

APPROXIMATION OPERATORS CONSTRUCTED  
BY MEANS OF SHEFFER SEQUENCES

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**Abstract.** In this paper we introduce a class of positive linear operators by using the "umbral calculus", and we study some approximation properties of it. Let  $Q$  be a delta operator, and  $S$  an invertible shift invariant operator. For  $f \in C[0, 1]$  we define

$$(L_n^{Q,S}f)(x) = \frac{1}{s_n(1)} \sum_{k=0}^n \binom{n}{k} p_k(x) s_{n-k}(1-x) f\left(\frac{k}{n}\right),$$

where  $(p_n)_{n \geq 0}$  is a binomial sequence which is the basic sequence for  $Q$ , and  $(s_n)_{n \geq 0}$  is a Sheffer set,  $s_n = S^{-1}p_n$ . These operators generalize the binomial operators of T. Popoviciu.

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## 1. INTRODUCTION

Let  $P$  be the linear space of all polynomials with real coefficients, and  $P_n$  the linear space of all polynomials of degree at most  $n$ .

We will consider some linear operators defined on  $P$ . We will denote by  $I$  the identity and by  $D$  the derivative. The shift operator  $E^a : P \rightarrow P$  is defined by  $E^a p(x) = p(x + a)$ .

A linear operator  $T$  which commutes with all shift operators is called a *shift invariant operator*. In symbols,  $E^a T = T E^a$ , for all real  $a$ .

Let us remind that if  $T_1$  and  $T_2$  are shift invariant operators, then  $T_1 T_2 = T_2 T_1$ .

**DEFINITION 1.** A *shift invariant operator for which  $Qx = \text{const} \neq 0$  is called a delta operator*.

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By a *polynomial sequence* we shall denote a sequence of polynomials  $p_n(x)$ ,  $n = 0, 1, 2, \dots$  where  $p_n(x)$  is of degree exactly  $n$  for all  $n$ .

A sequence of *binomial type* is a polynomial sequence  $(p_n)_{n \geq 0}$  with  $p_0(x) = 1$  and satisfying the identities

$$p_n(x+y) = \sum_{k=0}^n \binom{n}{k} p_k(x) p_{n-k}(y),$$

for all  $x, y$  and  $n = 0, 1, 2, \dots$

DEFINITION 2. Let  $Q$  be a delta operator and  $(p_n(x))_{n \geq 0}$  a polynomial sequence. If

- i)  $p_0(x) = 1$ ,
- ii)  $p_n(0) = 0$ ,  $n = 1, 2, \dots$ ,
- iii)  $Qp_n = np_{n-1}$ ,  $n = 1, 2, \dots$ ,

then  $(p_n)$  is called the sequence of basic polynomials for  $Q$ .

PROPOSITION 1. [8].

- i) Every delta operator has a unique sequence of basic polynomials.
- ii) If  $p_n(x)$  is a basic sequence for some delta operator  $Q$ , then it is binomial.
- iii) If  $p_n(x)$  is a binomial sequence, then it is a basic sequence for some delta operator  $Q$ .

Let  $X$  be the multiplication operator defined as  $(Xp)(x) = xp(x)$  for every polynomial  $p$ .

For any operator  $T$  defined on  $P$ , the operator  $T' = TX - XT$  is called the Pincherle derivative of the operator  $T$ .

PROPOSITION 2. [8].

- i) If  $T$  is a shift invariant operator, then its Pincherle derivative is also a shift invariant operator.
- ii) If  $Q$  is a delta operator, then its Pincherle derivative  $Q'$  is an invertible operator.

PROPOSITION 3. [8], [11]. If  $(p_n(x))_{n \geq 0}$  is a sequence of basic polynomials for the delta operator  $Q$  then

- i)  $p_n(x) = X(Q')^{-1}p_{n-1}(x)$ ,  $n = 1, 2, \dots$ ,

$$\text{ii) } p_n(x) = x \sum_{k=0}^{n-1} \binom{n-1}{k} p_{n-1-k}(x) p'_{k+1}(0), \quad n = 1, 2, \dots$$

DEFINITION 3. A polynomial sequence  $(s_n(x))_{n \geq 0}$  is called a Sheffer set relative to the delta operator  $Q$  if:

- i)  $s_0(x) = \text{const} \neq 0$
- ii)  $Qs_n = ns_{n-1}, n = 1, 2, \dots$

An Appel set is a Sheffer set relative to the derivative  $D$ .

PROPOSITION 4. [11]. Let  $Q$  be a delta operator with basic polynomial set  $(p_n(x))_{n \geq 0}$  and  $(s_n(x))_{n \geq 0}$  a polynomial sequence. The next statements are equivalent:

- i)  $s_n(x)$  is a Sheffer set relative to  $Q$ .
- ii) There exists an invertible shift invariant operator  $S$  such that  $s_n(x) = S^{-1}p_n(x)$ .
- iii) For all  $x, y \in \mathbb{R}$  and  $n = 0, 1, 2, \dots$ , the following identity holds:

$$s_n(x+y) = \sum_{k=0}^n \binom{n}{k} p_k(x) s_{n-k}(y).$$

From the previous Proposition it results that the pair  $(Q, S)$  gives us a unique Sheffer set.

