Analysis of a History-dependent Frictionless Contact Problem

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Abstract

We consider a mathematical model which describes the quasistatic contact between a viscoelastic body and an obstacle, the so-called foundation. The contact is frictionless and is modelled with a new and nonstandard condition which involves both normal compliance, unilateral constraint and memory effects. We derive a variational of the problem which is the form of a history history-dependent variational inequality for the displacement field. Then, using a recent result obtained in [26], we prove the unique weak solvability of the problem. Next, we study the continuous dependence of the weak solution with respect the data and prove a first convergence result. Finally, we prove that the weak solution of the problem converges to the weak solution of a contact problem with normal compliance and memory term, as the stiffness coefficient of the foundation converges to infinity.

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1 Introduction

Phenomena of contact between deformable bodies abound in industry and everyday life. Contact of braking pads with wheels, tires with roads, pistons with skirts are just few simple examples. Common industrial processes such as metal forming, metal extrusion, involve contact evolutions. Owing to their inherent complexity, contact phenomena lead to mathematical models expressed in terms of strongly nonlinear elliptic or evolutionary boundary value problems.

An early attempt to study frictional contact problems within the framework of variational inequalities was made in [3]. An excellent reference on analysis and numerical approximations of contact problems involving elastic materials with or without friction is [7]. The variational analysis of various contact problems, including existence and uniqueness results, can be found in the monographs [4, 6, 15, 22]. The state of the art in the field can also be found in the proceedings [10, 18, 29] and in the special issue [21], as well.

To construct a mathematical model which describes a specific contact process we need to precise the material's behavior and the contact conditions, among others. In this paper we assume that the material is viscoelastic and we describe its behavior with a constitutive law with long memory of the form

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,ds. \tag{1.1}$$

Here $\boldsymbol{u}$ denotes the displacement field, $\boldsymbol{\sigma}$ represents the stress, $\boldsymbol{\varepsilon}(\boldsymbol{u})$ is the linearized strain tensor and, finally, $\mathcal{A}$ and $\mathcal{B}$ are the elasticity operator and the relaxation tensor, respectively. Results and mechanical interpretations in the study of viscoelastic materials of the form (1.1) can be found in [3, 17, 28], for instance. The analysis of various contact problems which such kind of materials was provided in [19, 20, 24]. There, the unique solvability of the problems was proved by using existence and uniqueness results for evolutionary variational inequalities involving a Volterra-type integral term; fully discrete schemes for the numerical approximation of the models were considered and error estimates were derived; finally, the schemes were implemented on a computer code and numerical simulations were presented. The analysis of models of antiplane frictional contact problems with viscoelastic materials of the form (1.1), including existence, uniqueness and convergence results, was performed in [25].

We turn now to describe some representative contact conditions used in the literature and, to this end, we denote by u_ν and σ_ν the normal displacement and the normal stress on the contact surface, respectively.

The so-called normal compliance contact condition describes a deformable foundation. It assigns a reactive normal pressure that depends on the interpenetration of the asperities on the body's surface and those of the foundation. A general expression for this condition is

$$-\sigma_\nu = p(u_\nu) \tag{1.2}$$

where p is a nonnegative prescribed function which vanishes for negative argument. Indeed, when $u_\nu < 0$ there is no contact and the normal pressure vanishes. When there is contact then u_ν is positive and represents a measure of the interpenetration of the asperities. Then, condition (1.2) shows that the foundation exerts a pressure on the body, which depends on the penetration.

A commonly used example of the normal compliance function p is

$$p(r) = c_\nu r^+. \tag{1.3}$$

Here the constant $c_\nu > 0$ is the surface stiffness coefficient and $r^+ = \max\{r, 0\}$ denotes the positive part of r. A second example is provided by the truncated normal compliance function

$$p(r) = \begin{cases} c_\nu r^+ & \text{if } r \le \alpha, \\ c_\nu \alpha & \text{if } r > \alpha, \end{cases} \tag{1.4}$$

where α is a positive coefficient related to the wear and hardness of the surface. In this case the contact condition (1.2) means that when the penetration is too large, i.e., when it exceeds α, the obstacle offers no additional resistance to penetration.

The normal compliance contact condition was first introduced in [14] and since then used in many publications, see, e.g., [7, 8, 9, 11] and references therein. The term *normal compliance* was first introduced in [8, 9]. An idealization of the normal compliance, which is used often in engineering literature, and can also be found in mathematical publications, is the Signorini contact condition, in which the foundation is assumed to be perfectly rigid. It is obtained, formally, from the normal compliance condition (1.2), (1.3), in the limit when the surface stiffness coefficient becomes infinite, i.e., $c_\nu \to \infty$, and thus interpenetration is not allowed. This leads to the idea of regarding contact with a rigid support as a limiting case of contact with deformable support, whose resistance to compression increases. The Signorini contact condition can be stated in the following complementarity form:

$$u_\nu \le 0, \qquad \sigma_\nu \le 0, \qquad \sigma_\nu u_\nu = 0. \tag{1.5}$$

This condition was first introduced in [23] and then used in many papers, see, e.g., [22] for further details and references. Assume now that there is an initial gap $g > 0$ between the body and the foundation. Then the Signorini contact condition in a form with a gap function is given by

$$u_\nu \le g, \qquad \sigma_\nu \le 0, \qquad \sigma_\nu (u_\nu - g) = 0. \tag{1.6}$$

In various situations the reaction of the foundation at the moment t depends on the history of the penetration and, therefore, it cannot be determinate as a function of the current value $u_\nu(t)$. In this case one can assume that the normal stress satisfies a condition of the form

$$-\,\sigma_\nu(t) = \int_0^t b(t-s)\, u_\nu^+(s)\, ds, \tag{1.7}$$

in which b represents a given function, the so-called surface memory function. Contact conditions of the form (1.7) have a simple physical interpretation if there are no cycles of contact and separation during the time interval of interest. For instance, assume in what follows that b is a positive function. Moreover, assume that in the time interval

$[0;t]$ there is only penetration (i.e. $u_\nu(s) \geq 0$ for all $s \in [0,t]$). Then (1.7) shows that the reaction of the foundation at t is towards the body (since $\sigma_\nu(t) \leq 0$). Also, if in the time interval $[0;t]$ there is separation (i.e. $u_\nu(s) < 0$ for all $s \in [0,t]$) then there is no reaction at the moment t (since $\sigma_\nu(t) = 0$). Now, assume a situation in which u_ν is positive in time interval $[0,t_0]$ and negative on the time interval $[t_0,t]$. Then, following (1.7) we have

$$-\sigma_\nu(t) = \int_0^{t_0} b(t-s)\, u_\nu^+(s)\, ds,$$

since the integral on the remaining interval $[t_0,t]$ vanishes. Assume, in addition, that the support of the function b is included in the interval $[0,\delta]$ with $\delta > 0$. Two possibilities arise. First, if $t - t_0 > \delta$ it follows that $b(t-s) = 0$ for all $s \in [0,t_0]$ and (1.7) shows the normal stress $\sigma_\nu(t)$ vanishes. Second, if $t - t_0 \leq \delta$ (1.7) implies that $\sigma_\nu(t) \leq 0$ i.e. a residual pression exists at the moment t on the body's surface. We interpret this as a memory effect in which the foundation prevents the separation, moves towards the body and exerts a pression on a short interval of time of length δ. Various other mechanical interpretation of the condition (1.7) could be obtained if b is assumed to be a negative function or if this condition is associated to the normal compliance condition (1.2), as shown in Section 3 below. Note that conditions of the form (1.7) were considered in [13] in the study of a lumped model with contact and friction.

