On a Newton-Steffensen type method

I. Păvăloiu and E. Cătinaş*

"T. Popoviciu" Institute of Numerical Analysis, Romanian Academy, P.O. Box 68-1,
Cluj-Napoca, Romania, e-mail: `pavaloiu@ictp.acad.ro,ecatinas@ictp.acad.ro`

Abstract

In this paper we study the convergence of a Newton-Steffensen type method for solving nonlinear equations in $\mathbb{R}$, introduced by Sharma [J.R. Sharma, A composite third order Newton-Steffensen method for solving nonlinear equations, Appl. Math. Comput. 169 (2005), 242–246].

Under simplified assumptions regarding the smoothness of the nonlinear function, we show that the q-convergence order of the iterations is 3. The efficiency index of the method is $\sqrt[3]{3}$, and is larger than $I_2 = \sqrt{2}$, which corresponds to the Newton method or the Steffensen method.

Moreover, we show that if the nonlinear function maintains the same monotony and convexity on an interval containing the solution, and the initial approximation satisfies the Fourier condition, then the iterations converge monotonically to the solution.

We also obtain a posteriori formulas for controlling the errors.

The numerical examples confirm the theoretical results.

Keywords: Nonlinear equations, Newton method, Steffensen method, convergence order.
2000 MSC: 65H05

1. Introduction

Consider the equation

$$f(x) = 0 \tag{1}$$

$f : [a, b] \to \mathbb{R}$, $a < b$, and another equivalent equation:

$$x - g(x) = 0 \tag{2}$$

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*Corresponding author.

$g : [a, b] \to [a, b]$.

The Steffensen type methods are interpolatory methods (see, e.g., [8], [5]), with the nodes controlled at each step by the auxiliary function g [1]–[4], [6]–[7], [9]–[11], [14]. The method generates a sequence $\{x_n\}_{n \geq 0}$ which approximates a solution x^* of (1):

$$x_{n+1} = x_n - \frac{f(x_n)}{[x_n, g(x_n); f]}, \quad n = 0, 1, ..., x_0 \in [a, b]. \tag{3}$$

In the papers [9], [10] is studied the convergence of the Steffensen method when g is given by

$$g(x) = x - \lambda f(x)$$

$\lambda \in \mathbb{R}$. The parameter λ is determined such that the sequences $\{x_n\}_{n \geq 0}$ and $\{g(x_n)\}_{n \geq 0}$ approximate bilaterally the solution, and the convergence order of these methods is 2.

In [11] Sharma introduces a Steffensen type method of order three, with g given by

$$g(x) = x - \frac{f(x)}{f'(x)}, \tag{4}$$

which we shall call as the Newton-Steffensen (N-S) method. Since it requires 3 function evaluations at each iteration step, its efficiency index is $I_3 = \sqrt[3]{3}$, and is larger than $I_2 = \sqrt{2}$, which corresponds to the Newton method or the Steffensen method (see, e.g., [9], [8], [12]), and therefore the method is worth to be considered.

In this paper we shall show in Section 2 that the convergence order 3 can be obtained under some weaker smoothness assumptions on f than in [11] and [13]. Moreover, in Section 3 we shall study the convergence of the (N-S) method in hypotheses regarding the monotony and convexity of f on $[a, b]$. Under some simple assumptions on f, including the Fourier conditions, we shall show that the (N-S) method leads to monotonic sequences which approximate the solution. In Section 4 we illustrate the theory by some numerical examples, which confirm the theoretical results.

2. Local convergence of the method

Consider the following hypotheses:
α) equation (1) has a solution $x^* \in]a, b[$.
β) $f \in C^2[a, b]$,
γ) $f'(x) \neq 0$ on $[a, b]$.
Sharma proved the following result.

Theorem 1. *[11]If f satisfies $\alpha)$–$\gamma)$ and moreover, there exists $f'''(x^*)$, then the (N-S) method has the q-convergence order at least 3, with the asymptotic constant given by relation:*

$$|x_{n+1} - x^*| = \left(\frac{f''(x^*)}{2f'(x^*)}\right)^2 |x_n - x^*|^3 + o\left(|x_n - x^*|^4\right), \ \ as \ n \to \infty. \quad (5)$$

Of course that the fact that the sequence is well defined must be implicitly assumed.

We show that the requirement that f is three times differentiable can be dropped.

In [13], the authors studied the semilocal and local convergence of the (N-S) iterates in the setting of Banach spaces. Local convergence with order 3 was established under the assumption of Lipschitz continuity of f''. We show here that even the Lipschitz continuity may be relaxed, and we consider the following assumption:

$\beta')$ $f \in C^1[a,b]$, and $\exists f''(x), \forall x \in (a,b)$, which is continuous at x^*.