## 2. THE OPERATORS CONSTRUCTED BY MEANS OF SHEFFER POLYNOMIALS AND THEIR CONVERGENCE

In 1931 in [9] Tiberiu Popoviciu has used binomial sequences in order to construct some operators of the form

$$(1) \quad (L_n f)(x) = \frac{1}{p_n(1)} \sum_{k=0}^n \binom{n}{k} p_k(x) p_{n-k}(1-x) f\left(\frac{k}{n}\right)$$

where  $f \in C[0, 1]$  and  $x \in [0, 1]$ . These operators are called *binomial operators*.

Such operators and their generalizations have been studied by the Romanian mathematicians as: D. D. Stancu, A. Lupaş, L. Lupaş, G. Moldovan, C. Manole, O. Agratini, A. Vernescu, and others.

Let  $Q$  be a delta operator and  $S$  an invertible shift invariant operator. Let  $(p_n(x))_{n \geq 0}$  be the sequence of basic polynomials for  $Q$ , and  $(s_n(x))_{n \geq 0}$  a

Sheffer set relative to  $Q$ ,  $s_n = S^{-1}p_n$  with  $s_n(1) \neq 0$  for any positive integer  $n$ .

In this note we want to study the operators  $L_n^{Q,S} : C[0, 1] \rightarrow C[0, 1]$ ,

$$(2) \quad (L_n^{Q,S} f)(x) = \frac{1}{s_n(1)} \sum_{k=0}^n \binom{n}{k} p_k(x) s_{n-k}(1-x) f\left(\frac{k}{n}\right)$$

Because  $p_k(0) = \delta_{k,0}$  (from the definition of basic polynomials), we have  $(L_n^{Q,S} f)(0) = f(0)$ .

In order to evaluate expression  $(L_n^{Q,S} e_m)(x)$ , where  $e_m(x) = x^m$  we shall make use of C. Manole's method for binomial operators (see [5]) which we have adapted to our purposes.

Let us introduce the polynomials

$$(3) \quad S_m(x, y, n) = \sum_{k=0}^n \binom{n}{k} p_k(x) s_{n-k}(y) \left(\frac{k}{n}\right)^m$$

From Proposition 4 iii) we have  $S_0(x, y, n) = s_n(x+y)$ .

In the following we consider that  $x$  is the variable. Let us denote  $\theta = X(Q')^{-1}$ .

From Proposition 3 i) it results that  $\theta p_k(x) = p_{k+1}(x)$  and consequently the linear operator  $\theta$  is called the *shift operator* for the sequence  $(p_n)_{n \geq 0}$  (see [10]). Therefore  $\theta Q p_k(x) = \theta(k p_{k-1}(x)) = k p_k(x)$ ; consequently  $k$  is an eigenvalue for the operator  $\theta Q$ , with its eigenvector  $p_k(x)$ . We have

$$(4) \quad (\theta Q)^m = k^m p_k(x)$$

for every positive integer  $m$ , and then

$$\begin{aligned} S_m(x, y, n) &= \frac{1}{n^m} (\theta Q)^m \sum_{k=0}^n \binom{n}{k} p_k(x) s_{n-k}(y) \\ &= \frac{1}{n^m} (\theta Q)^m S_0(x, y, n) = \frac{1}{n^m} (\theta Q)^m s_n(x+y). \end{aligned}$$

In this way we obtain

$$(5) \quad S_m(x, y, n) = \frac{1}{n^m} (\theta Q)^m E^y s_n(x)$$

Using the operational formula (see for instance [10])

$$(\theta Q)^m = \sum_{k=0}^n S(m, k) \theta^k Q^k,$$

where  $S(m, k) = [0, 1, \dots, k; e_m]$  are the Stirling numbers of the second kind, relation (5) becomes:

$$(6) \quad S_m(x, y, n) = \frac{1}{n^m} \sum_{k=0}^n S(m, k) \theta^k Q^k E^y s_n(x).$$

Because  $Q$  is shift invariant and  $Q^k s_n(x) = n(n-1) \dots (n-k+1) s_{n-k}(x) = n^{[k]} s_{n-k}(x)$  we obtain

$$(7) \quad S_m(x, y, n) = \frac{1}{n^m} \sum_{k=0}^n \binom{n}{k} k! S(m, k) \theta^k E^y s_{n-k}(x), \quad \forall m \in \mathbb{N}^*$$

**THEOREM 1.** *If  $L_n^{Q,S}$  is the linear operator defined by (2) then*

$$(8) \quad \begin{aligned} (L_n^{Q,S} e_0)(x) &= e_0(x) \\ (L_n^{Q,S} e_1)(x) &= a_n e_1(x) \\ (L_n^{Q,S} e_2)(x) &= b_n x^2 + x(a_n - b_n - c_n), \end{aligned}$$

where

$$(9) \quad \begin{aligned} a_n &= \frac{[(Q')^{-1} s_{n-1}](1)}{s_n(1)}, \\ b_n &= \frac{n-1}{n} \frac{[(Q')^{-2} s_{n-2}](1)}{s_n(1)}, \\ c_n &= \frac{n-1}{n} \frac{[(Q')^{-2} (S^{-1})' S s_{n-2}](1)}{s_n(1)} \end{aligned}$$

*Proof.* Using the notation (3) we can write

$$(10) \quad (L_n^{Q,S} e_m)(x) = S_m(x, 1-x, n) / s_n(1).$$

Because  $S_0(x, 1-x, n) = s_n(1)$  we have  $(L_n^{Q,S} e_0)(x) = e_0(x)$ .