In the present paper we study a quasistatic frictionless contact problem for viscoelastic materials of the form (1.1). The novelty consists in the fact that the contact condition we use describes a deformable foundation which becomes rigid when the penetration reaches a critical bound and which developes memory effects. This contact condition includes as particular cases both the normal compliance condition (1.2), the Signorini condition (1.6) and the history-dependent condition (1.7). Considering such condition leads to a new and nonstandard mathematical model which, in a variational formulation, is given by a history-dependent variational inequality for the displacement field. We prove the unique weak solvability of the problem then we establish two convergence results.

The rest of the paper is structured as follows. In Section 2 we present the notation we shall use as well as some preliminary material. In Section 3 we describe the model of the contact process. In Section 4 we list the assumptions on the data and derive the variational formulation of the problem. Then we state and prove our main existence and uniqueness result, Theorem 4.1. In Section 5 we state and prove our first converge result, Theorem 5.1. It states the continuous dependence of the solution with respect to the data. Finally, in Section 6 we state and prove our second converge result, Theorem 6.1. It states that the weak solution of the problem with normal compliance, memory term and unilateral constraint can be approached by the weak solution of a problem with normal compliance and memory term, as the stiffness coefficient of the foundation converges to infinity.

2 Notation and preliminaries

Everywhere in this paper we use the notation $\mathbb{N}^*$ for the set of positive integers and $\mathbb{R}_+$ will represent the set of non negative real numbers, i.e. $\mathbb{R}_+ = [0, +\infty)$. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. The inner product and norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ are defined by

$$\begin{aligned}
&\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i\ , \qquad \|\boldsymbol{v}\| = (\boldsymbol{v} \cdot \boldsymbol{v})^{\frac{1}{2}} \qquad \forall\, \boldsymbol{u} = (u_i),\ \boldsymbol{v} = (v_i) \in \mathbb{R}^d,\\
&\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij} \tau_{ij}\ , \qquad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{\frac{1}{2}} \qquad \forall\, \boldsymbol{\sigma} = (\sigma_{ij}),\ \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d.
\end{aligned}$$

Let $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ be a bounded domain with Lipschitz continuous boundary Γ and let Γ_1, Γ_2, and Γ_3 be three measurable parts of Γ such that $\mathrm{meas}\,(\Gamma_1) > 0$. We use the notation $\boldsymbol{x} = (x_i)$ for a typical point in Ω and we denote by $\boldsymbol{\nu} = (\nu_i)$ the outward unit normal at Γ. Here and below the indices i, j, k, l run between 1 and d and, unless stated otherwise, the summation convention over repeated indices is used. An index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, e.g. $u_{i,j} = \partial u_i / \partial x_j$. We use standard notation for the Lebesgue and Sobolev spaces associated to Ω and Γ. In particular, we recall that the inner products on the Hilbert spaces $L^2(\Omega)^d$ and $L^2(\Gamma_3)^d$ are given by

$$(\boldsymbol{u}, \boldsymbol{v})_{L^2(\Omega)^d} = \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v}\, dx, \qquad (\boldsymbol{u}, \boldsymbol{v})_{L^2(\Gamma_2)^d} = \int_{\Gamma_2} \boldsymbol{u} \cdot \boldsymbol{v}\, da,$$

and the associated norms will be denoted by $\|\cdot\|_{L^2(\Omega)^d}$ and $\|\cdot\|_{L^2(\Gamma_2)^d}$, respectively. Moreover, we consider the spaces

$$\begin{aligned}
V &= \{\, \boldsymbol{v} \in H^1(\Omega)^d : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1 \,\},\\
Q &= \{\, \boldsymbol{\tau} = (\tau_{ij}) \in L^2(\Omega)^d : \tau_{ij} = \tau_{ji} \,\},\\
Q_1 &= \{\, \boldsymbol{\tau} = (\tau_{ij}) \in Q : \tau_{ij,j} \in L^2(\Omega) \,\}.
\end{aligned}$$

These are real Hilbert spaces endowed with the inner products

$$\begin{aligned}
(\boldsymbol{u}, \boldsymbol{v})_V &= \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx,\\
(\boldsymbol{\sigma}, \boldsymbol{\tau})_Q &= \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau}\, dx,\\
(\boldsymbol{\sigma}, \boldsymbol{\tau})_{Q_1} &= \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau}\, dx + \int_\Omega \mathrm{Div}\, \boldsymbol{\sigma} \cdot \mathrm{Div}\, \boldsymbol{\tau}\, dx,
\end{aligned}$$

and the associated norms $\|\cdot\|_V$, $\|\cdot\|_Q$ and $\|\cdot\|_{Q_1}$, respectively. Here $\boldsymbol{\varepsilon}$ and Div are the deformation and divergence operators given by

$$\begin{aligned}
&\boldsymbol{\varepsilon}(\boldsymbol{v}) = (\varepsilon_{ij}(\boldsymbol{v})), \quad \varepsilon_{ij}(\boldsymbol{v}) = \frac{1}{2}\,(v_{i,j} + v_{j,i}) \quad \forall\, \boldsymbol{v} \in H^1(\Omega)^d,\\
&\mathrm{Div}\, \boldsymbol{\tau} = (\tau_{ij,j}) \quad \forall\, \boldsymbol{\tau} \in Q_1.
\end{aligned}$$

Completeness of the space $(V, \|\cdot\|_V)$ follows from the assumption meas$(\Gamma_1) > 0$, which allows the use of Korn's inequality.