Theorem 2. *If f satisfies $\alpha), \beta'), \gamma)$, the (N-S) method is well defined and converges to x^*, then the method has the q-convergence order at least 3, with the same asymptotic constant given in (5).*

Proof. From the Newton identity and $\alpha)$ we have

$$f(x^*) = f(x_n) + [x_n, g(x_n); f](x^* - x_n) +$$
$$+ [x^*, x_n, g(x_n); f](x^* - x_n)(x^* - g(x_n)) = 0$$

whence, by (3) we get

$$x^* - x_{n+1} = -\frac{[x^*, x_n, g(x_n); f]}{[x_n, g(x_n); f]}(x^* - x_n)(x^* - g(x_n)), \ \ n = 0, 1, ... \quad (6)$$

By $\alpha)$, $\beta)$ and the Taylor formula we obtain

$$f(x^*) = f(x_n) + f'(x_n)(x^* - x_n) + \frac{f''(\xi_n)}{2}(x^* - x_n)^2 = 0, \quad (7)$$

where ξ_n belongs to the open interval determined by x_n and x^*.

Taking into account that $g(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}$, by (7) we infer

$$x^* - g(x_n) = -\frac{f''(\xi_n)}{2f'(x_n)}(x^* - x_n)^2, \ \ n = 0, 1, ..., \quad (8)$$

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which together with (6) attracts

$$x^* - x_{n+1} = \frac{f''(\xi_n)[x^*, x_n, g(x_n); f]}{2f'(x_n)[x_n, g(x_n); f]}(x^* - x_n)^3, \ n = 0, 1, ..., \quad (9)$$

which shows that the (N-S) method has q-order 3. The Mean Value Theorem for divided differences ensures that $[x^*, x_n, g(x_n); f] \to \frac{1}{2}f''(x^*)$ and $[x_n, g(x_n); f] \to f'(x^*)$, as $n \to \infty$, so we are led to the same asymptotic constant as in (5). ∎

3. Monotone convergence of the method

As it is well known, when $f \in C^2[a, b]$ preserves its monotonicity and convexity on $[a, b]$, if the initial approximation $x_0 \in [a, b]$ verifies the Fourier condition, then the Newton method yields a monotone sequence which converges to the solution $x^* \in [a, b]$. We shall show that under same such hypotheses, we obtain the same conclusions, but for a method with superior efficiency index.

We assume that the initial approximation x_0 verifies the Fourier condition (see [8]):

δ) $f''(x_0)f(x_0) > 0, \ x_0 \in [a, b]$.

We obtain the following results.

Theorem 3. *If f and x_0 verify $\alpha), \beta'), \delta)$ and, moreover,*

$\mathbf{i_1}$. *$f'(x) > 0, \forall x \in [a, b]$;*

$\mathbf{ii_1}$. *$f''(x) > 0, \forall x \in [a, b]$,*

then the elements of $\{x_n\}_{n \geq 0}$ and $\{g(x_n)\}_{n \geq 0}$ generated by the (N-S) method remain in $[a, b]$, converge to x^* and obey

$$x_n > g(x_n) > x_{n+1} > x^*, \qquad n = 0, 1, ...; \quad (10)$$

Proof. From $\beta)$ and i_1 it follows that $x^* \in]a, b[$ is a unique solution. Conditions $\alpha), \gamma), i_1$ and ii_1 imply that $x_0 > x^*$. Let $x_n \in [a, b]$ such that $x_n > x^*$. From i_1 we get that $f(x_n) > 0$ and $g(x_n) = x_n - \frac{f(x_n)}{f'(x_n)} < x_n$. Conditions i_1, ii_1 and (8) imply $x^* < g(x_n)$.

Using the mean value formulas for divided differences and taking into account i_1, ii_1 and (6), it follows that $x^* < x_{n+1}$.

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We notice that relation (3) can be written as

$$x_{n+1} = g(x_n) - \frac{f(g(x_n))}{[x_n, g(x_n); f]}, \qquad n = 0, 1, ..., \qquad (11)$$

whence, since $f(g(x_n)) > 0$ by i_1 it follows $x_{n+1} < g(x_n)$.

