REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Tome 33, N° 1, 2004, pp. 51–60

ON COMPOUND OPERATORS DEPENDING ON s PARAMETERS*

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Abstract. In this note we introduce a compound operator depending on s parameters using binomial sequences. We compute the values of this operator on the test functions, we give a convergence theorem and a representation of the remainder in the corresponding approximation formula. We also mention some special cases of this operator.

MSC 2000. 41A36, 41A80, 05A40.

Keywords. Compound linear and positive approximation operators, polynomial sequences of binomial type, integral representation of remainder.

1. INTRODUCTION

In this note we introduce a compound operator using polynomial sequences of binomial type. We begin by defining these sequences and their link with delta operators.

Definition 1. *A sequence of polynomials* $(p_m(x))_{m\geq 0}$ *is called a sequence of binomial type if* $\deg p_m = m$, $\forall m \in \mathbb{N}$ *and it satisfies the relations*

$$p_m(x+y) = \sum_{k=0}^{m} \binom{m}{k} p_k(x)\, p_{m-k}(y)$$

for every real numbers x *and* y *and every positive integer* m.

In the following we will consider linear operators defined on the algebra of polynomials.

A linear operator T is a *shift invariant operator* if $E^a T = T E^a$, for every a, where E^a is the shift operator defined by $E^a p(x) = p(x+a)$.

A linear operator Q is called a *delta operator* if Q is shift invariant and $Qx = \text{const.} \neq 0$. Some examples of delta operators are: the derivative D, the forward and backward difference operators $\nabla_\alpha = E^\alpha - I$ and $\Delta_\alpha = I - E^{-\alpha}$, the Touchard operator $T = \ln(I+D) = D - \frac{1}{2}D^2 + \frac{1}{3}D^3 - \frac{1}{4}D^4 + \ldots$ and the Laguerre operator $L = \frac{D}{I+D} = D - D^2 + D^3 - D^4 + \ldots$.

Definition 2. *We say that a sequence of polynomials* $(p_m(x))_{m\geq 0}$ *is the basic sequence for the delta operator* Q *if:*

i) $p_0(x) = 1$,

*This work has been supported by the Romanian Academy under grant GAR 13/2004.

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ii) $p_m(0) = 0$, $\forall m \geq 1$,
iii) $Qp_m = mp_{m-1}$, $\forall m \geq 1$.

It is known that every delta operator has a unique basic sequence (see [19]).

Proposition 3. [19]. *If $(p_m(x))_{m\geq 0}$ is a basic sequence for a delta operator then it is a sequence of binomial type; if $(p_m(x))_{m\geq 0}$ is a sequence of binomial type then there exists a delta operator for which $(p_m(x))_{m\geq 0}$ is the basic sequence.*

Definition 4. *If T is a linear operator, then its Pincherle derivative T' is defined by $T' = TX - XT$, where the linear operator X is defined by $(Xp)(x) = xp(x)$ for all x and all polynomials p.*

We mention that Umbral calculus allows a unified and simple study of sequences of binomial type. More details about these sequences can be found in [8], [9], [10], [16], [18], [19].

The use of binomial sequences in order to construct approximation operators was proposed by T. Popoviciu in [17], where he introduced a class of approximation operators of the form

$$(1)\qquad \left(T_m^Q f\right)(x) = \frac{1}{p_m(1)} \sum_{k=0}^{m} \binom{m}{k} p_k(x)\, p_{m-k}(1-x)\, f\left(\frac{k}{m}\right).$$

These operators and their generalizations were studied in [2], [5]–[7], [11]–[15], [20], [29], [31]–[36].