For an element $\boldsymbol{v} \in V$ we still write $\boldsymbol{v}$ for the trace of V and we denote by v_ν and $\boldsymbol{v}_\tau$ the normal and tangential components of $\boldsymbol{v}$ on Γ given by $v_\nu = \boldsymbol{v}\cdot\boldsymbol{\nu}$, $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu\boldsymbol{\nu}$. By the Sobolev trace theorem, there exists a positive constant c_0, depending on Ω, Γ_1, and Γ_3, such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^d} \le c_0\,\|\boldsymbol{v}\|_V \qquad \forall\,\boldsymbol{v} \in V. \tag{2.1}$$

For an regular function $\boldsymbol{\sigma} : \Omega \cup \Gamma \to \mathbb{S}^d$ we denote by σ_ν and $\boldsymbol{\sigma}_\tau$ the normal and the tangential components of the vector $\boldsymbol{\sigma\nu}$ on Γ, respectively, and we recall that $\sigma_\nu = \boldsymbol{\sigma\nu}\cdot\boldsymbol{\nu}$, $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma} - \sigma_\nu\boldsymbol{\nu}$. Moreover, the following Green's formula holds:

$$\int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx + \int_\Omega \mathrm{Div}\,\boldsymbol{\sigma}\cdot\boldsymbol{v}\,dx = \int_\Gamma \boldsymbol{\sigma\nu}\cdot\boldsymbol{v}\,da \quad \forall\,\boldsymbol{v} \in V. \tag{2.2}$$

Finally, we denote by $\mathbf{Q}_\infty$ the space of fourth order tensor fields given by

$$\mathbf{Q}_\infty = \{\ \mathcal{E} = (\mathcal{E}_{ijkl}) \mid \mathcal{E}_{ijkl} = \mathcal{E}_{jikl} = \mathcal{E}_{klij} \in L^\infty(\Omega), \quad 1 \le i,j,k,l \le d\ \}.$$

We note that $\mathbf{Q}_\infty$ is a real Banach space with the norm

$$\|\mathcal{E}\|_{\mathbf{Q}_\infty} = \sum_{0\le i,j,k,l\le d} \|\mathcal{E}_{ijkl}\|_{L^\infty(\Omega)}.$$

Moreover, a simple calculation shows that

$$\|\mathcal{E}\boldsymbol{\tau}\|_Q \le d\,\|\mathcal{E}\|_{\mathbf{Q}_\infty}\|\boldsymbol{\tau}\|_Q \qquad \forall\,\mathcal{E} \in \mathbf{Q}_\infty,\ \boldsymbol{\tau} \in Q. \tag{2.3}$$

For each Banach space X we use the notation $C(\mathbb{R}_+;X)$ for the space of continuously functions defined on $\mathbb{R}_+$ with values on X. It is well known that $C(\mathbb{R}_+;X)$ can be organized in a canonical way as a Fréchet space, i.e. as a complete metric space in which the corresponding topology is induced by a countable family of seminorms. Details can be found in [2] and [12], for instance. Here we restrict ourseleves to recall that the convergence of a sequence $(x_m)_m$ to the element x, in the space $C(\mathbb{R}_+;X)$, can be described as follows:

$$\begin{cases} x_m \to x \quad \text{in } C(\mathbb{R}_+;X) \ \text{ as } \ m \to \infty \ \text{ if and only if} \\[2ex] \displaystyle\max_{r\in[0,n]} \|x_m(r) - x(r)\|_X \to 0 \ \text{ as } m \to \infty, \ \text{ for all } \ n \in \mathbb{N}^*. \end{cases} \tag{2.4}$$

Equivalence (2.4) will be used several times in Section 5 of the paper.

Consider now a real Hilbert space X with inner product $(\cdot,\cdot)_X$ and associated norm $\|\cdot\|_X$. Also, assume given a set $K \subset X$, the operators $A : K \to X$, $\mathcal{S} : C(\mathbb{R}_+;X) \to C(\mathbb{R}_+;X)$ and a function $f : \mathbb{R}_+ \to X$ such that:

$$K \text{ is a closed, convex, nonempty subset of } X. \tag{2.5}$$

$$\left\{\begin{array}{l}\text{(a) There exists } m>0 \text{ such that} \\ \qquad (Au_1 - Au_2, u_1 - u_2)_X \geq m\,\|u_1-u_2\|_X^2 \quad \forall\, u_1, u_2 \in K. \\ \text{(b) There exists } L>0 \text{ such that} \\ \qquad \|Au_1 - Au_2\|_X \leq L\,\|u_1 - u_2\|_X \quad \forall\, u_1, u_2 \in K.\end{array}\right. \tag{2.6}$$

$$\left\{\begin{array}{l}\text{For every } n\in\mathbb{N}^* \text{ there exists } r_n>0 \text{ such that} \\ \quad \|\mathcal{S}u_1(t) - \mathcal{S}u_2(t)\|_Y \leq r_n \displaystyle\int_0^t \|u_1(s)-u_2(s)\|_X\,ds \\ \quad \forall\, u_1,\, u_2 \in C(\mathbb{R}_+;X),\ \forall\, t\in[0,n].\end{array}\right. \tag{2.7}$$

$$f \in C(\mathbb{R}_+;X). \tag{2.8}$$

We have following result, which represents a particular case of a more general existence and uniqueness result proved in [26].

Theorem 2.1 *Assume that* (2.5)–(2.8) *hold. Then there exists a unique function* $u \in C(\mathbb{R}_+;X)$ *such that, for all* $t\in\mathbb{R}_+$, *the inequality below holds:*

$$\begin{aligned} u(t)\in K, \quad (Au(t), v-u(t))_X + (\mathcal{S}u(t), v-u(t))_X \\ \geq (f(t), v-u(t))_X \qquad \forall\, v\in K. \end{aligned} \tag{2.9}$$

Following the terminology introduced in [26] we refer to (2.9) as a history-dependent variational inequality. To avoid any confusion, we note that here and below the notation $Au(t)$ and $\mathcal{S}u(t)$ are short hand notation for $A(u(t))$ and $(\mathcal{S}u)(t)$, i.e. $Au(t) = A(u(t))$ and $\mathcal{S}u(t) = (\mathcal{S}u)(t)$, for all $t\in\mathbb{R}_+$.

3 The model

The physical setting is as follows. A viscoelastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 1,2,3)$ with a Lipschitz continuous boundary Γ, divided into three measurable parts Γ_1, Γ_2 and Γ_3 such that meas$(\Gamma_1) > 0$. The body is subject to the action of body forces of density $\boldsymbol{f}_0$. We also assume that the body is fixed on Γ_1 and surfaces tractions of density $\boldsymbol{f}_2$ act on Γ_2. On Γ_3, the body is in frictionless contact with a obstacle, the so-called foundation. We assume that the foundation is deformable and, therefore, the penetration is allowed. Nevertheless, when the penetration reaches a given bound g, the foundation becomes rigid. And, finally, there are memory effects during the contact process. The process is quasistatic, and it is studied in the interval of time $\mathbb{R}_+ = [0,\infty)$. With these assumption, the classical formulation of the problem is the following.