As we have

$$(11) \quad \theta E^y s_{n-1}(x) = X(Q')^{-1} E^y s_{n-1}(x) = X E^y (Q')^{-1} s_{n-1}(x),$$

we obtain from (7):  $S_1(x, 1-x, n) = x[(Q')^{-1} s_{n-1}](1)$ ; consequently we get:

$$(L_n^{Q,S} e_1)(x) = \frac{[(Q')^{-1} s_{n-1}](1)}{s_n(1)} x.$$

Using the Pincherle derivative of the shift operator  $E^y$

$$(12) \quad (E^y)' = y E^y = E^y X - X E^y$$

we can write

$$\theta E^y s_{n-k}(x) = E^y X(Q')^{-1} s_{n-k}(x) - y E^y (Q')^{-1} s_{n-k}(x)$$

Then

$$(13) \quad \theta^2 E^y s_{n-k}(x) = X E^y (Q')^{-1} E^y X(Q')^{-1} s_{n-k}(x) - y X E^y (Q')^{-2} s_{n-k}(x)$$

Because  $s_{n-k} = S^{-1} p_{n-k}$ ,  $X S^{-1} = S^{-1} X - (S^{-1})'$  (from the definition of Pincherle derivative) and  $(Q')^{-1} p_{n-k}(x) = p_{n-k+1}(x)/x$  (from Proposition 3 i), we obtain

$$(14) \quad \theta^2 E^y s_{n-k}(x) = X E^y (Q')^{-1} s_{n-k+1}(x) - X E^y (Q')^{-2} (S^{-1})' S s_{n-k}(x) - y X E^y (Q')^{-2} s_{n-k}(x).$$

Replacing (11) and (14) in (7) we can write

$$S_2(x, y, n) = x E^y (Q')^{-1} s_{n-1}(x) - \frac{n-1}{n} [x E^y (Q')^{-2} (S^{-1})' S s_{n-2}(x) - y x E^y (Q')^{-2} s_{n-2}(x)]$$

From (10) and the previous relation one obtains expression  $L_n^{Q,S} e_2$  from theorem's conclusion.  $\square$

LEMMA 1. *Let  $Q$  be a delta operator and  $S$  an invertible shift invariant operator. Let  $(p_n(x))_{n \geq 0}$  be the sequence of basic polynomials for  $Q$  and  $(s_n(x))_{n \geq 0}$  a Sheffer set relative to  $Q$ ,  $s_n = S^{-1} p_n$  with  $s_n(1) \neq 0$  for any positive integer  $n$ . If  $p'_k(0) \geq 0$  and  $s_k(0) \geq 0$  for  $n = 0, 1, 2, \dots$  then the operator  $L_n^{Q,S}$  defined by (2) is positive.*

*Proof.* If  $p'_k(0) \geq 0$  using Proposition 3 ii), it is easy to prove by induction that  $p_k(x) \geq 0$ ,  $\forall k \in \mathbb{N}$  and  $\forall x \in [0, 1]$ .

If we consider  $x = 0$  in Proposition 4 iii) we obtain

$$s_n(x) = \sum_{k=0}^n \binom{n}{k} p_k(x) s_{n-k}(0);$$

accordingly, for  $s_k(0) \geq 0$  and  $p_k(x) \geq 0$ ,  $\forall k \in \mathbb{N}$ ,  $\forall x \in [0, 1]$ , we have  $s_k(x) \geq 0$ ,  $\forall k \in \mathbb{N}$  and  $\forall x \in [0, 1]$ . Therefore the operator  $L_n^{Q,S}$  is positive.  $\square$

LEMMA 2. *If the operator  $L_n^{Q,S}$  is positive, then  $a_n \in [0, 1]$ ,  $b_n \leq 1$  and  $0 \leq c_n \leq \min\{\frac{1-b_n}{2}, a_n - a_n^2\}$ ,  $\forall n \in \mathbb{N}$ , where  $a_n, b_n$  and  $c_n$  are defined by (9).*

*Proof.* Since  $0 \leq e_1(t) \leq 1, \forall t \in [0, 1]$  and the operator  $L_n^{Q,S}$  is positive, we have  $0 \leq (L_n^{Q,S} e_1)(x) \leq 1, \forall x \in [0, 1]$ , and as  $(L_n^{Q,S} e_1)(x) = a_n x$ , we get  $a_n \in [0, 1]$ .

From  $t(1-t) \geq 0$  it results that  $(L_n^{Q,S} e_1)(x) - (L_n^{Q,S} e_2)(x) \geq 0$ , which leads to  $x(1-x)b_n + xc_n \geq 0, \forall x \in [0, 1]$  and choosing  $x = 1$ , we get  $c_n \geq 0$ .

