# FUNCTIONAL-DIFFERENTIAL EQUATIONS WITH "MAXIMA", VIA WEAKLY PICARD OPERATORS THEORY

DIANA OTROCOL* AND IOAN A. RUS†

ABSTRACT. The purpose of this paper is to present a differential equation with "maxima". Existence, uniqueness, inequalities of Čaplygin type and data dependence (monotony, continuity) results for the solution of the Cauchy problem of this equation are obtained using weakly Picard operators theory.
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## 1. INTRODUCTION

Differential equations with maximum arise naturally when solving practical problems, in particular, in those which appear in the study of systems with automatic regulation. The existence and uniqueness of solutions of equation with maxima is considered in [1], [3]-[5], [8], [13]. The asymptotic stability of the solution of this equations and other problems concerning equations with maxima are investigated in [2], [6], [7], [9], [14].

The purpose of this paper is to study the following Cauchy problem

$$(1.1) \qquad x'(t) = f(t, x(t), \max_{a \leq \xi \leq t} x(\xi)), \ t \in [a, b]$$

$$(1.2) \qquad x(a) = \alpha$$

where

(C$_1$) $\alpha \in \mathbb{R}$ and $f \in C([a, b] \times \mathbb{R}^2)$ are given;

(C$_2$) there exists $L_f > 0$ such that

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \leq L_f \max(|u_1 - v_1|, |u_2 - v_2|)$$

for all $t \in [a, b]$ and $u_i, v_i \in \mathbb{R}, i = 1, 2$.

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*"Tiberiu Popoviciu" Institute of Numerical Analysis, P.O.Box. 68-1, 400110, Cluj-Napoca, Romania, e-mail: `dotrocol@ictp.acad.ro`.

†Department of Applied Mathematics, "Babeş-Bolyai" University, Cluj-Napoca, Romania, e-mail: `iarus@math.ubbcluj.ro`.

In the condition $(C_1)$ the problem (1.1)–(1.2), $x \in C^1[a,b]$ is equivalent with the fixed point equation

$$(1.3) \qquad x(t) = \alpha + \int_a^t f(s, x(s), \max_{a \leq \xi \leq s} x(\xi)) ds, \ t \in [a,b],$$

$x \in C[a,b]$, and the equation (1.1) is equivalent with

$$(1.4) \qquad x(t) = x(a) + \int_a^t f(s, x(s), \max_{a \leq \xi \leq s} x(\xi)) ds, \ t \in [a,b],$$

$x \in C[a,b]$.

Let us consider the following operators:

$$B_f, E_f : C[a,b] \to C[a,b]$$

defined by

$$B_f(x)(t) := \text{second part of (1.3)}$$

and

$$E_f(x)(t) := \text{second part of (1.4)}.$$

For $\alpha \in \mathbb{R}$, we consider $X_\alpha := \{x \in C[a,b] | \ x(a) = \alpha\}$.

We remark that

$$C[a,b] = \bigcup_{\alpha \in \mathbb{R}} X_\alpha$$

is a partition of $C[a,b]$.

We have

**Lemma 1.1.** *If $(C_1)$ is satisfied, then*
   (a) $B_f(C[a,b]) \subset X_\alpha$ *and* $E_f(X_\alpha) \subset X_\alpha, \ \forall \alpha \in \mathbb{R}$;
   (b) $B_f|_{X_\alpha} = E_f|_{X_\alpha}, \ \forall \alpha \in \mathbb{R}$.

In this paper we shall prove that if $(C_1)$ and $(C_2)$ are satisfied and if $L_f$ is small enough, then the operator $E_f$ is weakly Picard operator ([11]), in $(C[a,b], \|\cdot\|)$ where $\|x\| := \max_{a \leq t \leq b} x(t)$, and we study the equation (1.1) in the terms of the weakly Picard operator theory.

## 2. Weakly Picard operators

Let $(X, d)$ be a metric space and $A : X \to X$ an operator. We shall use the following notations:

$F_A := \{x \in X \mid A(x) = x\}$ - the fixed point set of $A$;

$I(A) := \{Y \subset X \mid A(Y) \subset Y, Y \neq \emptyset\}$ - the family of the nonempty invariant subsets of $A$;

By $H$ we denote the Pompeiu-Housdorff functional, $H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$ defined by:

$$H(Y, Z) := \max\{\sup_{y \in Y}\inf_{z \in Z} d(y, z), \sup_{z \in Z}\inf_{y \in Y} d(y, z)\}$$

**Definition 2.1.** *([11], [12]) Let $(X, d)$ be a metric space. An operator $A : X \to X$ is a Picard operator (PO) if there exists $x^* \in X$ such that:*
   (i) $F_A = \{x^*\}$;

(ii) *the sequence* $(A^n(x_0))_{n\in\mathbb{N}}$ *converges to* $x^*$ *for all* $x_0 \in X$.