2. COMPOUND OPERATORS DEPENDING ON S PARAMETERS

Let Q be a delta operator with the basic sequence $(p_k(x))_{k\geq 0}$, which satisfy $p_m(1) \neq 0$ and $p_m'(0) \geq 0$ for every positive integer m. For every function $f \in C[0,1]$ we introduce the compound operator

$$(2)\qquad \left(L^Q_{m,r_1,\ldots,r_s} f\right)(x) = \sum_{k=0}^{m-r_1\ldots-r_s} p^Q_{m-r_1\ldots-r_s,k}(x) \sum_{j=0}^{s} \frac{p_j(x)p_{s-j}(1-x)}{p_s(1)} F^{r_1,\ldots,r_s}_{m,k,j}(f),$$

where $p^Q_{n,k}(x) = \binom{n}{k} \frac{p_k(x)p_{n-k}(1-x)}{p_n(1)}$,

$$\begin{aligned} F^{r_1,\ldots,r_s}_{m,k,j}(f) &= f\left(\tfrac{k+r_1+r_2+\ldots+r_j}{m}\right) + f\left(\tfrac{k+r_2+r_3+\ldots+r_{j+1}}{m}\right) \\ &\quad + f\left(\tfrac{k+r_1+r_3+\ldots+r_{j+1}}{m}\right) + \ldots + f\left(\tfrac{k+r_{s-j+1}+\ldots+r_{s-1}+r_s}{m}\right) \end{aligned}$$

and $r_1, \ldots, r_s$ are s non-negative integer parameters, independent of the number m and such that $0 \leq r_1 \leq \ldots \leq r_s$ and $r_1 + \ldots + r_s < m$.

If $p_m'(0) \geq 0$ for every positive integer m then $p_m(x) \geq 0$, $\forall x \in [0,1]$ so this condition assures the positivity of the operator $\left(L^Q_{m,r_1,\ldots,r_s} f\right)(x)$.

From Definition 2 ii), it results that

$$p_{n,k}^{Q}(0)=\begin{cases}1, & \text{if } k=0\\ 0, & \text{if } k\neq 0\end{cases} \quad \text{and} \quad p_{n,k}^{Q}(1)=\begin{cases}1, & \text{if } k=n\\ 0, & \text{if } k\neq n\end{cases}$$

so the expression $\left(L_{m,r_1,\ldots,r_s}^{Q} f\right)(0)$ contains only a nonzero term, for $k=j=0$, while the only nonzero term in $\left(L_{m,r_1,\ldots,r_s}^{Q} f\right)(1)$ appears for $k=m-r_1-\ldots-r_s$ and $j=s$. Consequently, it is easy to see that this approximation operator interpolates the function f at both ends of the interval $[0,1]$, that is

$$\left(L_{m,r_1,\ldots,r_s}^{Q} f\right)(0)=f(0), \quad \left(L_{m,r_1,\ldots,r_s}^{Q} f\right)(1)=f(1).$$

We remark that for $s=0$ the operator $L_{m,r_1,\ldots,r_s}^{Q}$ reduces to the binomial operator of T. Popoviciu T_m^Q.

In the following we will compute the values of this operator for the test functions $e_n(x)=x^n$, for $n=0,1,2$. For this we need Manole's results contained in the next

Proposition 5. [13], [14]. *The values of the binomial operators of T. Popoviciu type on the test functions are:*

$$T_m^Q e_i = e_i, \quad \textit{for } i=0,1 \textit{ and} \tag{3}$$
$$\left(T_m^Q e_2\right)(x)=x^2+x(1-x)\,d_m^Q,$$

where

$$d_m^Q = 1-\tfrac{m-1}{m}\tfrac{(Q')^{-2}p_{m-2}(1)}{p_m(1)} \tag{4}$$

and Q' is the Pincherle derivative of delta operator Q.

