps.

In the following result we show that under some reasonable assumptions on functions f, p and q, the sequence (10) has the q-convergence order at least n.

Proposition 1.2 *Let* $M = \sup_{y\in F} \left|(f^{-1}(y))^{(n)}\right|,\ m_1 = \sup_{x\in[a,b]} |f'(x)|$ *and assume there exist* $\ell_1, \ell_2 \in \mathbb{R}$, $\ell_1 > 0,\ \ell_2 > 0$ *such that* p *and* q *obey the center Lipschitz conditions*

$$(12)\qquad \begin{aligned} |p(x) - p(x^*)| &\le \ell_1 |x - x^*|,\ \forall x \in [a,b]; \\ |q(x) - q(x^*)| &\le \ell_2 |x - x^*|,\ \forall x \in [a,b]. \end{aligned}$$

Then

$$(13)\qquad |x^* - x_{m+1}| \le \frac{M m_1^n \ell_1^n \ell_2^k}{n!} |x^* - x_m|^n,\ m = 1, 2, ...,$$

whence the q-convergence order of the sequence is at least n.

The proof is immediately obtained from (11).

Next we shall study the convergence of the method (10) in the particular cases $s = 1$, $k = 2$ resp. $s = 2$, $k = 1$, which both lead by (13) to q-convergence orders at least 3.

Under reasonable assumptions regarding mainly the monotony and convexity of f on $[a,b]$, we shall show that the functions p and q may be determined such that the convergence order of (10) is 3 and, moreover, we obtain sequences which approximate bilaterally the solution.

If $x, y, z \in [a,b]$ and $u = f(x)$, $v = f(y)$, $w = f(z)$ then it is known that the following relations hold for the divided differences of f and f^{-1}

$$(14)\qquad \begin{aligned} [u, v; f^{-1}] &= \frac{1}{[x, y; f]}; \\ [u, v, w; f^{-1}] &= -\frac{[x, y, z; f]}{[x, y; f][x, z; f][y, z; f]} \end{aligned}$$

where

$$[x, y; f] := \frac{f(y) - f(x)}{y - x},\ [x, y, z; f] := \frac{[y, z; f] - [x, y; f]}{z - x},$$

and $[x, x; f] := f'(x)$.

Using the divided differences for the Hermite polynomial in the case $s = 1$ and $k = 2$, from (10) we get for x_{m+1} the following expression

$$x_{m+1} = p(x_m) - \frac{f(p(x_m))}{[p(x_m), h(x_m); f]} - \frac{[p(x_m), h(x_m), h(x_m); f]}{[p(x_m), h(x_m); f]^2 f'(h(x_m))} f(p(x_m)) f(h(x_m)) \tag{15}$$

$x_1 \in [a, b]$, $m = 1, 2, \ldots$.

In this case, the error verifies

$$x^* - x_{m+1} = -[0, f(p(x_m)),\ f(h(x_m)), f(h(x_m)); f^{-1}] f(p(x_m)) f^2(h(x_m)). \tag{16}$$

The mean value formula for the divided differences attracts the existence of a point $\theta_m \in \mathrm{int}(F)$ such that:

$$[0, f(p(x_m)), f(h(x_m)), f(h(x_m)); f^{-1}] = \frac{[f^{-1}(\theta_m)]'''}{6}. \tag{17}$$

Since f is bijective there exists $\xi_m \in [a, b]$ such that $\theta_m = f(\xi_m)$. Taking into account (5), by (16) and (17) it follows

$$x^* - x_{m+1} = -\frac{3[f''(\xi_m)]^2 - f'(\xi_m) f'''(\xi_m)}{6[f'(\xi_m)]^5} f(p(x_m)) f^2(h(x_m)),\ m = 1, 2, \ldots. \tag{18}$$

Analogously, for $s = 2$ and $k = 1$ we obtain the iterations

$$x_{m+1} = h(x_m) - \frac{f(h(x_m))}{[h(x_m), p(x_m); f]} - \frac{[h(x_m), p(x_m), p(x_m); f]}{[h(x_m), p(x_m); f]^2 f'(p(x_m))} f(p(x_m)) f(h(x_m)), \tag{19}$$

$m = 1, 2, \ldots,\ x_1 \in [a, b]$.