Since  $t^2 - t + 1/4 \geq 0$ , we obtain  $(L_n^{Q,S} e_2)(x) - (L_n^{Q,S} e_1)(x) + (L_n^{Q,S} e_0)(x)/4 \geq 0, \forall x \in [0, 1]$ , relation equivalent to  $x^2 b_n - x b_n - x c_n + 1/4 \geq 0, \forall x \in [0, 1]$ . If we consider  $x = 1/2$ , it results that  $c_n \leq (1 - b_n)/2$  and because  $c_n \geq 0$  we get  $b_n \leq 1$ .

Finally, from the Schwarz's inequality,

$$[(L_n^{Q,S} e_1)(x)]^2 \leq (L_n^{Q,S} e_2)(x)(L_n^{Q,S} e_0)(x),$$

we have  $a_n^2 x^2 \leq b_n x^2 + x(a_n - b_n - c_n), \forall x \in [0, 1]$ . For  $x = 1$  that implies  $c_n \leq a_n - a_n^2$ .  $\square$

**THEOREM 2.** *Let  $Q$  be a delta operator and  $S$  an invertible shift invariant operator. Let  $(p_n(x))_{n \geq 0}$  be the sequence of basic polynomials for  $Q$ , with  $p'_n(0) \geq 0, \forall n \in \mathbb{N}$ , and  $(s_n(x))_{n \geq 0}$  a Sheffer set relative to  $Q$ ,  $s_n = S^{-1} p_n$  with  $s_n(1) \neq 0$  and  $s_n(0) \geq 0, \forall n \in \mathbb{N}$ . If  $f \in C[0, 1]$  and  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = 1$ , where  $a_n$  and  $b_n$  are defined by (9), then the operator  $L_n^{Q,S}$  converges to the function  $f$ , uniformly on the interval  $[0, 1]$ .*

*Proof.* If  $\lim_{n \rightarrow \infty} a_n = 1$  then  $\lim_{n \rightarrow \infty} (L_n^{Q,S} e_1)(x) = e_1(x)$ . From Lemma 2,  $c_n \leq a_n - a_n^2$  so we have  $\lim_{n \rightarrow \infty} c_n = 0$ , and as  $\lim_{n \rightarrow \infty} b_n = 1$ , we get  $\lim_{n \rightarrow \infty} (L_n^{Q,S} e_2)(x) = e_2(x)$ . Therefore  $\lim_{n \rightarrow \infty} (L_n^{Q,S} e_i)(x) = e_i(x)$  for  $i = 0, 1, 2$  so we can use the convergence criterion of Bohman–Korokvin.  $\square$

### 3. REPRESENTATIONS OF THE OPERATOR $L_N^{Q,S}$

**THEOREM 3.** *The operator  $L_n^{Q,S}$  can be represented in the form*

$$(15) \quad (L_n^{Q,S} f)(x) = \sum_{k=0}^n \frac{k!}{n^k} \binom{n}{k} \left[0, \frac{1}{n}, \dots, \frac{k}{n}; f\right] d_{k,n}(x),$$

where

$$d_{k,n}(x) = \frac{1}{s_n(1)} (\theta^k E^{1-x} s_{n-k})(x).$$

Moreover  $L_n^{Q,S}(P_m) \subseteq P_m, \forall m \in \mathbb{N}$ .

*Proof.* From the Newton interpolation formula we have

$$f\left(\frac{k}{n}\right) = \sum_{j=0}^k \frac{j!}{n^j} \binom{k}{j} \left[0, \frac{1}{n}, \dots, \frac{j}{n}; f\right]$$

If we denote  $w_{k,n}(x, y) = \binom{n}{k} p_k(x) s_{n-k}(y)$  then

$$\sum_{k=0}^n w_{k,n}(x, y) f\left(\frac{k}{n}\right) = \sum_{k=0}^n \frac{k!}{n^k} \left[0, \frac{1}{n}, \dots, \frac{k}{n}; f\right] \sum_{j=k}^n \binom{j}{k} w_{j,n}(x, y).$$

But

$$\begin{aligned} \sum_{j=k}^n \binom{j}{k} w_{j,n}(x, y) &= \binom{n}{k} \sum_{j=k}^n \binom{n-k}{j-k} p_j(x) s_{n-j}(y) \\ &= \binom{n}{k} \sum_{j=0}^{n-k} \binom{n-k}{j} p_{j+k}(x) s_{n-k-j}(y) \\ &= \binom{n}{k} \theta^k \sum_{j=0}^{n-k} \binom{n-k}{j} p_j(x) s_{n-k-j}(y) \\ &= \binom{n}{k} \theta^k E^y s_{n-k}(x) \end{aligned}$$

Since  $(L_n^{Q,S} f)(x) = \frac{1}{s_n(1)} \sum_{k=0}^n w_{k,n}(x, 1-x) f\left(\frac{k}{n}\right)$  we obtain (15).

In order to show that  $L_n^{Q,S}(P_m) \subseteq P_m$  we shall prove that  $\deg(d_{k,n}(x)) = k$ .