Lemma 6. *If $L_{m,r_1,\ldots,r_s}^{Q}$ is the approximation operator defined by* (2) *then we have the following relations*

$$\begin{aligned} L_{m,r_1,\ldots,r_s}^{Q} e_i &= e_i, \ \textit{for } i=0,1 \textit{ and} \\ \left(L_{m,r_1,\ldots,r_s}^{Q} e_2\right)(x) &= x^2+x(1-x)\,A_{m,r_1,\ldots,r_s}^{Q}, \end{aligned}$$

where

$$A_{m,r_1,\ldots,r_s}^{Q} = \tfrac{1}{m^2}\cdot \tag{5}$$

$$\cdot\left[(m-r_1-\ldots-r_s)^2\, d_{m-r_1\ldots-r_s}^{Q} + r_1^2+\ldots+r_s^2+\tfrac{2}{s-1}(sd_s-1)\sum_{\substack{u,v=1\\ u\neq v}}^{s} r_u r_v\right].$$

Proof. First we make the convention that $\binom{s}{j}=0$, if $s<0$ or $j<0$.

Because $(p_m(x))$ is a basic sequence for the delta operator Q according to Proposition 3 it is a polynomial sequence of binomial type and using Definition 2 we have $\sum_{k=0}^{m} p_{m,k}^{Q}(x) = 1$ so we can write

$$\left(L_{m,r_1,\dots,r_s}^{Q} e_0\right)(x) = \sum_{k=0}^{m-r_1\dots-r_s} p_{m-r_1\dots-r_s,k}^{Q}(x) \sum_{j=0}^{s} p_{s,j}^{Q}(x) = 1 = e_0(x).$$

In the case of the next test function e_1 we have

$$\begin{aligned}
&\left(L_{m,r_1,\dots,r_s}^{Q} e_1\right)(x) = \\
&= \tfrac{1}{m} \sum_{k=0}^{m-r_1\dots-r_s} p_{m-r_1\dots-r_s,k}^{Q}(x) \sum_{j=0}^{s} \tfrac{p_j(x)p_{s-j}(1-x)}{p_s(1)} \left[\tbinom{s}{j}k + (r_1+\dots+r_s)\tbinom{s-1}{j-1}\right] \\
&= \tfrac{1}{m}\Big[(m-r_1-\dots-r_s)\left(T_{m-r_1\dots-r_s}^{Q} e_1\right)(x)\left(T_s^{Q} e_0\right)(x) + \\
&\qquad + (r_1+\dots+r_s)\left(T_{m-r_1\dots-r_s}^{Q} e_0\right)(x)\left(T_s^{Q} e_1\right)(x)\Big] \\
&= \tfrac{(m-r_1-\dots-r_s)x+(r_1+\dots+r_s)x}{m} \\
&= x.
\end{aligned}$$

Finally, for e_2 we can write

$$\begin{aligned}
&\left(L_{m,r_1,\dots,r_s}^{Q} e_2\right)(x) = \\
&= \tfrac{1}{m^2} \sum_{k=0}^{m-r_1\dots-r_s} p_{m-r_1\dots-r_s,k}^{Q}(x) \sum_{j=0}^{s} \tfrac{p_j(x)p_{s-j}(1-x)}{p_s(1)} \cdot \\
&\quad \cdot \left[\tbinom{s}{j}k^2 + \left(r_1^2+\dots+r_s^2\right)\tbinom{s-1}{j-1} + 2k(r_1+\dots+r_s)\tbinom{s-1}{j-1} + 2\tbinom{s-2}{j-2}\sum_{\substack{u,v=1\\u\neq v}}^{s} r_u r_v\right].
\end{aligned}$$

Using the relation $\binom{s-2}{j-2} = \frac{j(j-1)}{s(s-1)}\binom{s}{j} = \frac{s}{s-1}\binom{s}{j}\frac{j^2}{s^2} - \frac{1}{s-1}\binom{s}{j}\frac{j}{s}$ in the last expression we obtain