Problem $\mathcal{P}$. *Find a displacement field* $\boldsymbol{u} : \Omega\times\mathbb{R}_+ \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega\times\mathbb{R}_+ \to \mathbb{S}^d$ *such that, for all* $t\in\mathbb{R}_+$,

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\,ds \qquad \text{in} \quad \Omega, \tag{3.1}$$

$$\mathrm{Div}\,\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \qquad \text{in} \quad \Omega, \tag{3.2}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \qquad \text{on} \quad \Gamma_1, \tag{3.3}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \qquad \text{on} \quad \Gamma_2, \tag{3.4}$$

$$\left.\begin{array}{l} u_\nu(t) \le g,\ \sigma_\nu(t) + p(u_\nu(t)) + \displaystyle\int_0^t b(t-s)\,u_\nu^+(s)\,ds \le 0, \\ (u_\nu(t) - g)\Big(\sigma_\nu(t) + p(u_\nu(t) + \displaystyle\int_0^t b(t-s)\,u_\nu^+(s)\,ds\Big) = 0 \end{array}\right\} \qquad \text{on} \quad \Gamma_3, \tag{3.5}$$

$$\boldsymbol{\sigma}_\tau(t) = \boldsymbol{0} \qquad \text{on} \quad \Gamma_3. \tag{3.6}$$

Here and below, in order to simplify the notation, we do not indicate explicitly the dependence of various functions on the spatial variable $\boldsymbol{x} \in \Omega \cup \Gamma$. Equation (3.1) represents the viscoelastic constitutive law of the material introduced in Section 1 and equation (3.2) is the equilibrium equation. Conditions (3.3) and (3.4) are the displacement and traction boundary conditions, respectively, and condition (3.6) shows that the tangential stress on the contact surface, denoted $\boldsymbol{\sigma}_\tau$, vanishes. We use it here since we assume that the contact process is frictionless.

We now describe the contact condition (3.5) in which our main interest is. Here σ_ν denotes the normal stress, u_ν is the normal displacement and p is a Lipschitz continuous increasing function which vanishes for a negative argument. Moreover, b is the surface memory function and $g > 0$ is a given bound for the normal displacement. This condition can be derived in the following way. First, we assume that the penetration is limited by the bound g and, therefore, at each time moment $t \in \mathbb{R}_+$, the normal displacement satisfies the inequality

$$u_\nu(t) \le g \quad \text{on } \Gamma_3. \tag{3.7}$$

Next, we assume that the normal stress has an additive decomposition of the form

$$\sigma_\nu(t) = \sigma_\nu^D(t) + \sigma_\nu^R(t) + \sigma_\nu^M(t) \quad \text{on } \Gamma_3. \tag{3.8}$$

in which the functions σ_ν^D, σ_ν^R and σ_ν^M describe the deformability, the rigidity and the memory properties of the foundation, at each $t \in \mathbb{R}_+$. We assume that the function σ_ν^D satisfies the normal compliance contact condition (1.2), that is

$$-\,\sigma_\nu^D(t) = p(u_\nu(t)) \quad \text{on } \Gamma_3. \tag{3.9}$$

The part σ_ν^R of the normal stress satisfies the Signorini condition in the form with a gap function (1.6), i.e.

$$\sigma_\nu^R(t) \le 0, \quad \sigma_\nu^R(t)(u_\nu(t) - g) = 0 \quad \text{on } \Gamma_3. \tag{3.10}$$

And, finally, the function σ_ν^M satisfies the memory condition (1.7), that is

$$-\,\sigma_\nu^M(t) = \int_0^t b(t-s)\, u_\nu^+(s)\, ds \quad \text{on } \Gamma_3. \tag{3.11}$$

We combine equalities (3.8), (3.10) and (3.11) to see that

$$\sigma_\nu^R(t) = \sigma_\nu(t) + p(u_\nu(t)) + \int_0^t b(t-s)\, u_\nu^+(s)\, ds \quad \text{on } \Gamma_3. \tag{3.12}$$

Then we substitute equality (3.12) in (3.10) and use inequality (3.7) to obtain the contact condition (3.5).

Not that (3.5) describes a condition with unilateral constraint, since inequality (3.7) holds at each time moment. Assume now that at a given moment t there is penetration which did not reach the bound g, i.e. $0 < u_\nu(t) < g$. Then (3.5) yields

$$-\,\sigma_\nu(t) = p(u_\nu(t)) + \int_0^t b(t-s) u_\nu^+(s)\, ds. \tag{3.13}$$

This equality shows that at the moment t, the reaction of the foundation depend both on the current value of the penetration (represented by the term $p(u_\nu(t))$) as well as on the history of the penetration (represented by the integral term in (3.13)). When b is a positive function the reaction of the foundation is larger than that given by the term $p(u_\nu(t))$ and we conclude that equality (3.13) models the hardening phenomenon of the surface. When b is a negative function the reaction of the foundation is smaller than that given by the term $p(u_\nu(t))$ and we conclude that equality (3.13) models the softening phenomenon of the surface. Hardening and softening of contact surfaces represent an important phenomenon which appear in various industrial applications applications, see for instance [16] and references therein.

In conclusion, condition (3.5) shows that the contact follows a normal compliance condition with memory term of the form (3.13) but up to the limit g and then, when this limit is reached, the contact follows a Signorini-type unilateral condition with the gap g. For this reason we refer to this condition as to a normal compliance contact condition with memory term and unilateral constraint. It can be interpreted physically as follows. The foundation is assumed to be made of a hard material covered by a thin layer of a soft material with thickness g. The soft material has a viscoelastic behaviour, i.e. is deformable, allows penetration and presents memory effects; the contact with this layer is modelled with normal compliance and memory term. The hard material is perfectly rigid and, therefore, it does not allow penetration; the contact with this material is modelled with the Signorini contact condition. To resume, the foundation has a rigid-viscoelastic behavior; its viscoelastic behavior is given by the layer of the soft material while its rigid behavior is given by the hard material.

In the particular case when $b = 0$ the contact condition (3.5) was introduced in [5], in the study of a dynamic frictionless contact problem with elastic-visco-plastic materials. Then, it was used in [1] and [27] in the study of various quasistatic contact problems. Also, note that when $g > 0$, $p = 0$ and $b = 0$ condition (3.5) becomes the Signorini contact condition in a form with a gap function, (1.7). And, finally, if $b = 0$ and $g \to \infty$, we recover the normal compliance contact condition with a zero gap function, (1.2).