**Definition 2.2.** *([11], [12]) Let* $(X, d)$ *be a metric space. An opera-tor* $A : X \to X$ *is a weakly Picard operator (WPO) if the sequence* $(A^n(x))_{n\in\mathbb{N}}$ *converges for all* $x \in X$, *and its limit (which may depend on x) is a fixed point of A.*

**Definition 2.3.** *([11], [12]) If A is weakly Picard operator then we consider the operator* $A^\infty$ *defined by*

$$A^\infty : X \to X, \ A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

**Remark 2.4.** *([11], [12]) It is clear that* $A^\infty(X) = F_A$.

**Definition 2.5.** *([11], [12]) Let A be a weakly Picard operator and* $c > 0$. *The operator A is c -weakly Picard operator if*

$$d(x, A^\infty(x)) \le cd(x, A(x)), \ \forall x \in X.$$

For some examples of WPOs see [10], [11], [12].

## 3. CAUCHY PROBLEM

Relative to problem (1.1)–(1.2) we have

**Theorem 3.1.** *We suppose that:*
  (a) *the condition* $(C_1)$ *and* $(C_2)$ *are satisfied;*
  $(C_3)$ $L_f(b - a) < 1$.
  *Then the problem (1.1)–(1.2) has, in* $C[a, b]$, *a unique solution and this solution is the uniform limit of the successive approximations.*

*Proof.* The problem (1.1)–(1.2) is equivalent with the fixed point equation

$$B_f(x) = x, \ x \in C[a, b].$$

On the other hand we have that

$$|B_f(x)(t) - B_f(y)(t)| \le L_f \int_a^t \max\left(|x(s) - y(s)|, \left|\max_{a\le\xi\le s} x(\xi) - \max_{a\le\xi\le s} y(\xi)\right|\right) ds.$$

But

$$\max_{a\le s\le b}\left|\max_{a\le\xi\le s} x(\xi) - \max_{a\le\xi\le s} y(\xi)\right| \le \max_{a\le s\le b} |x(s) - y(s)|.$$

So,

$$\|B_f(x) - B_f(y)\| \le L_f(b - a) \|x - y\|, \ \forall x, y \in C[a, b],$$

i.e., $B_f$ is a contraction w.r.t. Chebyshev norm on $C[a, b]$. The proof follows from the contraction principle.    □

**Remark 3.2.** *In the conditions of Theorem 3.1, the operator* $B_f$ *is PO. But*

$$B_f|_{X_\alpha} = E_f|_{X_\alpha}, \ \forall\alpha \in \mathbb{R}.$$

*Hence, the operator* $E_f$ *is WPO and* $F_{E_f} \cap X_\alpha = \{x_\alpha^*\}, \forall\alpha \in \mathbb{R}$, *where* $x_\alpha^*$ *is the unique solution of the problem (1.1)–(1.2).*

## 4. Inequalities of Čaplygin type

We have

**Theorem 4.1.** *We suppose that:*
  (a) *the conditions $(C_1)$, $(C_2)$ and $(C_3)$ are satisfied;*
  (b) *$f(x, \cdot, \cdot) : \mathbb{R}^2 \to \mathbb{R}^2$ is increasing, i.e., $u_1 \le v_1, u_2 \le v_2 \Rightarrow f(x, u_1, u_2) \le f(x, v_1, v_2).$*

*Let $x$ be a solution of equation (1.1) and $y$ a solution of the inequality*

$$y'(t) \le f(t, y(t), \max_{a \le \xi \le t} y(\xi)), \ t \in [a, b].$$

*Then*

$$y(a) \le x(a) \ \text{implies that } y \le x.$$

*Proof.* In the terms of the operator $E_f$, we have

$$x = E_f(x) \text{ and } y \le E_f(y),$$

and $x(a) \le y(a)$.