We remind that if  $(p_n)$  is a basic sequence for  $Q = q(D)$  and  $h(t)$  is the compositional inverse of  $q(t)$ , then the generating function for  $(p_n)$  is

$$(16) \quad \sum_{k=0}^{\infty} p_k(x) \frac{t^k}{k!} = e^{xh(t)}$$

and if  $s_n = S^{-1}p_n$ , with  $S = s(D)$  then

$$(17) \quad \sum_{k=0}^{\infty} s_k(x) \frac{t^k}{k!} = \frac{1}{s(h(t))} e^{xh(t)}$$

If we differentiate the relation (16)  $m$  times with respect to  $t$ , we get

$$(18) \quad \sum_{k=0}^{\infty} p_{k+m}(x) \frac{t^k}{k!} = \frac{d^m}{dx^m} (e^{xh(t)}) = (xh_1(t) + x^2h_1(t) + \dots + x^m h_m(t)) e^{xh(t)},$$

where every  $h_i(t)$  is a product of derivatives of  $h(t)$ .

Let us denote  $r(k, m, x) = \sum_{j=0}^k \binom{k}{j} p_{j+m}(x) s_{k-j}(1-x)$ . Expanding  $\frac{1}{s(h(t))} h_i(t) e^{h(t)} = \sum_{k \geq 0} \alpha_{ik} \frac{t^k}{k!}$ , from (17) and (18) we get  $r(k, m, x) = x\alpha_{1k} + x^2\alpha_{2k} + \dots + x^m\alpha_{mk}$ .



Because  $d_{k,n}(x) = r(n-k, k, x)/s_n(1)$  we obtain  $\deg(d_{k,n}(x)) = k$ .

Suppose that  $p \in P_m$ . Then  $[0, \frac{1}{n}, \dots, \frac{k}{n}; p] = 0$  for  $k \geq m+1$ , and using (15) we get  $L_n^{Q,S}(P_m) \subseteq P_m$ .  $\square$

REMARK 1. For  $Q = D$  (it means that  $s_n$  is an Appell set  $A_n$ ) we have  $\theta = X$ , therefore in this case  $d_{k,n} = \frac{A_{n-k}(1)}{A_n(1)}x^k$ . This representation for operators constructed with Appell sequences was given by C. Manole in [5].

THEOREM 4. *Suppose that all the assumptions of Theorem 2 are true, then there exists  $\theta_{1n}, \theta_{2n}, \theta_{3n} \in [0, 1]$  such that  $\forall x \in [0, 1]$  and  $\forall f \in C[0, 1]$  we have*

$$(L_n^{Q,S}f)(x) = f(a_nx) + \alpha(x, n)[\theta_{1n}, \theta_{2n}, \theta_{3n}; f]$$

where  $\alpha(x, n) = x^2(b_n - a_n^2) + x(a_n - b_n - c_n)$ .

*Proof.* First we shall prove that  $f(a_nx) \leq (L_n^{Q,S}f)(x)$  for every convex function  $f$ .

Let us denote  $c_k = \frac{1}{s_n(1)} \binom{n}{k} p_k(x) s_{n-k}(1-x)$  and  $x_k = \frac{k}{n}$ ,  $k = 0, 1, \dots, n$ .

We have  $c_k \geq 0$ ,  $\sum_{k=0}^n c_k = 1$  and  $x_k > 0$ ,  $\forall k \in \mathbb{N}$ . If  $f$  is a convex function then  $f(\sum_{k=0}^n c_k x_k) \leq \sum_{k=0}^n c_k f(x_k)$ ; but

$$\sum_{k=0}^n c_k x_k = (L_n^{Q,S}e_1)(x) = a_nx \quad \text{and} \quad \sum_{k=0}^n c_k f(x_k) = (L_n^{Q,S}f)(x)$$

therefore we get  $f(a_nx) \leq (L_n^{Q,S}f)(x)$ .

If we consider the formula

$$f(a_nx) = (L_n^{Q,S}f)(x) + (R_n f)(x)$$

we have  $(R_n f) \leq 0$  for every convex function  $f$ .

Since  $(R_n e_i)(x) = 0$  for  $i = 0, 1$ , the degree of exactness of the previous formula is one and then there exist  $\theta_{1n}, \theta_{2n}, \theta_{3n} \in [0, 1]$  such that the remainder can be represented in the following form

$$(R_n f)(x) = (R_n e_2)(x)[\theta_{1n}, \theta_{2n}, \theta_{3n}; f]$$

where  $(R_n e_2)(x) = x^2(a_n^2 - b_n) + x(b_n + c_n - a_n)$ , so we obtain the conclusion.  $\square$

## 4. EXAMPLES

1. If  $S = I$  then  $s_n = p_n$  and in this case the operator defined by (2) becomes the binomial operator (1) introduced by Tiberiu Popoviciu in [9].

