

DIFFERENTIABILITY WITH RESPECT TO A PARAMETER FOR A LOTKA-VOLTERRA SYSTEM WITH DELAYS, VIA STEP METHOD*

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Abstract. In this paper, using the step method, we establish the differentiability with respect to parameter for a Lotka-Volterra system with two delays.

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1. INTRODUCTION

The purpose of this paper is to study the λ -dependence of the solution of the Lotka-Volterra problem:

$$(1) \quad x'_i(t) = f_i(t, x_1(t), x_2(t), x_1(t - \tau_1), x_2(t - \tau_2); \lambda), \quad t \in [t_0, b], \lambda \in J$$

$$(2) \quad \begin{aligned} x_1(t) &= \varphi(t), \quad t \in [t_0 - \tau_1, t_0], \\ x_2(t) &= \psi(t), \quad t \in [t_0 - \tau_2, t_0]. \end{aligned}$$

There have been many studies on this subject. Differentiability with initial data for the functional differential equations was first established by Hale in [1], but differentiability with respect to delays for delay differential equations was proved by Hale and Ladeira in [2] and by A. Tămășan in [12]. The paper of Hokkanen and Moroșanu [4] gives a proof for delay differential equations case using the step method. The Picard operators technique proposed by I.A. Rus, [8], [9], [10], was used by V. Mureșanu [5] to prove continuity with respect to λ , M. Șerban [11] and us [7] using the theorem of fibre contraction.

In this paper we use the following theorem for the simple case of ordinary differential system.

THEOREM 1. [3] *Consider the initial value problem*

$$\begin{aligned} x'(t) &= g(t, x(t); \lambda), \quad t \in [a, b], \\ x(t_0) &= x_0, \end{aligned}$$

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where $g \in C^1([a, b] \times \mathbb{R} \times J, \mathbb{R})$, $\left\| \frac{\partial g(t, u, \lambda)}{\partial u} \right\|_{\mathbb{R}} \leq M_1$. Then the unique solution $x \in C^1([a, b] \times J)$.

2. MAIN RESULT

We consider the following Lotka-Volterra system with parameter:

$$(3) \quad x'_i(t) = f_i(t, x_1(t), x_2(t), x_1(t - \tau_1), x_2(t - \tau_2); \lambda), \quad t \in [t_0, b], \lambda \in J,$$

$$(4) \quad \begin{aligned} x_1(t) &= \varphi(t), \quad t \in [t_0 - \tau_1, t_0], \\ x_2(t) &= \psi(t), \quad t \in [t_0 - \tau_2, t_0]. \end{aligned}$$

We suppose that:

- (H₁) $\tau_1 \leq \tau_2$; $t_0 < b$; $J \subset \mathbb{R}$, a compact interval;
- (H₂) $f_i \in C^1([t_0, b] \times \mathbb{R}^4 \times J)$, $i = 1, 2$;
- (H₃) $\exists M_1 > 0$ such as $\left\| \frac{\partial f_i}{\partial u_j}(t, u_1, u_2, u_3, u_4; \lambda) \right\|_{\mathbb{R}} \leq M_1$,
for all $t \in [t_0, b]$, $u_j \in \mathbb{R}$, $j = \overline{1, 4}$, $\lambda \in J$, $i = 1, 2$;
- (H₄) $\varphi \in C([t_0 - \tau_1, t_0])$, $\psi \in C([t_0 - \tau_2, t_0])$.

In the above conditions, from Theorem 1 in [6] we have that the problem (3)–(4) has a unique solution, $(x_1(t; \lambda), x_2(t; \lambda))$.

We prove that

$$x_1(t; \cdot) \in C^1(J), \text{ for all } t \in [t_0 - \tau_1, b],$$

$$x_2(t; \cdot) \in C^1(J), \text{ for all } t \in [t_0 - \tau_2, b],$$

applying Theorem 1 and using the step method.

We consider the system

$$(5) \quad \begin{aligned} x'_i(t, \lambda) &= f_i(t, x_1(t, \lambda), x_2(t, \lambda), x_1(t - \tau_1, \lambda), x_2(t - \tau_2, \lambda); \lambda), \\ & \quad t \in [t_0, b], \lambda \in J \end{aligned}$$

with the initial conditions

$$(6) \quad \begin{aligned} x_1(t; \lambda) &= \varphi(t; \lambda), \\ x_2(t; \lambda) &= \psi(t; \lambda), \end{aligned}$$

where $x_1 \in C([t_0 - \tau_1, b] \times J) \cap C^1[t_0, b]$, $x_2 \in C([t_0 - \tau_2, b] \times J) \cap C^1[t_0, b]$.

THEOREM 2. *In conditions (H₁), (H₂), (H₄), the solution $(x_1(t; \lambda), x_2(t; \lambda))$ of the problem (5)–(6) is continuously with respect to λ .*

THEOREM 3. *In conditions (H₁)–(H₄), the solution $(x_1(t; \lambda), x_2(t; \lambda))$ of the problem (5)–(6) is differentiable with respect to λ .*

Proof. We prove that

$$x_1(t; \cdot) \in C^1(J), \text{ for all } t \in [t_0 - \tau_1, b],$$

$$x_2(t; \cdot) \in C^1(J), \text{ for all } t \in [t_0 - \tau_2, b],$$

applying Theorem 1 and using the step method.

For $t \in [t_0, t_0 + \tau_1]$, from Theorem 1 we have that $x_1(t; \cdot) \in C^1(J)$, $x_2(t; \cdot) \in C^1(J)$.