$$\begin{aligned}
&\left(L_{m,r_1,..,r_s}^{Q} e_2\right)(x) = \\
&= \tfrac{1}{m^2}\Big\{(m-r_1-\dots-r_s)^2\left(T_{m-r_1\dots-r_s}^{Q} e_2\right)(x)\left(T_s^{Q} e_0\right)(x) \\
&\qquad + \left(r_1^2+\dots+r_s^2\right)\left(T_{m-r_1\dots-r_s}^{Q} e_0\right)(x)\left(T_s^{Q} e_1\right)(x) \\
&\qquad + 2(m-r_1-\dots-r_s)(r_1+\dots+r_s)\left(T_{m-r_1\dots-r_s}^{Q} e_1\right)(x)\left(T_s^{Q} e_1\right)(x) \\
&\qquad + \tfrac{2}{s-1}\sum_{\substack{u,v=1\\u\neq v}}^{n} r_u r_v\left[s\left(T_s^{Q} e_2\right)(x) - \left(T_s^{Q} e_1\right)(x)\right]\Big\}.
\end{aligned}$$

If we use the relations (3) we can rewrite the last expression as

$$\left(L^Q_{m,r_1,..,r_s}e_2\right)(x) =$$
$$= \frac{1}{m^2}\Big\{ (m-r_1-...-r_s)^2\left[x^2+x\,(1-x)\,d^Q_{m-r_1...-r_s}\right] + \left(r_1^2+...+r_s^2\right)x$$
$$+\,2\,(m-r_1-...-r_s)\,(r_1+...+r_s)\,x^2$$
$$+\,\frac{2}{s-1}\sum_{\substack{u,v=1\\ u\neq v}}^{s} r_u r_v\left[s\left(x^2+x\,(1-x)\,d^Q_s\right)-x\right]\Big\}.$$

After some simple computations we obtain the expression from the conclusion of lemma. □

Using the well known theorem of Bohman-Korovkin and the expressions obtained in the above lemma for $L^Q_{m,r_1,...,r_s}e_i$, $i=0,1,2$, we can state the following convergence theorem

THEOREM 7. *Let $f\in C\,[0,1]$. Let Q be a delta operator having the basic sequence $p_m(x)$ with $p_m(1)\neq 0$ and $p'_m(0)\geq 0$ for every positive integer m. If $d^Q_m\to 0$, as $m\to\infty$, then the operator $L^Q_{m,r_1,..,r_s}f$ converges to the function f, uniformly on $[0,1]$.*

3. SPECIAL CASES

1. If $r_1=...=r_s=r$ the compound operator defined by (2) reduces to the operator which we have studied in [7]

$$\left(S^Q_{m,r,s}f\right)(x)=\sum_{k=0}^{m-sr}p^Q_{m-sr,k}(x)\sum_{j=0}^{s}p^Q_{s,j}(x)\,f\left(\tfrac{k+jr}{m}\right) \tag{6}$$

and $\left(S^Q_{m,r,s}e_2\right)(x)=x^2+\frac{x(1-x)}{m^2}\left[\,(m-rs)^2\,d^Q_{m-rs}+s^2r^2d^Q_s\right]$.

2. For $Q=D$ one obtains the operator introduced and studied by D.D. Stancu in [27]

$$\left(L^D_{m,r_1,...,r_s}f\right)(x)=$$
$$=\sum_{k=0}^{m-r_1...-r_s}\binom{m-r_1-...-r_s}{k}x^k\,(1-x)^{m-r_1-...-r_s-k}\sum_{j=0}^{s}x^j\,(1-x)^{s-j}\,F^{r_1,...,r_s}_{m,k,j}(f)\,.$$

Here we have $d^D_m=\frac{1}{m}$, so it results

$$\left(L^D_{m,r_1,...,r_s}e_2\right)(x)=x^2+\frac{x(1-x)}{m}\Big[1+\frac{1}{m}\sum_{j=1}^{s}r_j\,(r_j-1)\Big].$$