In this case, for the error one holds an equality analogous to formula (18):

$$x^* - x_{m+1} = -\frac{3[f''(\xi_m)]^2 - f'(\xi_m) f'''(\xi_m)}{6[f'(\xi_m)]^5} f^2(p(x_m)) f(h(x_m)), \tag{20}$$

$m = 1, 2, \ldots$

2 The convergence of the iterations (15) and (19)

In this section we shall provide some conditions under which methods (15) and (19) generate sequences which approximate bilaterally the solution.

We consider the following assumptions on f, p and q :

(a) equation (1) has at least one solution $x^* \in]a, b[$;

(b) $f \in C^3([a, b])$;

(c) equations (1) and (2) are equivalent;

(d) function p is derivable on $]a, b[$ and $0 < p'(x) < 1$, $\forall x \in]a, b[$;

(e) function q is decreasing and continuous on $[a, b]$;

Denote $E_f : [a, b] \to \mathbb{R}$,

$$E_f(x) = 3[f''(x)]^2 - f'(x)f'''(x). \tag{21}$$

It can be easily seen that

$$E_f(x) = E_{-f}(x). \tag{22}$$

We obtain the following result when f is increasing and convex.

Theorem 2.1 *If $x_1 \in [a, b]$ and f, p, q verify the following conditions*

i_1. *assumptions* (a)–(e) *are verified;*

ii_1. $f'(x) > 0$, $\forall x \in [a, b]$;

iii_1. $f''(x) \geq 0$, $\forall x \in [a, b]$;

iv_1. $x_1 < x^*$;

v_1. $h(x_1) \leq b$;

vi_1. $E_f(x) \geq 0$, $\forall x \in]a, b[$.

Then the elements of $(x_m)_{m\geq 1}$, $(p(x_m))_{m\geq 1}$ and $(h(x_m))_{m\geq 1}$ generated by (15) *remain in $[a, b]$ and, moreover the following relations hold:*

j_1. $x_m < p(x_m) < x_{m+1} < x^* < h(x_{m+1}) < h(x_m)$, $m = 1, 2, ...$;

jj_1. $x^* - x_{m+1} < h(x_{m+1}) - x_{m+1}$;

jjj$_1$. $\lim_{m\to\infty} x_m = \lim_{m\to\infty} p(x_m) = \lim_{m\to\infty}\ h(x_m) = x^*$.

Proof. By ii$_1$ it follows that $x^* \in]a, b[$ is the unique solution of eq (1). Let $x_m \in [a, b]$ be an approximation of x^* such that $x_m < x^*$ and $h(x_m) \le b$. By hypothesis (d) it follows that there exists $c_m \in]a, b[$ such that $p(x^*)-p(x_m) = p'(c_m)(x^* - x_m) < x^* - x_m$, whence $p(x_m) > x_m$. Since p is increasing, from $x_m < x^*$ it follows $p(x_m) < p(x^*) = x^*$. This last relation, together with (e) imply $h(x_m) = q(p(x_m)) > q(x^*) = x^*$, i.e., $h(x_m) > x^*$. From relations $p(x_m) < x^* < h(x_m)$ and ii$_1$ we get $f(p(x_m)) < 0$ and $f(h(x_m)) > 0$. If in relation (15) we take into account hypotheses ii$_1$ and iii$_1$ and we apply the mean value formulas for the divided differences, we get $x_{m+1} > p(x_m)$. If in (18) we take into account vi$_1$, ii$_1$ and the values of f at $p(x_m) and\ h(x_m)$, we get $x_{m+1} < x^*$. Finally, from hypotheses (d) and (e) for $x_m < x_{m+1}$ it follows $h(x_{m+1}) < h(x_m)$ and from $x_{m+1} < x^*$ we get $h(x_{m+1}) > x^*$, such that relation j$_1$ is proved by induction. Relation jj$_1$ is implied by j$_1$.