It is obvious that the elements of $\{x_n\}_{n \geq 0}$ and $\{g(x_n)\}_{n \geq 0}$ belong to $]x^*, x_0] \subset [a, b]$. It remains to show the convergence to the solution. Indeed, if $\ell = \lim x_n$ then by (10) $\lim g(x_n) = g(\ell) = \ell$, whence $f(\ell) = 0$ i.e., $\ell = x^*$. ∎

Theorem 4. *The statements of Theorem 3 hold true if assumptions i_1, ii_1 are replaced by*

i_2. $f'(x) < 0$, $\forall x \in [a, b]$;

ii_2. $f''(x) < 0$, $\forall x \in [a, b]$.

Proof. The result can be obtained as a consequence of Theorem 3, if instead of (1) we consider the equation $h(x) = 0$, with $h = -f$. ∎

Theorem 5. *If f and x_0 verify $\alpha), \beta'), \delta)$ and, moreover*

i_3. $f'(x) > 0$, $\forall x \in [a, b]$;

ii_3. $f''(x) < 0$, $\forall x \in [a, b]$,

then the elements of $\{x_n\}_{n \geq 0}$ and $\{g(x_n)\}_{n \geq 0}$ generated by the (N-S) method remain in $[a, b]$, converge to x^* and obey

$$x_n < g(x_n) < x_{n+1} < x^*, \qquad n = 0, 1, ... \qquad (12)$$

Proof. Hypotheses $\beta)$ and i_3 ensure that $x^* \in [a, b]$ is a unique solution. Assumptions $\alpha), \gamma), i_3$ and ii_3 lead to $f(x_0) < 0$ or $x_0 < x^*$. Let $x_n < x^*$ for some $x_n \in [a, b]$. By i_3 we get $f(x_n) < 0$ and $g(x_n) = x_n - \frac{f(x_n)}{f'(x_n)} > x_n$. By (8), i_3 and ii_3 it follows $g(x_n) < x^*$. Relation (3) implies $x_{n+1} > x_n$ and by (6) we get $x_{n+1} < x^*$. By (11) we get $x_{n+1} > g(x_n)$. ∎

Theorem 6. *The same conclusions hold if in Theorem 5 i_3 and ii_3 are replaced by*

i_4. $f'(x) < 0$, $\forall x \in [a, b]$;

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ii$_4$. $f''(x) > 0$, $\forall x \in [a, b]$.

Proof. This result can be obtained from Theorem 5 by taking $-f$ instead of f. ∎

Remark 1. *The hypotheses of Theorems 3–6 ensure the fact that the elements of $\{x_n\}_{n\geq 0}$ and $\{g(x_n)\}_{n\geq 0}$ remain in $[a, b]$. Since the hypotheses of Theorem 2 are also verified, it follows that the iterations converge with q-order 3.*

Remark 2. *From relation*

$$f(x_{n+1}) = f(x_n) + [x_n, g(x_n); f](x_{n+1} - x_n) + $$
$$+ [x_{n+1}, x_n, g(x_n); f](x_{n+1} - x_n)(x_{n+1} - g(x_n)),$$

taking into account (3) we get

$$x^* - x_{n+1} = -\frac{[x_{n+1}, x_n, g(x_n); f]}{[x^*, x_{n+1}; f]}(x_{n+1} - x_n)(x_{n+1} - g(x_n)).$$

whence it follows that in the assumptions of Theorems 3–6 we have

$$|x^* - x_{n+1}| = O(|x_{n+1} - x_n|^2).$$

This formula can be useful for a posteriori control of the error.

4. Numerical examples

We consider four numerical examples, and we compute the iterations using double precision in Matlab.

Example 1. *Consider the equation*

$$f(x) = x^2 - x \sin x + e^{x+1} - 3 = 0, \ \ x \in [0, 1].$$

One can verify that $f(0) = e - 3 < 0$, $f(1) = 1 + e^2 - \sin 1 - 3 = e^2 - \sin 1 - 2 > 0$, $f'(x) = 2x - \sin x - x \cos x + e^{x+1} > 2x + e^{x+1} - 2 > 0$, for all $x \in [0, 1]$. Also, $f''(x) = 2 - 2\cos x + x \sin x + e^{x+1} = 4\sin^2\frac{x}{2} + x \sin x + e^{x+1} > 0$ for all $x \in [0, 1]$. Therefore the assumptions of Theorem 3 apply for $x_0 = 1$.

We have $g(x) = x - \frac{x^2 - x \sin x + e^{x+1} - 3}{2x - \sin x - x \cos x + e^{x+1}} = \frac{x^2 - x^2 \cos x + (x-1)e^{x+1} + 3}{2x - \sin x - x \cos x + e^{x+1}}$

The obtained numerical results are shown in Table 1. The values of $f(x_n)$ are rounded to two decimals, since we are interested only in the magnitude of these values.