4 Existence and uniqueness results

To derive the variational formulation of the problem $\mathcal{P}$ we list the assumptions on the problem data. First, we assume that the elasticity operator $\mathcal{A}$ and the relaxation tensor $\mathcal{B}$ satisfy the following conditions.

$$\left\{\begin{array}{l}\text{(a) } \mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d.\\ \text{(b) There exists } L_{\mathcal{A}} > 0 \text{ such that}\\ \qquad \|\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)\| \le L_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|\\ \qquad\quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega.\\ \text{(c) There exists } m_{\mathcal{A}} > 0 \text{ such that}\\ \qquad (\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2\\ \qquad\quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega.\\ \text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega,\\ \qquad \text{for any } \boldsymbol{\varepsilon} \in \mathbb{S}^d.\\ \text{(e) The mapping } \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x}, \boldsymbol{0}) \text{ belongs to } Q.\end{array}\right. \tag{4.1}$$

$$\mathcal{B} \in C(\mathbb{R}_+; \mathbf{Q}_\infty). \tag{4.2}$$

The normal compliance and the surface memory function satisfy the conditions

$$\left\{\begin{array}{l}\text{(a) } p : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+.\\ \text{(b) There exists } L_p > 0 \text{ such that}\\ \qquad |p(\boldsymbol{x}, r_1) - p(\boldsymbol{x}, r_2)| \le L_p\, |r_1 - r_2|\\ \qquad\quad \forall\, r_1,\, r_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3.\\ \text{(c) } (p(\boldsymbol{x}, r_1) - p(\boldsymbol{x}, r_2))(r_1 - r_2) \ge 0\\ \qquad\quad \forall\, r_1,\, r_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3.\\ \text{(d) The mapping } \boldsymbol{x} \mapsto p(\boldsymbol{x}, r) \text{ is measurable on } \Gamma_3,\\ \qquad \text{for any } r \in \mathbb{R}.\\ \text{(e) } p(\boldsymbol{x}, r) = 0 \text{ for all } r \le 0, \text{ a.e. } \boldsymbol{x} \in \Gamma_3.\end{array}\right. \tag{4.3}$$

$$b \in C(\mathbb{R}_+; L^\infty(\Gamma_3)). \tag{4.4}$$

Finally, we assume that the densities of body forces and surface tractions have the regularity

$$\boldsymbol{f}_0 \in C(\mathbb{R}_+; L^2(\Omega)^d), \quad \boldsymbol{f}_2 \in C(\mathbb{R}_+; L^2(\Gamma_2)^d) \tag{4.5}$$

and, moreover, we introduce the set of admissible displacements fields defined by

$$U = \{\, \boldsymbol{v} \in V : v_\nu \le g \text{ on } \Gamma_3 \,\}. \tag{4.6}$$

Assume in what follows that $(\boldsymbol{u}, \boldsymbol{\sigma})$ are sufficiently regular functions which satisfy (3.1)–(3.6) and let $\boldsymbol{v} \in U$ and $t > 0$ be given. We use the Green formula (2.2) and the equilibrium equation (3.2) to obtain

$$\int_\Omega \boldsymbol{\sigma}(t) \cdot (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))\, dx = \int_\Omega \boldsymbol{f}_0(t) \cdot (\boldsymbol{v} - \boldsymbol{u}(t))\, dx + \int_\Gamma \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}(t))\, da.$$

We split the boundary integral over Γ_1, Γ_2 and Γ_3 and, since $\boldsymbol{v} - \boldsymbol{u}(t) = \boldsymbol{0}$ on Γ_1 and $\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t)$ on Γ_2, we obtain

$$\begin{aligned} \int_\Omega \boldsymbol{\sigma}(t) \cdot (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))\, dx &= \int_\Omega \boldsymbol{f}_0(t) \cdot (\boldsymbol{v} - \boldsymbol{u}(t))\, dx \\ &+ \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot (\boldsymbol{v} - \boldsymbol{u}(t))\, da + \int_{\Gamma_3} \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}(t))\, da. \end{aligned} \tag{4.7}$$

Moreover, since

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}(t)) = \sigma_\nu(t)(v_\nu - u_\nu(t)) + \boldsymbol{\sigma}_\tau(t)\boldsymbol{\nu} \cdot (\boldsymbol{v}_\tau - \boldsymbol{u}_\tau(t)) \quad \text{on } \Gamma_3,$$

condition (3.6) implies that

$$\int_{\Gamma_3} \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}(t))\, da = \int_{\Gamma_3} \sigma_\nu(t)(v_\nu - u_\nu)\, da. \tag{4.8}$$

We write now

$$\begin{aligned} \sigma_\nu(t)(v_\nu - u_\nu(t)) &= \Big(\sigma_\nu(t) + p(u_\nu(t)) + \int_0^t b(t-s)\, u_\nu^+(s)\, ds\Big)(v_\nu - g) \\ &+ \Big(\sigma_\nu(t) + p(u_\nu(t)) + \int_0^t b(t-s)\, u_\nu^+(s)\, ds\Big)(g - u_\nu(t)) \\ &- \Big(p(u_\nu(t)) + \int_0^t b(t-s)\, u_\nu^+(s)\, ds\Big)(v_\nu - u_\nu(t)) \quad \text{on } \Gamma_3, \end{aligned}$$

then we use the contact conditions (3.5) and the definition (4.6) of the set U to see that

$$\sigma_\nu(t)(v_\nu - u_\nu(t)) \ge -\Big(p(u_\nu(t)) + \int_0^t b(t-s)\, u_\nu^+(s)\, ds\Big)(v_\nu - u_\nu(t)) \quad \text{on } \Gamma_3,$$

and, therefore,

$$\int_{\Gamma_3} \sigma_\nu(t)(v_\nu - u_\nu(t))\, da \ge -\int_{\Gamma_3} \Big(p(u_\nu(t)) + \int_0^t b(t-s)\, u_\nu^+(s)\, ds\Big)(v_\nu - u_\nu(t))\, da. \tag{4.9}$$

We combine now equalities (4.7), (4.8) then we use inequality (4.9) to deduce that

$$
\begin{aligned}
&(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q + (p(u_\nu(t)), v_\nu - u_\nu(t))_{L^2(\Gamma_3)} \\
&\quad + \Big(\int_0^t b(t-s)\, u_\nu^+(s)\, ds, v_\nu - u_\nu(t) \Big)_{L^2(\Gamma_3)} \\
&\quad\quad \geq (\boldsymbol{f}_0(t), \boldsymbol{v} - \boldsymbol{u}(t))_{L^2(\Omega)^d} + (\boldsymbol{f}_2(t), \boldsymbol{v} - \boldsymbol{u}(t))_{L^2(\Gamma_2)^d}
\end{aligned}
\tag{4.10}
$$

In addition, we note that the boundary condition (3.3), the first inequality in (3.5) and notation (4.6) imply that $\boldsymbol{u}(t) \in U$. Therefore, using the constitutive law (3.1) and inequality (4.10) we derive the following variational formulation of Problem $\mathcal{P}$.