The fact that the elements of $(x_m)_{m\ge1}$, $(p(x_m))_{m\ge1}$ and $(h(x_m))_{m\ge1}$ remain in $[a, b]$ follows from j$_1$. Moreover, these sequences are monotone and bounded, and therefore they converge. Letting $\ell = \lim x_m$, then by (15) we have $\ell = x^*$. Since p and q are continuous functions, we obtain jjj$_1$.

The theorem is proved. ■

Next we consider the case when f is decreasing and concave. If instead of equation (1) we consider equation

$$-f(x) = 0$$

and we take into account (22) and the fact that x_{m+1} given by (15) is the same if we replace f by $-f$, then from Theorem 2.1 we deduce

Corollary 2.2 *If $x_1 \in [a, b]$ and the functions f, p, q verify*

i$_2$. *hypotheses* (a)–(e) *hold;*

ii$_2$. $f'(x) < 0,\ \forall x \in [a, b]$;

iii$_2$. $f''(x) \le 0,\ \forall x \in [a, b]$;

iv$_2$. $x_1 < x^*$;

v$_2$. $h(x_1) \le b$;

vi$_2$. $E_f(x) \ge 0,\ \forall x \in]a, b[$, *then the elements of the sequences* $(x_m)_{m\geq 1}$, $(p(x_m))_{m\geq 1}$ *and* $(h(x_m))_{m\geq 1}$ *generated by* (15) *remain in* $[a,b]$ *and the properties* j_1–jjj_1 *from Theorem* 2.1 *hold.*

The following result refers to the case when f is increasing and concave.

Theorem 2.3 *If* $x_1 \in [a,b]$ *and functions* f, p, q *verify*

i_3. *hypotheses* (a)–(e) *hold;*

ii_3. $f'(x) > 0$, $\forall x \in [a,b]$;

iii_3. $f''(x) \leq 0$, $\forall x \in [a,b]$;

iv_3. $x_1 > x^*$;

v_3. $h(x_1) \geq a$;

vi_3. $E_f(x) \geq 0$, $\forall x \in [a,b]$,

then the elements of the sequences $(x_m)_{m\geq 1}$, $(p(x_m))_{m\geq 1}$, *and* $(h(x_m))_{m\geq 1}$, *generated by* (15), *remain in* $[a,b]$ *and, moreover,*

j_3. $x_m > p(x_m) > x_{m+1} > x^* > h(x_{m+1}) > h(x_m)$, $m = 1, 2, ...$;

jj_3. $x_{m+1} - x^* < x_{m+1} - h(x_{m+1})$;

jjj_3. $\lim_{m\to\infty} x_m = \lim_{m\to\infty} p(x_m) = \lim_{m\to\infty} h(x_m) = x^*$.

Proof. Hypotheses i_3 and ii_3 ensure that $x^* \in]a,b[$ is the unique solution of (1). If $x_m \in [a,b]$, $m \geq 1$, obeys $x_m > x^*$ and $h(x_m) \geq a$ then it can be easily shown that hypotheses (d) and (e) lead to relations $x_m > p(x_m) > x^* > h(x_m)$. These inequalities, together with ii_3 imply $f(p(x_m)) > 0$ and $f(h(x_m)) < 0$. By (15), using the previous relations and assumptions ii_3 and iii_3 we get $x_{m+1} < p(x_m)$. From (18), ii_3 and vi_3 it follows $x_{m+1} > x^*$, and therefore j_3 is proved. Property jj_3 is an immediate consequence of j_3. Property j_3 also implies that the elements of $(x_m)_{m\geq 1}$, $(p(x_m))_{m\geq 1}$ and $h(x_m)_{m\geq 1}$ remain in $[a,b]$. The proof of jjj_3 is analogous to the corresponding one in Theorem 2.1. ■

We obtain the following consequence of the above theorem, which is similar to the one obtained for Theorem 2.1. Now f is decreasing and convex.