From the conditions $(C_1)$, $(C_2)$ and $(C_3)$ we have that the operator $E_f$ is WPO. From the condition (b), $E_f^\infty$ is increasing ([11]). If $\alpha \in \mathbb{R}$, then we denote by $\widetilde{\alpha}$ the following function

$$\widetilde{\alpha} : [a, b] \to \mathbb{R}, \ \widetilde{\alpha}(t) = \alpha, \ \forall t \in [a, b].$$

We have

$$y \le E_f(y) \le \ldots \le E_f^\infty(y) = E_f^\infty(\widetilde{y}(a)) \le E_f^\infty(\widetilde{x}(a)) = x.$$

$\square$

## 5. Data dependence: monotony

In this section we need the following abstract result.

**Lemma 5.1.** *(Comparison principle, [12]) Let $(X, d, \le)$ an ordered metric space and $A, B, C : X \to X$ be such that:*
  (a) *$A \le B \le C$;*
  (b) *the operator $A, B, C$, are WPOs;*
  (c) *the operator $B$ is increasing.*
  *Then $x \le y \le z$ imply that $A^\infty(x) \le B^\infty(y) \le C^\infty(z)$.*

From this abstract result we have

**Theorem 5.2.** *Let $f_i \in C([a, b] \times \mathbb{R}^2), i = 1, 2$, be as in Theorem 3.1. We suppose that:*
  (i) *$f_1 \le f_2 \le f_3$;*
  (ii) *$f_2(t, \cdot, \cdot) : \mathbb{R}^2 \to \mathbb{R}^2$ is increasing;*
  *Let $x_i \in C^1[a, b]$ be a solution of the equation*

$$x_i'(t) = f_i(t, x(t), \max_{a \le \xi \le t} x(\xi)), \ t \in [a, b] \text{ and } i = 1, 2, 3.$$

*If $x_1(a) \le x_2(a) \le x_3(a)$, then $x_1 \le x_2 \le x_3$.*

*Proof.* From Theorem 3.1 we have that the operator $E_{f_i}, i = 1, 2, 3$, are WPOs. From the condition (ii) the operator $E_{f_2}$ is monotone increasing. From the condition (i) it follows that

$$E_{f_1} \leq E_{f_2} \leq E_{f_3}.$$

Let $\widetilde{x}_i(a) \in C[a, b]$ be defined by $\widetilde{x}_i(a)(t) = x_i(a), \ \forall t \in [a, b]$. It is clear that

$$\widetilde{x}_1(a)(t) \leq \widetilde{x}_2(a)(t) \leq \widetilde{x}_3(a)(t), \ \forall t \in [a, b].$$

From Lemma 5.1 we have that

$$E_{f_1}^\infty(\widetilde{x}_1(a)) \leq E_{f_2}^\infty(\widetilde{x}_2(a)) \leq E_{f_3}^\infty(\widetilde{x}_3(a)).$$

But $x_i = E_{fi}^\infty(\widetilde{x}_i(a))$, and $x_1 \leq x_2 \leq x_3$. $\qquad\square$

## 6. DATA DEPENDENCE: CONTINUITY

Consider the Cauchy problem (1.1)–(1.2) and suppose the conditions of the Theorem 3.1 are satisfied. Denote by $x^*(\cdot; \alpha, f)$ the solution of this problem.

We need the following well known result (see [11]).

**Theorem 6.1.** *Let $(X, d)$ be a complete metric space and $A, B : X \to X$ two operators. We suppose that*

   (i) *the operator $A$ is a $\alpha$ -contraction;*
  (ii) *$F_B \neq \emptyset$;*
 (iii) *there exists $\eta > 0$ such that*

$$d(A(x), B(x)) \leq \eta, \ \forall x \in X.$$

*Then, if $F_A = \{x_A^*\}$ and $x_B^* \in F_B$, we have*

$$d(x_A^*, x_B^*) \leq \frac{\eta}{1 - \alpha}.$$

We can state the following result:

**Theorem 6.2.** *Let $\alpha_i, f_i, i = 1, 2$ be as in the Theorem 3.1. Furthermore, we suppose that there exists $\eta_i > 0, i = 1, 2$ such that*

  (i) *$|\alpha_1(t) - \alpha_2(t)| \leq \eta_1, \forall t \in [a, b]$;*
 (ii) *$|f_1(t, u_1, u_2) - f_2(t, u_1, u_2)| \leq \eta_2, \forall t \in [a, b], u_i \in \mathbb{R}, i = 1, 2$.*

*Then*

$$\|x_1^*(t; \alpha_1, f_1) - x_2^*(t; \alpha_2, f_2)\| \leq \frac{\eta_1 + (b - a)\eta_2}{1 - L_f(b - a)},$$

*where $x_i^*(t; \alpha_i, f_i), i = 1, 2$ are the solution of the problem (1.1)–(1.2) with respect to $\alpha_i, f_i$ and $L_f = \max(L_{f_1}, L_{f_2})$.*

*Proof.* Consider the operators $B_{\alpha_i, f_i}, i = 1, 2$. From Theorem 3.1 these operators are contractions.