2.1. For $s=1$ the above operator reduces to the following operator

$$\left(L^D_{m,r}f\right)(x)=\sum_{k=0}^{m-r}\binom{m-r}{k}x^k\,(1-x)^{m-r-k}\left[\,(1-x)\,f\left(\tfrac{k}{m}\right)+xf\left(\tfrac{k+r}{m}\right)\right] \tag{7}$$

which was constructed by D.D. Stancu in [26] using a probabilistic approach.
The above mentioned author have found the eigenvalues for this operator

$$\lambda_0(m,r)=\lambda_1(m,r)=1$$
$$\lambda_i(m,r)=\left(1-\tfrac{r}{m}\right)\left(1-\tfrac{r+1}{m}\right)\ldots\left(1-\tfrac{r+j-2}{m}\right)\left(1+\tfrac{(j-1)(r-1)}{m}\right),$$
$$\text{for } 2\le j\le m-r+i.$$

We mention also that D. D. Stancu in [25] obtained a quadrature formula using this operator

$$\int_0^1 f(x)\,\mathrm{d}x=$$
$$=\tfrac{1}{(m-r+1)(m-r+2)}\Bigg[\sum_{k=0}^{r-1}(m-r-k+1)\,f\left(\tfrac{k}{m}\right)+(m-2r+2)\sum_{k=r}^{m-r}f\left(\tfrac{k}{m}\right)$$
$$+\sum_{k=m-r+1}^{m}(k-r+1)\,f\left(\tfrac{k}{m}\right)\Bigg]+\rho_{m,r}(f),$$

where, if we suppose that $f\in C^2[0,1]$, the remainder has the following simple form

$$\rho_{m,r}(f)=-\tfrac{1}{2m}\left[1+\tfrac{r(r-1)}{m}\right]f''(\xi),\ 0<\xi<1.$$

For $f\in C^{(s+1)}[0,1]$ O. Agratini gave an estimate for the difference

$$\left|\left(L_{m,r}^{D,\alpha}f\right)^{(s)}-f^{(s)}(x)\right|,\ \ s\le m-r$$

in which appears the first modulus of continuity ω_1 for the derivatives of order s and $s+1$ of f (see [1]).
The bivariate analogue of the operator defined by (7), having as domain the square $[0,1]\times[0,1]$

$$\left(L_{m,n,r,s}^{D}f\right)(x)=\sum_{k=0}^{m-r}\sum_{j=0}^{n-s}\tbinom{m-r}{k}\tbinom{n-s}{j}x^k(1-x)^{m-r-k}y^j(1-y)^{n-s-j}$$
$$\cdot\Big[(1-x)(1-y)f\left(\tfrac{k}{m},\tfrac{j}{n}\right)+x(1-y)f\left(\tfrac{k+r}{m},\tfrac{j}{n}\right)+$$
$$+(1-x)yf\left(\tfrac{k}{m},\tfrac{j+s}{n}\right)+xyf\left(\tfrac{k+r}{m},\tfrac{j+s}{n}\right)\Big]$$

was studied by D.D. Stancu in [28]. In the same paper a cubature formula (using this operator) was constructed.

2.2. The operator obtained for $s=1$ and $r=2$, $L_{m,2}^{D}$ has been studied by H. Brass [4].

3. If we consider the delta operator $Q = \frac{\nabla_\alpha}{\alpha} = \frac{I-E^{-\alpha}}{\alpha}$ with the basic sequence $p_m(x) = x^{[m,-\alpha]} = x(x+\alpha)\ldots(x+(m-1)\alpha)$ then we obtain the following operator

$$\left(L_{m,r_1,\ldots,r_s}^{\frac{\nabla_\alpha}{\alpha}} f\right)(x) = \sum_{k=0}^{m-r_1\ldots-r_s} \binom{m-r_1-\ldots-r_s}{k} x^{[k,-\alpha]}(1-x)^{[m-r_1-\ldots-r_s-k,-\alpha]} \cdot \sum_{j=0}^{s} x^{[j,-\alpha]}(1-x)^{[s-j,-\alpha]} F_{m,k,j}^{r_1,\ldots,r_s}(f). \tag{8}$$