Corollary 2.4 *If* $x_1 \in [a,b]$ *and functions* f, p, q *verify*

i$_4$. *assumptions* (a)–(e) *hold;*

ii$_4$. $f'(x) < 0$, $\forall x \in [a,b]$;

iii$_4$. $f''(x) \geq 0$, $\forall x \in [a,b]$;

iv$_4$. $x_1 > x^*$;

v$_4$. $h(x_1) \geq a$;

vi$_4$. $E_f(x) \geq 0$, $\forall x \in]a,b[$.

Then the elements of $(x_m)_{m\geq 1,}$ $(p(x_m))_{m\geq 1}$ *and* $(h(x_m))_{m\geq 1}$*, generated by* (15)*, remain in* $[a,b]$ *and, moreover, properties* j$_3$–jjj$_3$ *of Theorem* 2.3 *hold.*

Now we study the convergence of the sequences $(x_m)_{m\geq 1}$ given by (19). We notice that in (19) x_{m+1} may also be written as:

$$(23) \qquad x_{m+1} = p(x_m) - \frac{f(p(x_m))}{[p(x_m), h(x_m); f]} - \frac{[h(x_m), p(x_m), p(x_m); f] f(p(x_m)) f(h(x_m))}{[h(x_m), p(x_m); f]^2 f'(p(x_m))}.$$

We have seen in the proof of the previous results that the hypothesis $E_f(x) \geq 0$, $\forall x \in [a,b]$ was essential. The iterates (15) and the results concerning their convergence cannot be applied if $E_f(x) \leq 0$, $\forall x \in [a,b]$. We shall see in the following that in the study of the convergence of iterations (19) or (23), the hypothesis $E_f(x) \leq 0$, $\forall x \in [a,b]$ turns out to be essential. Therefore the previous and the subsequent results allow us to choose in practice either method (15) or method (19), (23) depending on the sign of $E_f(x)$.

Theorem 2.5 *If* $x_1 \in [a,b]$ *and functions* f, p, q *obey conditions* i$_1$–v$_1$ *of Theorem* 2.1 *and, moreover,*

$$E_f(x) \leq 0, \ \forall x \in [a,b],$$

then the elements of sequences $(x_m)_{m\geq 1}$, $(p(x_m))_{m\geq 1}$ *and* $(h(x_m))_{m\geq 1}$ *generated by* (23) *remain in* $[a,b]$ *and, moreover, satisfy relations* j$_1$–jjj$_1$ *from Theorem* 2.1.

Proof. By i_1 and ii_1 it follows that the solution x^* is unique in $]a,b[$. Let $x_m \in [a,b]$ be an approximation for x^*, which satisfies iv_1 and v_1. From the proof of Theorem 2.1 it follows that $x_m < p(x_m) < x^* < h(x_m)$. These relations, together with ii_1 imply that $f(p(x_m)) < 0$ and $f(h(x_m)) > 0$. Applying the mean value formulas for divided differences and using ii_1 and iii_1, by (23) we get $x_{m+1} > p(x_m)$. Using $E_f(x) \leq 0$, the signs of f at $p(x_m)$ and $h(x_m)$ and assumption ii_1, by (20) we get $x_{m+1} < x^*$. Inequality $x_m < x_{m+1}$ implies $h(x_{m+1}) < h(x_m)$, which shows that j_1 is true. The proof of Theorem 2.1 shows that properties jj_1 and jjj_1 are obvious. ■

The following result is similar to Corollary 2.2, we state it without proof.

Corollary 2.6 *If $x_1 \in [a,b]$, f, p, q verify conditions i_2–v_2 from Corollary 2.2 and, moreover, $E_f(x) \leq 0$ $\forall x \in [a,b]$ then the elements of $(p(x_m))_{m\geq 1}$, $(x_m)_{m\geq 1}$ and $(h(x_m))_{m\geq 1}$, generated by (23), remain in $[a,b]$ and the conclusions j_1–jjj_1 of Theorem 2.1 are true.*

The next result is analogous to Theorem 2.3.