Additionally

$$\|B_{\alpha_1, f_1}(x) - B_{\alpha_2, f_2}(x)\| \leq \eta_1 + (b - a)\eta_2,$$

$\forall x \in C[a, b]$.

Now the proof follows from the Theorem 6.1, with $A := B_{\alpha_1, f_1}$, $B = B_{\alpha_2, f_2}$, $\eta = \eta_1 + (b-a)\eta_2$ and $\alpha := L_f(b - t_0)$, where $L_f = \max(L_{f_1}, L_{f_2})$.
□

In what follow we shall use the $c$-WPOs techniques to give some data dependence results.

**Theorem 6.3.** *([10], [12]) Let $(X, d)$ be a metric space and $A_i : X \to X$, $i = 1, 2$. Suppose that*

(i) *the operator $A_i$ is $c_i$-weakly Picard operator, $i = 1, 2$;*
(ii) *there exists $\eta > 0$ such that*

$$d(A_1(x), A_2(x)) \leq \eta, \ \forall x \in X.$$

*Then $H(F_{A_1}, F_{A_2}) \leq \eta \max(c_1, c_2)$.*

We have

**Theorem 6.4.** *Let $f_1$ and $f_2$ be as in the Theorem 3.1. Let $S_{E_{f_1}}, S_{E_{f_2}}$ be the solution set of system (1.1) corresponding to $f_1$ and $f_2$. Suppose that there exists $\eta > 0$, such that*

$$(6.1) \qquad\qquad |f_1(t, u_1, u_2) - f_2(t, u_1, u_2)| \leq \eta$$

*for all $t \in [a, b], u_i \in \mathbb{R}, i = 1, 2$.*
*Then*

$$H_{\|\cdot\|_C}(S_{E_{f_1}}, S_{E_{f_2}}) \leq \frac{(b-a)\eta}{1 - L_f(b-a)},$$

*where $L_f = \max(L_{f_1}, L_{f_2})$ and $H_{\|\cdot\|_C}$ denotes the Pompeiu-Housdorff functional with respect to $\|\cdot\|_C$ on $C[a, b]$.*

*Proof.* In the condition of Theorem 3.1, the operators $E_{f_1}$ and $E_{f_2}$ are $c_i$-weakly Picard operators, $i = 1, 2$.
Let

$$X_\alpha := \{x \in C[a, b] |\ x(a) = \alpha\}.$$

It is clear that $E_{f_1}|_{X_\alpha} = B_{f_1}$, $E_{f_2}|_{X_\alpha} = B_{f_2}$. Therefore,

$$\left|E_{f_1}^2(x) - E_{f_1}(x)\right| \leq L_{f_1}(b-a)\left|E_{f_1}(x) - x\right|,$$

$$\left|E_{f_2}^2(x) - E_{f_2}(x)\right| \leq L_{f_2}(b-a)\left|E_{f_2}(x) - x\right|,$$

for all $x \in C[a, b]$.
Now, choosing

$$\alpha_1 = L_{f_1}(b-a) \text{ and } \alpha_2 = L_{f_2}(b-a),$$

we get that $E_{f_1}$ and $E_{f_2}$ are $c_i$-weakly Picard operators, $i = 1, 2$ with $c_1 = (1 - \alpha_1)^{-1}$ and $c_2 = (1 - \alpha_2)^{-1}$. From (6.1) we obtain that

$$\|E_{f_1}(x) - E_{f_2}(x)\|_C \leq (b-a)\eta,$$

$\forall x \in C[a, b]$. Applying Theorem 6.3 we have that

$$H_{\|\cdot\|_C}(S_{E_{f_1}}, S_{E_{f_2}}) \leq \frac{(b - a)\eta}{1 - L_f(b - a)},$$

where $L_f = \max(L_{f_1}, L_{f_2})$ and $H_{\|\cdot\|_C}$ is the Pompeiu-Housdorff functional with respect to $\|\cdot\|_C$ on
$C[a, b]$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

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