Taking into account that $d_m^{\frac{\nabla_\alpha}{\alpha}} = \frac{1+\alpha m}{(1+\alpha)m}$, we obtain the following expression for this operator on e_2,

$$\left(L_{m,r_1,\ldots,r_s}^{\frac{\nabla_\alpha}{\alpha}} e_2\right)(x) = x^2 + \frac{x(1-x)}{m^2}\left[(m-r_1-\ldots-r_s)^2 \frac{1+\alpha(m-r_1-\ldots-r_s)}{1+\alpha} + r_1^2+\ldots+r_s^2 + \frac{2\alpha}{1+\alpha}\sum_{\substack{u,v=1\\u\neq v}}^{s} r_u r_v\right].$$

3.1. If $r_1 = \ldots = r_s = r$ in the relation (8) then this operator reduces to the operator studied by D.D. Stancu and J.W. Drane in [33] and the expression (5) reduces to $A_{m,r,s}^{\frac{\nabla_\alpha}{\alpha}} = \frac{sr^2(1+\alpha s)+(m-sr)(1+\alpha(m-sr))}{m^2(1+\alpha)}$.

4. For Q arbitrary and $s=1$ the operator defined by (2) reduces to the operator

$$\left(L_{m,r}^{Q} f\right)(x) = \sum_{k=0}^{m-r} p_{m-r,k}^{Q}\left[(1-x)f\left(\tfrac{k}{m}\right) + xf\left(\tfrac{k+r}{m}\right)\right]$$

and

$$\left(L_{m,r}^{Q} e_2\right)(x) = x^2 + \frac{x(1-x)}{m^2}\left[r^2 + (m-r)^2 d_{m-r}^{Q}\right].$$

4. AN INTEGRAL REPRESENTATION FOR THE REMAINDER

We consider the following approximation formula

$$f(x) = \left(L_{m,r_1,\ldots,r_s}^{Q} f\right)(x) + \left(R_{m,r_1,\ldots,r_s}^{Q} f\right)(x). \tag{9}$$

From Lemma 6 it results that the degree of exactness of this formula is 1.

If $f \in C^2[0,1]$, using the Peano's theorem, the remainder in the above formula can be represented under the form

$$\left(R_{m,r_1,\ldots,r_s}^{Q} f\right)(x) = \int_0^1 G_{m,r_1,\ldots,r_s}^{Q}(t;x)\, f''(t)\,\mathrm{d}t,$$

where $G_{m,r_1,\ldots,r_s}^{Q}(t;x) = \left(R_{m,r_1,\ldots,r_s}^{Q}\varphi_x\right)(t)$ and $\varphi_x(t) = (x-t)_+ = \frac{x-t+|x-t|}{2}$.

Because for a fixed value of x, $G^{Q}_{m,r_1,\dots,r_s}(t;x)$ is negative we can apply the mean value theorem and we obtain that it exists $\xi\in[0,1]$ such that

$$\left(R^{Q}_{m,r_1,\dots,r_s}f\right)(x)=f''(\xi)\int_0^1 G^{Q}_{m,r_1,\dots,r_s}(t;x)\,\mathrm{d}t.$$

Because the Peano kernel $G^{Q}_{m,r_1,\dots,r_s}(t;x)$ is independent of the function f we can take $f(x)=x^2$ in the previous relation and we obtain

$$\begin{aligned}\int_0^1 G^{Q}_{m,r_1,\dots,r_s}(t;x)\,\mathrm{d}t &= \tfrac{1}{2}\left(R^{Q}_{m,r_1,\dots,r_s}e_2\right)(x)\\ &= -\tfrac{1}{2}x(1-x)\,A^{Q}_{m,r_1,\dots,r_s},\end{aligned}$$

where $A^{Q}_{m,r_1,\dots,r_s}$ is defined by (5).

So, for every function $f\in C^2[0,1]$, we obtain a Cauchy-type form for the remainder in the approximation formula (9)

$$\left(R^{Q}_{m,r_1,\dots,r_s}f\right)(x)=\tfrac{x(x-1)}{2}A^{Q}_{m,r_1,\dots,r_s}f''(\xi),$$

where $\xi\in[0,1]$.

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Received by the editors: January 14, 2004.