1.1. For  $Q = D$  the basic sequence is  $p_n(x) = x^n$  and  $L_n^{D,I}$  is the Bernstein operator  $B_n$ .

1.2. If  $Q$  is Abel operator  $A = E^{-\beta}D$  we have  $p_n(x) = x(x + n\beta)^{n-1}$  and  $L_n^{A,I}$  is the second operator introduced by Cheney and Sharma in [1],

$$\begin{aligned} (L_n^{A,I}f)(x) &= \\ &= \frac{1}{(1+n\beta)^{n-1}} \sum_{k=0}^n \binom{n}{k} x(x+k\beta)^{k-1} (1-x)(1-x+(n-k)\beta)^{n-k-1} f\left(\frac{k}{n}\right) \end{aligned}$$

1.3. For Laguerre delta operator  $L = \frac{D}{D+I}$  the basic sequence is  $l_n(x) = \sum_{k=0}^n \binom{n}{k} \frac{(n-1)!}{(k-1)!} x^k$  and the corresponding binomial operator has been considered by T. Popoviciu.

1.4. The delta operator  $Q = \frac{1}{\alpha} \nabla_{\alpha} = \frac{1}{\alpha} (I - E^{-\alpha})$  has the basic sequence  $p_n(x) = x^{[n, -\alpha]} = x(x + \alpha) \dots (x + (n-1)\alpha)$  and in this case we obtain the operator

$$(S_n f)(x) = \frac{1}{1^{[n, -\alpha]}} \sum_{k=0}^n \binom{n}{k} x^{[k, -\alpha]} (1-x)^{[n-k, -\alpha]} f\left(\frac{k}{n}\right)$$

which has been introduced and investigated in detail by D. D. Stancu in [14], [16] and other papers.

1.5. The exponential polynomials  $t_n(x) = \sum_{k=0}^n S(n, k) x^k = e^{-x} \sum_{k=0}^{\infty} \frac{k^n x^k}{k!}$ , where  $S(n, k)$  denote the Stirling numbers of the second kind, are basic polynomials for the delta operator  $T = \ln(I+D)$ . The approximation operator construct by means of the exponential polynomials

$$(L_n^T f)(x) = \frac{1}{t_n(1)} \sum_{k=0}^n \binom{n}{k} t_k(x) t_{n-k}(1-x) f\left(\frac{k}{n}\right)$$

was studied by C. Manole in [5].

1.6. If we take the delta operator  $Q = G = \frac{1}{\alpha}E^{-\beta}\nabla_{\alpha} = \frac{1}{\alpha}(E^{-\beta} - E^{-\alpha-\beta})$  its basic sequence is  $p_n(x) = x(x + \alpha + n\beta)^{[n-1, -\alpha]}$  and the operator

$$(L_n^G f)(x) = \frac{1}{(1 + n\beta)^{[n, -\alpha]}} \cdot \sum_{k=0}^n \binom{n}{k} x(x + \alpha + k\beta)^{[k-1, -\alpha]} (1-x)(1-x + (n-k)\beta)^{[n-k, -\alpha]} f\left(\frac{k}{n}\right)$$

was investigated by D. D. Stancu, G. Moldovan. In [18] D. D. Stancu and M. R. Occorsio have studied this operator with the nodes  $\frac{k+\gamma}{n+\delta}$ ,  $0 \leq \gamma \leq \delta$ .

2. If  $Q = D$  and  $S$  is an invertible shift invariant operator then  $p_n(x) = x^n$  and  $s_n = A_n = S^{-1}x^n$  is an Appell set. The operator of the form

$$(L_n^{D,S} f)(x) = \frac{1}{A_n(1)} \sum_{k=0}^n \binom{n}{k} x^k A_{n-k} (1-x) f\left(\frac{k}{n}\right)$$

was introduced and investigated by C. Manole in [5].

2.1. If  $S = (I+D)^{-1}$  the corresponding Appell set is  $A_n(x) = x^n + nx^{n-1}$  and then

$$(L_n^{D,(I+D)^{-1}} f)(x) = \frac{1}{n+1} \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} (n-k+1-x) f\left(\frac{k}{n}\right).$$

3. If we take  $Q = A = E^{-\beta}D$  and  $S = E^{\beta}Q' = I - \beta D$  then  $p_n(x) = x(x + n\beta)^{n-1}$  is the basic sequence for  $Q$  and  $s_n(x) = (x + n\beta)^n$  a Sheffer set for  $Q$  we obtain the first operator introduced by Cheney and Sharma in [1]:

$$(L_n^{A,I-\beta D} f)(x) = \frac{1}{(1+n\beta)^n} \sum_{k=0}^n \binom{n}{k} x(x + k\beta)^{k-1} (1-x + (n-k)\beta)^{n-k} f\left(\frac{k}{n}\right)$$

4. For  $Q = \frac{1}{\alpha}E^{-\beta}\nabla_{\alpha} = \frac{1}{\alpha}(E^{-\beta} - E^{-\alpha-\beta})$  and  $S = E^{\alpha+\beta}Q' = \frac{1}{\alpha}((\alpha + \beta)I - \beta E^{\alpha})$  we have  $p_n(x) = x(x + \alpha + n\beta)^{[n-1, -\alpha]}$  and  $s_n(x) = (x + n\beta)^{[n, -\alpha]}$  therefore the operator  $L_n^{Q,S}$  in this case is

$$(L_n^{[\alpha, \beta]} f)(x) = \frac{1}{(1+n\beta)^{[n, -\alpha]}} \sum_{k=0}^n \binom{n}{k} x(x + \alpha + k\beta)^{[k-1, -\alpha]} (1-x + (n-k)\beta)^{[n-k, -\alpha]} f\left(\frac{k}{n}\right)$$

If we replace  $x$  with  $s(x)$  we obtain a operator which has been studied by G. Moldovan in [6]. He has found the value of this operator for the monomials  $e_i$  for  $i = 1, 2$  using some generalized identities of Vandermonde type.