It can be seen that the (N-S) method generates decreasing iterates for $\{x_n\}_{n\geq 0}$.

n	x_n	$g(x_n)$	$f(x_n)$
0	1.000000000000000e+0	4.320688774181047e-1	4.5e+00
1	2.300692760447372e-1	1.070409169425782e-1	4.2e-01
2	9.915547164564892e-2	9.860719010016147e-2	1.6e-03
3	9.86703883247032e-2	9.860703879072202e-2	1.3e-10
4	9.860703879072187e-2	9.860703879072202e-2	-4.4e-16

Table 1: Numerical results for solving $x^2 - x \sin x + e^{x+1} - 3 = 0, \ x \in [0, 1]$.

Example 2. *Consider the following equation*

$$f(x) = x^2 + \cos x - xe^x = 0, \ x \in [0, 1].$$

We have: $f(0) = 1 > 0$, $f(1) = -e + 1 + \cos 1 < 0$, $f'(x) = 2x - \sin x - e^x - xe^x$, $f''(x) = 2 - \cos x - 2e^x - xe^x < 0$, $f'''(x) = \sin x - (3 + x)e^x$, which imply that $f'(x) < 0, f''(x) < 0, \ x \in [0, 1]$.

If $x_0 = 1$, the assumptions of Theorem 4 are verified.

We obtain the results presented in Table 2. As expected, the (N-S) iterates are decreasing.

n	x_n	$g(x_n)$	$f(x_n)$
0	1.000000000000000e+0	7.246446975670946e-1	-1.2e+00
1	6.607648584752154e-1	6.395167806664399e-1	-5.3e-02
2	6.391602133769920e-1	6.391540963613613e-1	-1.5e-05
3	6.391540963320078e-1	6.391540963320076e-1	-6.7e-16

Table 2: Numerical results for solving $x^2 + \cos x - xe^x = 0, \ x \in [0, 1]$.

Example 3. *Consider*

$$f(x) = \sin x + 2x - 2, \ x \in [0, \frac{\pi}{2}].$$

We have $f(0) = -2 < 0$, $f(\frac{\pi}{2}) = \pi - 1 > 0$, $f'(x) = \cos x + 2 > 0, \ x \in [0, \frac{\pi}{2}]$, while $f''(x) = -\sin x < 0, \ x \in [0, \frac{\pi}{2}]$.

Taking $x_0 = 0$, hypothesis of Theorem 5 apply.

The obtained results are presented in Table 3. The (N-S) iterates are increasing.

Example 4. *Consider*

$$f(x) = 3e^{-x} - x + 1, \ x \in [1, 2].$$

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n	x_n	$g(x_n)$	$f(x_n)$
0	0.000000000000000e+0	6.666666666666666e-1	-2.0e+00
1	6.831640060745233e-1	6.840365700507293e-1	-2.4e-03
2	6.840366566692261e-1	6.840366566778295e-1	-2.4e-11
3	6.840366566778295e-1	6.840366566778295e-1	0.0e+00

Table 3: Numerical results for solving $\sin x + 2x - 2 = 0$, $x \in [0, \frac{\pi}{2}]$.

We have $f(1) = \frac{3}{e} > 0$, $f(2) = \frac{3}{e^2} - 1 < 0$, $f'(x) = -3e^{-x} - 1 < 0$, $x \in [1, 2]$, while $f''(x) = 3e^{-x} > 0$, $x \in [1, 2]$.

Taking $x_0 = 1$, hypothesis of Theorem 6 apply.

The obtained results are presented in Table 4. The (N-S) iterates are increasing.

n	x_n	$g(x_n)$	$f(x_n)$
0	1.000000000000000e+0	1.524633113581329e+0	1.1e+00
1	1.593748766088184e+0	1.603527625548530e+0	1.6e-02
2	1.603545706091483e+0	1.603545739535836e+0	5.4e-08
3	1.603545739535836e+0	1.603545739535836e+0	-2.2e-16

Table 4: Numerical results for solving $3e^{-x} - x + 1 = 0$, $x \in [1, 2]$.

It is interesting to note that we get $f(x_3) = 0$ in Example 3, which suggests that $x_3 = x^*$. However, if instead of the standard double precision we use higher precision (e.g., variable precision arithmetic) for representing x_3 from Table 3, the computation of $f(x_3)$ no longer yields 0 (of course, we obtain values with the magnitude comparable to the machine epsilon from double precision, $\epsilon \approx 2.22\mathrm{e}{-16}$). This means that x_3 from Table 3 is just a very good approximation (in double precision arithmetic) to the solution.

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