For $t \in [t_0 + \tau_1, t_0 + 2\tau_1]$, from Theorem 1 we have that $x_1(t; \cdot) \in C^1(J)$, $x_2(t; \cdot) \in C^1(J)$. From (5) and (H_2) it follows that $x'_1(t_0 + \tau_1 - 0) = x'_1(t_0 + \tau_1 + 0)$ and $x'_2(t_0 + \tau_1 - 0) = x'_2(t_0 + \tau_1 + 0)$. Then $(x_1(t; \lambda), x_2(t; \lambda))$ is differentiable at $t_0 + \tau_1$.

For $t \in [t_0 + n\tau_1, t_0 + \tau_2]$, from Theorem 1 we have that $x_1(t; \cdot) \in C^1(J)$, $x_2(t; \cdot) \in C^1(J)$. From (5) and (H_2) it follows that $x'_1(t_0 + n\tau_1 - 0) = x'_1(t_0 + n\tau_1 + 0)$ and $x'_2(t_0 + n\tau_1 - 0) = x'_2(t_0 + n\tau_1 + 0)$. Then $(x_1(t; \lambda), x_2(t; \lambda))$ is differentiable at $t_0 + n\tau_1$.

So $(x_1(t; \lambda), x_2(t; \lambda))$ is C^1 at the knots. \square

We present below a simple example to illustrate the procedures of applying our results.

EXAMPLE 1. Consider the Lotka-Volterra type predator-prey system with two delays with parameter:

$$(7) \quad \begin{aligned} x'_1(t, \lambda) &= \lambda x_1(t; \lambda) [2 - x_1(t; \lambda) - x_2(t; \lambda)] \\ x'_2(t, \lambda) &= \lambda x_2(t; \lambda) [2 - x_1(t - 1; \lambda) - x_2(t - 2; \lambda)] \end{aligned}, \quad t \in [0, 2]$$

with the initial conditions

$$(8) \quad \begin{aligned} x_1(t; \lambda) &= t + 1, & t \in [t_0 - 1, t_0], \\ x_2(t; \lambda) &= t + 2, & t \in [t_0 - 2, t_0]. \end{aligned} \quad \square$$

Equation (7) is of the form (5). Therefore, the assumptions (H_1) – (H_4) are satisfied by system (7)–(8). Moreover, Theorem 2 holds. Applying Theorem 3, we draw the following conclusions.

For $t \in [t_0, t_0 + 1]$, from Theorem 1 we have that $x_1(t; \cdot) \in C^1(J)$, $x_2(t; \cdot) \in C^1(J)$.

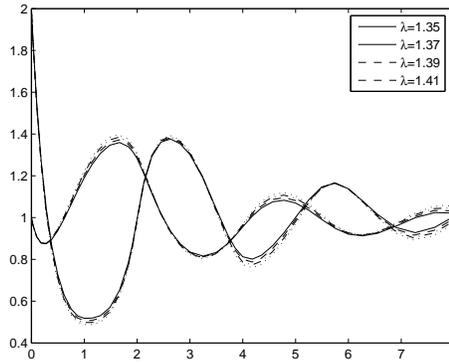


Fig. 1. Comparison between Lotka-Volterra systems with different parameters.

For $t \in [t_0 + 1, t_0 + 2]$, from Theorem 1 we have that $x_1(t; \cdot) \in C^1(J)$, $x_2(t; \cdot) \in C^1(J)$. From (5) and (H_2) it follows that $x'_1(t_0 + 1 - 0) = x'_1(t_0 + 1 + 0)$ and $x'_2(t_0 + 1 - 0) = x'_2(t_0 + 1 + 0)$. Then $(x_1(t; \lambda), x_2(t; \lambda))$ is differentiable

at $t_0 + 1$. So $(x_1(t; \lambda), x_2(t; \lambda))$ is C^1 at the knots. Therefore Theorem 3 is applicable.

Here we give a portrait of the trajectory of (7)–(8) drawn using MATLAB facilities. The results from numerical computation are plotted for $\lambda = 1.35, 1.37, 1.39, 1.41$ in Fig. 1.

REFERENCES

- [1] HALE, J., *Functional differential equations*, Springer-Verlag, New York, Heidelberg, Berlin, 1977.
- [2] HALE, J. and LADEIRA, L., *Differentiability with respect to delays*, J. Differential Equations, **92**, pp. 14–26, 1991.
- [3] HARTMAN, P., *Ordinary differential equations*, John Wiley & Sons, Inc., New York, London, Sydney, 1964.
- [4] HOKKANEN, V. M. and MOROȘANU, GH., *Differentiability with respect to delay*, Differential Integral Equations, **11**, no. 4, pp. 589–603, 1998.
- [5] MUREȘAN, V., *Differential equations with linear modified argument*, Ph. D. Thesis, “Babeș-Bolyai” University, 1997.
- [6] OTROCOL, D., *Data dependence for the solution of a Lotka-Volterra system with two delays*, Mathematica (to appear).
- [7] OTROCOL, D., *Smooth dependence on parameters for some Lotka-Volterra system with delays* (to appear).
- [8] RUS, I. A., *Generalized contractions*, Seminar of Fixed Point Theory, “Babeș-Bolyai” University, pp. 1–130, 1983.
- [9] RUS, I. A., *Functional-differential equation of mixed type, via weakly Picard operators*, Seminar of Fixed Point Theory, Cluj-Napoca, **3**, pp. 335–346, 2002.
- [10] RUS, I. A., *Generalized Contractions and Applications*, Cluj University Press, 2001.
- [11] ȘERBAN, M. A., *Fiber φ -contractions*, Studia Univ. “Babeș-Bolyai”, Mathematica, **44**, no. 3, pp. 99–108, 1999.
- [12] TĂMĂȘAN, AL., *Note on Differentiability with respect to delay*, P.U.M.A., **9**, nos. 1–2, pp. 215–220, 1998.

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