Theorem 2.7 *If $x_1 \in [a,b]$, and functions f, p, q verify hypotheses i_3–v_3 of Theorem 2.3 and moreover $E_f(x) \leq 0$, $\forall x \in [a,b]$, then $(x_m)_{m\geq 1}$, $(p(x_m))_{m\geq 1}$, $(h(x_m))_{m\geq 1}$, generated by (23) remain in $[a,b]$ and the conclusions j_3–jjj_3 of Theorem 2.3 hold true.*

Proof. The uniqueness of the solution x^* is obvious by assumption ii_3. If $x_m \in [a,b]$, for some $m \geq 1$, obeys iv_3 then from the properties of functions p and q one obtains $x_m > p(x_m) > x^* > h(x_m)$, which, together with ii_3 lead to $f(h(x_m)) < 0$ and $f(p(x_m)) > 0$. By considerations analogous to those in the proof of Theorem 2.3, by (23) we obtain $x_{m+1} < p(x_m)$. Using hypothesis $E_f(x) \leq 0$, by (20) it follows that $x^* - x_{m+1} < 0$, i.e., $x_{m+1} > x^*$. Since $x_m > x_{m+1}$ we get $h(x_m) < h(x_{m+1})$, which proves j_3. The conclusions jj_3 and jjj_3 are immediately obtained as in the previous results. Conclusion j_3 also attracts that $(x_m)_{m\geq 1}, (p(x_m))_{m\geq 1}, (h(x_m))_{m\geq 1} \subset [a,b]$. ■

The following result is analogously obtained as Corollary 2.4.

Corollary 2.8 *If $x_1 \in [a,b]$ and the functions f, p, q verify hypotheses i_4–v_4 of Corollary 2.4, and moreover $E_f(x) \leq 0$ $\forall x \in [a,b]$ then $(x_m)_{m\geq 1}$, $(p(x_m))_{m\geq 1}$, $(h(x_m))_{m\geq 1}$, generated by (23), remain in $[a,b]$ and conclusions j_3–jjj_3 of Theorem 2.3 hold true.*

3 Determining the auxiliary functions

In this section we present a concrete way how one can construct the auxiliary functions p and q such that hypotheses (c), (d), (e) as well as, depending on the case, conditions $h(x_1) \leq b$ or $h(x_1) \geq a$ to be verified.

Let p and q be given by

$$\begin{aligned} p(x) &= x - \lambda_1 f(x),\ \lambda_1 > 0; \\ q(x) &= x - \lambda_2 f(x),\ \lambda_2 > 0. \end{aligned} \tag{24}$$

Obviously, p and q verify (c). We shall determine the parameters λ_1 and λ_2 such that the other conditions are verified.

Assume that f verifies the hypotheses of Theorem 2.1, i.e. $f'(x) > 0$ and $f''(x) \geq 0\ \forall x \in [a, b]$. This means that f' is increasing, i.e.

$$f'(a) \leq f'(x) \leq f'(b), \forall x \in [a, b]. \tag{25}$$

The above relation, for $\lambda_1 > 0$ implies

$$1 - \lambda_1 f'(b) \leq 1 - \lambda_1 f'(x) \leq 1 - \lambda_1 f'(a),\ \forall x \in [a, b].$$

In order that condition (d) is verified, it suffices that $a < \lambda_1 < \frac{1}{f'(b)}$.

Since $0 < p(x) < 1$, it follows that for $x \in [a, x^*]$, $p(x) \in]x, x^*[$, while for $x \in [x^*, b]$, $p(x) \in]x^*, x[$. Since $p(x^*) = x^*$, we have $p(x) \in [a, b]$, $\forall x \in [a, b]$.

In order that q verifies condition (e) it suffices that $q'(x) < 0\ \forall x \in [a, b]$:

$$q'(x) = 1 - \lambda_2 f'(x) < 0.$$

Since $q''(x) \leq 0$ it follows that q' is decreasing and therefore $q'(x) \leq q'(a)$. From $q'(a) < 0$ we get $\lambda_2 > \frac{1}{f'(a)}$. From such a value of λ_2 we have that function q decreases: $q(x) \leq q(a)$, $\forall x \in [a, b]$.