Problem $\mathcal{P}^V$. *Find a displacement field $\boldsymbol{u} : \mathbb{R}_+ \to V$ such that, for all $t \in \mathbb{R}_+$, the inequality below holds:*

$$
\begin{aligned}
&\boldsymbol{u}(t) \in U, \quad (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q \\
&\quad + \Big(\int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s)\, ds, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \Big)_Q \\
&\quad\quad + (p(u_\nu(t)), v_\nu - u_\nu(t))_{L^2(\Gamma_3)} + \Big(\int_0^t b(t-s) u_\nu^+(s)\, ds, v_\nu - u_\nu(t)) \Big)_{L^2(\Gamma_3)} \\
&\quad\quad\quad \geq (\boldsymbol{f}_0(t), \boldsymbol{v} - \boldsymbol{u}(t))_{L^2(\Omega)^d} + (\boldsymbol{f}_2(t), \boldsymbol{v} - \boldsymbol{u}(t))_{L^2(\Gamma_2)^d} \quad \forall\, \boldsymbol{v} \in U.
\end{aligned}
\tag{4.11}
$$

In the study of the problems $\mathcal{P}^V$ we have the following existence and uniqueness result.

Theorem 4.1 *Assume that* (4.1)–(4.5) *hold. Then, Problem $\mathcal{P}^V$ has a unique solution which satisfies $\boldsymbol{u} \in C(\mathbb{R}_+; V)$.*

Proof. We start with by providing an equivalent form of Problem $\mathcal{P}^V$. To this end we use the Riesz representation theorem to define the operators $P : V \to V$, $\mathcal{S} : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; V)$ and the function $\boldsymbol{f} : \mathbb{R}_+ \to V$ by equalities

$$
(P\boldsymbol{u}, \boldsymbol{v})_V = \int_{\Gamma_3} p(u_\nu) v_\nu\, da \quad \forall\, \boldsymbol{u},\, \boldsymbol{v} \in V,
\tag{4.12}
$$

$$
\begin{aligned}
(\mathcal{S}\boldsymbol{u}(t), \boldsymbol{v})_V &= \Big(\int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\, ds, \boldsymbol{\varepsilon}(\boldsymbol{v}) \Big)_Q \\
&\quad + \Big(\int_0^t b(t-s) u_\nu^+(s)\, ds, v_\nu \Big)_{L^2(\Gamma_3)} \quad \forall\, \boldsymbol{u} \in C(\mathbb{R}_+; V),\ \boldsymbol{v} \in V,
\end{aligned}
\tag{4.13}
$$

$$
(\boldsymbol{f}(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v}\, da \quad \forall\, \boldsymbol{u},\, \boldsymbol{v} \in V,\ t \in \mathbb{R}_+.
\tag{4.14}
$$

Then, it is easy to see that Problem $\mathcal{P}^V$ is equivalent to the problem of finding a function $\boldsymbol{u} : \mathbb{R}_+ \to V$ such that the inequality below holds, for all $t \in \mathbb{R}_+$:

$$\begin{aligned} &\boldsymbol{u}(t) \in U, \quad (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q + (P\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \\ &\quad +(\mathcal{S}\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \quad \forall\, \boldsymbol{v} \in U. \end{aligned} \tag{4.15}$$

To solve the variational inequality (4.15) we use Theorem 2.1 with $X = V$ and $K = U$. To this end we consider the operator $A : V \to V$ defined by

$$(A\boldsymbol{u}, \boldsymbol{v})_V = (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (P\boldsymbol{u}, \boldsymbol{v})_V \quad \forall\, \boldsymbol{u},\, \boldsymbol{v} \in V, \tag{4.16}$$

It is easy to see that condition (2.5) holds. Next, we use (4.1), (4.3) and (2.1) to see that the operator A verifies condition (2.6). Let $n \in \mathbb{N}^*$. Then, a simple calculation based on assumptions (4.2), (4.4) and inequalities (2.1), (2.3) shows that

$$\left\{ \begin{aligned} &\|\mathcal{S}\boldsymbol{u}_1(t) - \mathcal{S}\boldsymbol{u}_2(t)\|_V \leq d \max_{r\in[0,n]} \|\mathcal{B}(r)\|_{\mathbf{Q}_\infty} \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V \, ds \\ &\qquad + c_0^2 \max_{r\in[0,n]} \|b(r)\|_{L^\infty(\Gamma_3)} \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V \, ds \\ &\qquad\qquad \forall\, \boldsymbol{u}_1,\, \boldsymbol{u}_2 \in C(\mathbb{R}_+; V),\ \forall\, t \in [0, n]. \end{aligned} \right. \tag{4.17}$$

This inequality implies that the operator (4.13) satisfies condition (2.7) with

$$r_n = d \max_{r\in[0,n]} \|\mathcal{B}(r)\|_{\mathbf{Q}_\infty} + c_0^2 \max_{r\in[0,n]} \|b(r)\|_{L^\infty(\Gamma_3)}. \tag{4.18}$$

Finally, using (4.5) and (4.14) we deduce that $\boldsymbol{f} \in C(\mathbb{R}_+; V)$ and, therefore, (2.8) holds. It follows now from Theorem 2.1 that there exists a unique function $\boldsymbol{u} \in C(\mathbb{R}_+; V)$ which satisfies the inequality

$$\begin{aligned} &\boldsymbol{u}(t) \in U, \quad (A\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V + (\mathcal{S}\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \\ &\qquad \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \quad \forall\, \boldsymbol{v} \in U, \end{aligned} \tag{4.19}$$

for all $t \in \mathbb{R}_+$. And, using (4.16) we deduce that that there exists a unique function $\boldsymbol{u} \in C(\mathbb{R}_+; V)$ such that (4.15) holds for all $t \in \mathbb{R}_+$, which concludes the proof. □

Let $\boldsymbol{\sigma}$ be the function defined by (3.1). Then, it follows (4.1) and (4.2) that $\boldsymbol{\sigma} \in C(\mathbb{R}_+; Q)$. Moreover, it is easy to see that (4.10) holds for all $t \in \mathbb{R}_+$ and, using standard arguments, it result from here that

$$\operatorname{Div} \boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \mathbf{0} \qquad \forall\, t \in \mathbb{R}_+. \tag{4.20}$$

Therefore, using the regularity $\boldsymbol{f}_0 \in C(\mathbb{R}_+; L^2(\Omega)^d)$ in (4.5) we deduce that $\operatorname{Div} \boldsymbol{\sigma} \in C(\mathbb{R}_+; L^2(\Omega)^d)$ which implies that $\boldsymbol{\sigma} \in C(\mathbb{R}_+; Q_1)$. A couple of functions $(\boldsymbol{u}, \boldsymbol{\sigma})$ which satisfies (3.1), (4.11) for all $t \in \mathbb{R}_+$ is called a weak solution to the contact problem $\mathcal{P}$. We conclude that Theorem 4.1 provides the unique weak solvability of Problem $\mathcal{P}$. Moreover, the regularity of the weak solution is $\boldsymbol{u} \in C(\mathbb{R}_+; V)$, $\boldsymbol{\sigma} \in C(\mathbb{R}_+; Q_1)$.