We want to find the sequences  $a_n$ ,  $b_n$ ,  $c_n$ , which appears in  $(L_n^{[\alpha, \beta]} e_i)(x)$ , using relations (9).

The Pincherle derivative of  $Q$  is

$$Q' = -\frac{\beta}{\alpha} E^{-\beta} + \left(1 + \frac{\beta}{\alpha}\right) E^{-\alpha-\beta} = E^{-\alpha-\beta} \left(I - \frac{\beta}{\alpha} \Delta_\alpha\right)$$

so

$$(Q')^{-1} = E^{\alpha+\beta} \sum_{k \geq 0} \beta^k \left(\frac{\Delta_\alpha}{\alpha}\right)^k.$$

Since  $s_{n-1}(x) = (x + (n-1)\beta)^{[n-1, -\alpha]} = E^{(n-2)\alpha + (n-1)\beta} x^{[n-1, \alpha]}$  and  $x^{[n, \alpha]} = x(x-1)\dots(x-(n-1)\alpha)$  is the basic sequence for the delta operator  $\frac{\Delta_\alpha}{\alpha}$ , we have  $\left(\frac{\Delta_\alpha}{\alpha}\right)^k s_{n-1}(x) = E^{(n-2)\alpha + (n-1)\beta} \left(\frac{\Delta_\alpha}{\alpha}\right)^k x^{[n-1, \alpha]} = (n-1)^{[k]} E^{(n-2)\alpha + (n-1)\beta} \cdot x^{[n-1-k, \alpha]}$ . Because  $a_n = \frac{[(Q')^{-1} s_{n-1}](1)}{s_n(1)}$  we get

$$a_n = \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{k! \beta^k}{(1+n\beta)^{[k+1, -\alpha]}}.$$

Since  $b_n = \frac{n-1}{n} \frac{[(Q')^{-2} s_{n-2}](1)}{s_n(1)}$  and

$$(Q')^{-2} = E^{2\alpha+2\beta} \left(I - \frac{\beta}{\alpha} \Delta_\alpha\right)^{-2} = E^{2\alpha+2\beta} \sum_{k \geq 0} (k+1) \beta^k \left(\frac{\Delta_\alpha}{\alpha}\right)^k$$

we obtain

$$b_n = \frac{n-1}{n} \sum_{k=0}^{n-2} \binom{n-2}{k} \frac{(k+1)! \beta^k}{(1+n\beta)^{[k+2, -\alpha]}}.$$

The Pincherle derivative of  $S^{-1}$  may be written in the form

$$\begin{aligned} (S^{-1})' &= -(\alpha + \beta) E^{-\alpha-\beta} (Q')^{-1} - E^{-\alpha-\beta} (Q')^{-2} Q'' \\ &= -E^{-\alpha-\beta} (Q')^{-2} ((\alpha + \beta) Q' + Q''). \end{aligned}$$

Because  $Q'' = \frac{1}{\alpha} (\beta^2 E^{-\beta} - (\alpha + \beta)^2 E^{-\alpha-\beta})$  and  $(\alpha + \beta) Q' + Q'' = -\beta E^{-\beta}$  this implies

$$\begin{aligned} (Q')^{-2} (S^{-1})' S &= \beta (Q')^{-3} E^{-\beta} \\ &= \beta E^{3\alpha+2\beta} \left(I - \frac{\beta}{\alpha} \Delta_\alpha\right)^{-3} \\ &= E^{3\alpha+2\beta} \sum_{k \geq 0} \frac{(k+1)(k+2)}{2} \beta^{k+1} \left(\frac{1}{\alpha} \Delta_\alpha\right)^k. \end{aligned}$$

From (9) and the previous relation we get

$$c_n = \frac{n-1}{2n} \left( (1+n\alpha+n\beta) \sum_{k=0}^{n-3} \binom{n-2}{k} \frac{(k+2)! \beta^{k+1}}{(1+n\beta)^{[k+3, -\alpha]}} + \frac{n! \beta^{n-1}}{(1+n\beta)^{[n, -\alpha]}} \right).$$

### 5. EVALUATION OF THE ORDERS OF APPROXIMATION

Now we establish some estimates of the order of approximation of a function  $f \in C[0, 1]$  by means of the operator  $L_n^{Q,S}$ , defined by (2).