Since for $x \in [a, b]$ we have $p(x) \in]a, b[$, then $h(x) = q(p(x)) < q(a)$. Therefore, if $q(a) \leq b$ then for $x_1 < x^*$ we have $h(x_1) \leq b$.

Relation $q(a) \leq b$ leads us to

$$a - \lambda_2 f(a) \leq b$$

where $f(a) < 0$. The above relation implies in turn

$$\lambda_2 \leq \frac{a - b}{f(a)}.$$

If $\frac{1}{f'(a)} < \frac{a-b}{f(a)}$ then any value $\lambda_2 \in]\frac{1}{f'(a)}, \frac{a-b}{f(a)}]$ can be taken such that function q defined by (24) verifies hypothesis of Theorem 2.1.

Theorem 3.1 *If f obeys hypotheses of Theorem* 2.1 *and moreover,*

$$f(b) - f(a) < f'(a)(b - a)$$

then one can choose $\lambda_1 \in]0, \frac{1}{f'(b)}]$ and $\lambda_2 \in]\frac{1}{f'(a)}, \frac{a-b}{f(a)}]$ such that functions p and q defined in (24) *verify the hypotheses of Theorem* 2.1, *respectively Theorem* 2.5.

We consider in the following the functions $p,\ q$ given by

$$(26)\qquad \begin{aligned} p(x) &= x + \lambda_1 f(x), \quad \lambda_1 > 0; \\ q(x) &= x + \lambda_2 f(x), \quad \lambda_2 > 0. \end{aligned}$$

If we take $g(x) = -f(x)$ we obtain

$$\begin{aligned} p(x) &= x - \lambda_1 g(x); \\ q(x) &= x - \lambda_2 g(x). \end{aligned}$$

The following consequence of Theorem 3.1 can be easily proved:

Corollary 3.2 *If f obeys hypotheses of Corollary* 3.2 *and, moreover,*

$$f(a) < -f'(a)(b - a),$$

then there exist $\lambda_1 \in]0, \frac{1}{-f'(b)}],\ \lambda_2 \in [\frac{1}{-f'(a)}, \frac{a-b}{-f(a)}]$ such that functions p and q given by (26) *verify hypotheses of Corollary* 2.2, *resp. Corollary* 2.6.

Similarly to Theorem 3.1, we can prove the following results.

Theorem 3.3 *If f obeys hypotheses of Theorem* 3.1 *and, moreover,*

$$f(b) < (b - a)f'(b)$$

then there exist $\lambda_1 \in]0, \frac{1}{f'(a)}[$ and $\lambda_2 \in]\frac{1}{f'(b)}, \frac{b-a}{f(b)}]$ such that functions $p,\ q$ defined by (24) *verify hypotheses of Theorem* 2.3, *respectively Theorem* 2.7.

Corollary 3.4 *If f verifies the hypotheses of Corollary* 3.4 *and, moreover,*

$$f(b) \geq (b - a)f'(b)$$

then there exist $\lambda_1 \in]0, -\frac{1}{f'(a)}[$ and $\lambda_2 \in] - \frac{1}{f'(b)}, -\frac{b-a}{f(b)}]$ such that functions p and q given by (26) *obey hypotheses of Corollary* 2.4 *resp. Corollary* 2.8.

4 Numerical examples

Example 4.1 Consider

$$f(x) = e^x - 4x^2 = 0, \quad x \in [\tfrac{1}{2}, 1]. \tag{27}$$

One can easily see that $f'(x) < 0$, $f''(x) < 0$, $\forall \in x[\frac{1}{2}, 1]$. Since $f(\frac{1}{2}) > 0$ and $f(1) < 0$ if follows that equation (27) has a unique solution $x^* \in]\frac{1}{2}, 1[$. We also have that $E_f(x) > 0$, $\forall x \in [\frac{1}{2}, 1]$, and therefore the hypotheses of Corollary 2.2 are verified. Corollary 3.2 shows that one can take $\lambda_1 = \frac{1}{4}$ and $\lambda_2 = \frac{1}{2}$, such that $p(x) = x + \frac{1}{4}(e^x - 4x^2)$ and $q(x) = x + \frac{1}{2}(e^x - 4x^2)$, with $0 < p'(x) < 1$, $q(\frac{1}{2}) < 1$ and $q'(x) < 0$ $\forall x \in [\frac{1}{2}, 1]$.