5 A first convergence result

We study now the dependence of the solution of Problem $\mathcal{P}^V$ with respect to perturbations of the data. To this end, we assume in what follows that (4.1)–(4.5) hold and we denote by $\boldsymbol{u}$ the solution of Problem $\mathcal{P}^V$ obtained in Theorem 4.1. For each $\rho > 0$ let $\mathcal{B}_\rho$, p_ρ, b_ρ, $\boldsymbol{f}_{0\rho}$ and $\boldsymbol{f}_{2\rho}$ be perturbations of $\mathcal{B}$, p, b, $\boldsymbol{f}_0$ and $\boldsymbol{f}_2$ which satisfy conditions (4.2), (4.3), (4.4) and (4.5), respectively. We consider the following variational problem.

Problem $\mathcal{P}^V_\rho$. *Find a displacement field $\boldsymbol{u}_\rho : \mathbb{R}_+ \to V$ such that, for all $t \in \mathbb{R}_+$, the inequality below holds :*

$$\begin{aligned}
&\boldsymbol{u}_\rho(t) \in U, \quad (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)))_Q \\
&\quad + \Big(\int_0^t \mathcal{B}_\rho(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(s))\, ds, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) \Big)_Q \\
&\qquad + (p_\rho(u_{\rho\nu}(t)), v_\nu - u_{\rho\nu}(t))_{L^2(\Gamma_3)} + \Big(\int_0^t b_\rho(t-s) u^+_{\rho\nu}(s)\, ds, v_\nu - u_{\rho\nu}(t) \Big)_{L^2(\Gamma_3)} \\
&\qquad \geq (\boldsymbol{f}_{0\rho}(t), \boldsymbol{v} - \boldsymbol{u}_\rho(t))_{L^2(\Omega)^d} + (\boldsymbol{f}_{2\rho}(t), \boldsymbol{v} - \boldsymbol{u}_\rho(t))_{L^2(\Gamma_2)^d} \quad \forall\, \boldsymbol{v} \in U.
\end{aligned} \tag{5.1}$$

Note that, here and below, $u_{\rho\nu}$ represents the normal component of the function $\boldsymbol{u}_\rho$.

It follows from Theorem 4.1 that, for each $\rho > 0$ Problem $\mathcal{P}^V_\rho$ has a unique solution $\boldsymbol{u}_\rho \in C(\mathbb{R}_+; V)$. Consider now the following assumptions:

$$\mathcal{B}_\rho \to \mathcal{B} \quad \text{in} \quad C(\mathbb{R}_+; \mathbf{Q}_\infty) \quad \text{as} \quad \rho \to 0. \tag{5.2}$$

$$b_\rho \to b \quad \text{in} \quad C(\mathbb{R}_+; L^\infty(\Gamma_3)) \quad \text{as} \quad \rho \to 0. \tag{5.3}$$

$$\boldsymbol{f}_{0\rho} \to \boldsymbol{f}_0 \quad \text{in} \quad C(\mathbb{R}_+; L^2(\Omega)^d) \quad \text{as} \quad \rho \to 0. \tag{5.4}$$

$$\boldsymbol{f}_{2\rho} \to \boldsymbol{f}_2 \quad \text{in} \quad C(\mathbb{R}_+; L^2(\Gamma_2)^d) \quad \text{as} \quad \rho \to 0. \tag{5.5}$$

$$\left\{ \begin{array}{l} \text{There exists } G : \mathbb{R}_+ \to \mathbb{R}_+ \text{ and } \beta \in \mathbb{R}_+ \text{ such that} \\ \text{(a) } |p_\rho(\boldsymbol{x}, r) - p(\boldsymbol{x}, r)| \leq G(\rho)(|r| + \beta) \\ \qquad \forall\, r \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \Gamma_3,\ \text{for each } \rho > 0. \\ \text{(b) } G(\rho) \to 0 \quad \text{as} \quad \rho \to 0. \end{array} \right. \tag{5.6}$$

We have the following convergence result.

Theorem 5.1 *Under assumptions* (5.2)–(5.6), *the solution $\boldsymbol{u}_\rho$ of Problem $\mathcal{P}^V_\rho$ converges to the solution $\boldsymbol{u}$ of Problem $\mathcal{P}^V$, i.e.,*

$$\boldsymbol{u}_\rho \to \boldsymbol{u} \quad \text{in} \quad C(\mathbb{R}_+; V) \quad \text{as} \quad \rho \to 0. \tag{5.7}$$

Proof. Let $\rho > 0$. We use the Riesz representation theorem to define the operators $P_\rho : V \to V$, $\mathcal{S}_\rho : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; V)$ and the function $\boldsymbol{f}_\rho : \mathbb{R}_+ \to V$ by equalities

$$(P_\rho \boldsymbol{u}, \boldsymbol{v})_V = \int_{\Gamma_3} p_\rho(u_\nu) v_\nu \, da \quad \forall \, \boldsymbol{u}, \, \boldsymbol{v} \in V, \tag{5.8}$$

$$\begin{aligned}(\mathcal{S}_\rho \boldsymbol{u}(t), \boldsymbol{v})_V &= \Big(\int_0^t \mathcal{B}_\rho(t-s) \boldsymbol{\varepsilon}(\boldsymbol{u}(s)) \, ds, \boldsymbol{\varepsilon}(\boldsymbol{v}) \Big)_Q \\ &+ \Big(\int_0^t b_\rho(t-s) u_\nu^+(s) \, ds, v_\nu \Big)_{L^2(\Gamma_3)} \quad \forall \, \boldsymbol{u} \in C(\mathbb{R}_+; V), \ \boldsymbol{v} \in V,\end{aligned} \tag{5.9}$$

$$(\boldsymbol{f}_\rho(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_{0\rho}(t) \cdot \boldsymbol{v} \, dx + \int_{\Gamma_2} \boldsymbol{f}_{2\rho}(t) \cdot \boldsymbol{v} \, da \quad \forall \, \boldsymbol{u}, \, \boldsymbol{v} \in V, \ t \in \mathbb{R}_+. \tag{5.10}$$