According to a result of O. Shisha and B. Mond [13], we can write

$$|f(x) - (L_n^{Q,S} f)(x)| \leq \left[ 1 + \frac{1}{\delta^2} L_n^{Q,S} \left( (t-x)^2; x \right) \right] \omega_1(f; \delta), \quad \delta \in \mathbb{R}^+$$

Using the relations (8) we have

$$L_n^{Q,S} \left( (t-x)^2; x \right) = x^2(b_n - 2a_n + 1) + x(a_n - b_n - c_n)$$

so we get

$$|f(x) - (L_n^{Q,S} f)(x)| \leq \left[ 1 + \frac{1}{\delta^2} [x^2(b_n - 2a_n + 1) + x(a_n - b_n - c_n)] \right] \omega_1(f; \delta).$$

One observes that if  $b_n - 2a_n + 1 < 0$  then  $x^2(b_n - 2a_n + 1) + x(a_n - b_n - c_n) \leq \frac{(a_n - b_n - c_n)^2}{4(2a_n - b_n - 1)}$ ,  $\forall x \in [0, 1]$ .

By choosing  $\delta = \frac{1}{\sqrt{n}}$  we can state

**THEOREM 5.** *If  $f \in C[0, 1]$  and  $\exists k \in \mathbb{N}$  such as  $b_n - 2a_n + 1 < 0$ ,  $\forall n \geq k$ , then we can give the following estimation of the order of approximation, by means of the first modulus of continuity*

$$\|f - L_n^{Q,S} f\| \leq \left( 1 + \frac{n}{4} \frac{(a_n - b_n - c_n)^2}{(2a_n - b_n - 1)} \right) \omega_1 \left( f; \frac{1}{\sqrt{n}} \right), \quad n \geq k,$$

where  $a_n, b_n, c_n$  are defined by (9).

In the case of binomial operators of positive type defined by (1), since  $S' = I' = O$  we have

$$(19) \quad a_n = 1, \quad c_n = 0, \quad b_n = \frac{n-1}{n} \frac{[(Q')^{-2} p_{n-2}](1)}{p_n(1)}.$$

Then  $b_n - 2a_n + 1 = b_n - 1 < 0$ ,  $\forall n \in \mathbb{N}$  therefore the previous inequality reduces to

$$\|f - L_n f\| \leq \left( \frac{5}{4} + \frac{n}{4} d_n \right) \omega_1 \left( f; \frac{1}{\sqrt{n}} \right),$$

where

$$(20) \quad d_n = \frac{n-1}{n} - b_n = \frac{n-1}{n} \left( 1 - \frac{[(Q')^{-2} p_{n-2}](1)}{p_n(1)} \right)$$

We mention that this inequality was established by D. D. Stancu in [18].

In order to find an evaluation of the order of approximation using both moduli of smoothness  $\omega_1$  and  $\omega_2$  we can use a result of H. H. Gonska and R. K. Kovacheva included in the following

LEMMA 3. [2]. *If  $I = [a, b]$  is a compact interval of the real axis and  $I_1 = [a_1, b_1]$  is a subinterval of it, and if we assume that  $L : C(I) \rightarrow C(I_1)$  is a positive operator, such that  $Le_0 = e_0$  and  $0 \leq \delta \leq \frac{1}{2}(b-a)$ , then we have*

$$|f(x) - L(f(t); x)| \leq \frac{2}{\delta} |L(t-x; x)| \omega_1(f; \delta) + \frac{3}{2} \left[ 1 + \frac{1}{\delta} |L(t-x; x)| + \frac{1}{2\delta^2} L((t-x)^2; x) \right] \omega_2(f; \delta).$$

Using the relations (8) we obtain the inequality

$$|f(x) - (L_n^{Q,S} f)(x)| \leq \frac{2}{\delta} |(a_n - 1)x| \omega_1(f; \delta) + \frac{3}{2} \left[ 1 + \frac{1}{\delta} |(a_n - 1)x| + \frac{1}{2\delta^2} [x^2(b_n - 2a_n + 1) + x(a_n - b_n - c_n)] \right] \omega_2(f; \delta).$$

If  $b_n - 2a_n + 1 < 0$  the previous inequality implies

$$\|f - L_n^{Q,S} f\| \leq \frac{2}{\delta} (1 - a_n) \omega_1(f; \delta) + \frac{3}{2} \left[ 1 + \frac{1}{\delta} (1 - a_n) + \frac{1}{8\delta^2} \frac{(a_n - b_n - c_n)^2}{(2a_n - b_n - 1)} \right] \omega_2(f; \delta).$$

By choosing  $\delta = \frac{1}{\sqrt{n}}$  we get

$$\|f - L_n^{Q,S} f\| \leq 2\sqrt{n}(1 - a_n) \omega_1(f; \frac{1}{\sqrt{n}}) + \frac{3}{2} \left[ 1 + \sqrt{n}(1 - a_n) + \frac{n}{8} \frac{(a_n - b_n - c_n)^2}{(2a_n - b_n - 1)} \right] \omega_2(f; \frac{1}{\sqrt{n}}).$$

If we consider the binomial operator introduced by Tiberiu Popoviciu, using (19) and the previous relation, we arrive at an inequality which has found by D. D. Stancu (see [18])

$$\|f - L_n f\| \leq \frac{3}{16} (9 + nd_n) \omega_2(f; \frac{1}{\sqrt{n}}),$$

where  $d_n$  is defined by (20).

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