Taking $x_1 = \frac{1}{2}$ and using (15) we obtain the following values for x_n, $p(x_n)$ and $q(x_n)$.

Table 1

n	x_n	$p(x_n)$	$h(x_n)$	$h(x_n) - x_n$
1	5.000000000000000e-1	6.621803176750321e-1	7.547224706745652e-1	2.547224706745652e-01
2	7.146918975140570e-1	7.147966292104280e-1	7.148136852840175e-1	1.217877699604131e-04
3	7.148059123627770e-1	7.148059123627778e-1	7.148059123627780e-1	9.992007221626409e-16

Example 4.2 Consider

$$f(x) = x^2 - 2\cos(x) = 0 \tag{28}$$

$x \in [\frac{\pi}{6}, \frac{\pi}{2}]$. One can easily see that $f'(x) > 0$, $f''(x) > 0$, $E_f(x) > 0$, $\forall x \in [\frac{\pi}{6}, \frac{\pi}{2}]$, and therefore the hypotheses of Theorem 2.1 are verified. The auxiliary functions p and q from (24) for $x_1 = \frac{1}{6}$, $\lambda_2 = \frac{1}{2}$, are given by

$$p(x) = \frac{6x - x^2 + 2\cos(x)}{6}$$
$$q(x) = \frac{2x - x^2 + 2\cos(x)}{2}.$$

Since $f'(x) > 0$ $\forall x \in [\frac{\pi}{6}, \frac{\pi}{2}]$, $f(\frac{\pi}{6}) < 0$, $f(\frac{\pi}{2}) > 0$ it follows that equation (28) has a unique solution $x^* \in [\frac{\pi}{6}, \frac{\pi}{2}]$.

Consider in (15) $x_1 = \frac{\pi}{6}$ and we obtain the values in Table 2.

n	x_n	$p(x_n)$	$h(x_n)$	$h(x_n) - x_n$
1	5.235987755982988e-1	7.665812972251055e-1	1.193044203747889e+0	6.694454281495906e-01
2	1.018804247227570e+0	1.020605393992001e+0	1.022637703168053e+0	3.833455940482455e-03
3	1.021689953697528e+0	1.021689953944147e+0	1.021689954221672e+0	5.241440614867088e-10
4	1.021689954092185e+0	1.021689954092185e+0	1.021689954092185e+0	-2.220446049250313e-16

Example 4.3 Consider

$$f(x) := e^x + 6x - 5 = 0 \tag{29}$$

$x \in [0,1]$, having a unique solution in $]0,1[$. Since $f'(x) > 0$, $f''(x) > 0$ and $E_f(x) = 2e^x(e^x - 3) < 0$, $\forall x \in [0,1]$, the hypotheses of Theorem 2.5 are verified. Applying Theorem 3.1, we can take $\lambda_1 = \frac{1}{10}, \lambda_2 = \frac{1}{5}$ and by (24) we get

$$p(x) = \frac{4x - e^x + 5}{10}$$
$$q(x) = \frac{5 - x - e^x}{5}.$$

Let $x_1 = 0$, which implies $q(x_1) = \frac{4}{5} < 1$. By (23) we obtain the results presented in Table 3.

n	x_n	$p(x_n)$	$h(x_n)$	$h(x_n) - x_n$
1	0	4.000000000000000e-1	6.216350604717459e-1	6.216350604717459e-1
2	5.456771482503846e-1	5.456931999594989e-1	5.457005009495495e-1	2.335269916486915e-5
3	5.456979250249538e-1	5.456979250249538e-1	5.456979250249538e-1	0

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