It follows from the proof of Theorem 4.1 that $\boldsymbol{u}$ is a solution of Problem $\mathcal{P}^V$ iff $\boldsymbol{u}$ solves inequality (4.15), for all $t \in \mathbb{R}_+$. In a similar way, $\boldsymbol{u}_\rho$ is a solution of Problem $\mathcal{P}_\rho^V$ iff, for all $t \in \mathbb{R}_+$, the inequality below holds:

$$\begin{aligned}&\boldsymbol{u}_\rho(t) \in U, \quad (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)))_Q + (P_\rho \boldsymbol{u}_\rho(t), \boldsymbol{v} - \boldsymbol{u}_\rho(t))_V \\ &\quad + (\mathcal{S}_\rho \boldsymbol{u}_\rho(t), \boldsymbol{v} - \boldsymbol{u}_\rho(t)_V \geq (\boldsymbol{f}_\rho(t), \boldsymbol{v} - \boldsymbol{u}_\rho(t))_V \quad \forall \, \boldsymbol{v} \in U.\end{aligned} \tag{5.11}$$

Let $n \in \mathbb{N}^*$ and let $t \in [0, n]$. We take $\boldsymbol{v} = \boldsymbol{u}(t)$ in (5.11) and $\boldsymbol{v} = \boldsymbol{u}_\rho(t)$ in (4.15) and add the resulting inequalities to obtain

$$\begin{aligned}&(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q \\ &\quad \leq (P_\rho \boldsymbol{u}_\rho(t) - P\boldsymbol{u}(t), \boldsymbol{u}(t) - \boldsymbol{u}_\rho(t))_V \\ &\qquad + (\mathcal{S}_\rho \boldsymbol{u}_\rho(t) - \mathcal{S}\boldsymbol{u}(t), \boldsymbol{u}(t) - \boldsymbol{u}_\rho(t))_V + (\boldsymbol{f}_\rho(t) - \boldsymbol{f}(t), \boldsymbol{u}_\rho(t) - \boldsymbol{u}(t))_V.\end{aligned} \tag{5.12}$$

Next, we use the definitions (5.8) and (4.12), the monotonicity of the function p_ρ and assumption (5.6) to see that

$$\begin{aligned}(P_\rho \boldsymbol{u}_\rho(t) - P\boldsymbol{u}(t), \boldsymbol{u}(t) - \boldsymbol{u}_\rho(t))_V &= \int_{\Gamma_3} \big(p_\rho(u_{\rho\nu}) - p(u_\nu)\big)(u_\nu(t) - u_{\rho\nu}(t)) \, da \\ \leq \int_{\Gamma_3} \big(p_\rho(u_\nu(t)) - p(u_\nu(t))\big)&(u_\nu(t) - u_{\rho\nu}(t)) \, da \\ \leq \int_{\Gamma_3} \big|p_\rho(u_\nu(t)) - p(u_\nu(t))\big|& \, |u_\nu(t) - u_{\rho\nu}(t)| \, da \\ \leq \int_{\Gamma_3} G(\rho)(|u_\nu(t)| + \beta)& \, |u_\nu(t) - u_{\rho\nu}(t)| \, da.\end{aligned}$$

Therefore, using the trace inequality (2.1), after some elementary calculus we find that

$$(P_\rho \boldsymbol{u}_\rho(t) - P\boldsymbol{u}(t), \boldsymbol{u}(t) - \boldsymbol{u}_\rho(t))_V \tag{5.13}$$
$$\le G(\rho)\big(c_0^2\,\|\boldsymbol{u}(t)\|_V + c_0\beta\,\mathrm{meas}\,(\Gamma_3)^{\frac{1}{2}}\big)\,\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V.$$

On the other hand, using assumptions (4.2), (4.4) and arguments similar to those used in the proof of (4.17) we find that

$$\|\mathcal{S}_\rho \boldsymbol{u}_\rho(t) - \mathcal{S}\boldsymbol{u}(t)\|_V \le \|\mathcal{S}_\rho \boldsymbol{u}_\rho(t) - \mathcal{S}_\rho \boldsymbol{u}(t)\|_V + \|\mathcal{S}_\rho \boldsymbol{u}(t) - \mathcal{S}\boldsymbol{u}(t)\|_V$$
$$\le d \max_{r\in[0,n]} \|\mathcal{B}_\rho(r)\|_{\mathbf{Q}_\infty} \int_0^t \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|_V\,ds$$
$$+c_0^2 \max_{r\in[0,n]} \|b_\rho(r)\|_{L^\infty(\Gamma_3)} \int_0^t \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|_V\,ds$$
$$+d \max_{r\in[0,n]} \|\mathcal{B}_\rho(r) - \mathcal{B}(r)\|_{\mathbf{Q}_\infty} \int_0^t \|\boldsymbol{u}(s)\|_V\,ds$$
$$+ c_0^2 \max_{r\in[0,n]} \|b_\rho(r) - b(r)\|_{L^\infty(\Gamma_3)} \int_0^t \|\boldsymbol{u}(s)\|_V\,ds.$$

Therefore,

$$(\mathcal{S}_\rho \boldsymbol{u}_\rho(t) - \mathcal{S}\boldsymbol{u}(t), \boldsymbol{u}(t) - \boldsymbol{u}_\rho(t))_V \tag{5.14}$$
$$\le \|\mathcal{S}_\rho \boldsymbol{u}_\rho(t) - \mathcal{S}\boldsymbol{u}(t)\|_V \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V$$
$$\le \Big(\theta_{\rho n} \int_0^t \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|_V\,ds + \omega_{\rho n} \int_0^t \|\boldsymbol{u}(s)\|_V\,ds\Big) \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V$$

where

$$\theta_{\rho n} = d \max_{r\in[0,n]} \|\mathcal{B}_\rho(r)\|_{\mathbf{Q}_\infty} + c_0^2 \max_{r\in[0,n]} \|b_\rho(r)\|_{L^\infty(\Gamma_3)}, \tag{5.15}$$

$$\omega_{\rho n} = d \max_{r\in[0,n]} \|\mathcal{B}_\rho(r) - \mathcal{B}(r)\|_{\mathbf{Q}_\infty} + c_0^2 \max_{r\in[0,n]} \|b_\rho(r) - b(r)\|_{L^\infty(\Gamma_3)}. \tag{5.16}$$

Finally, we note that

$$(\boldsymbol{f}_\rho(t) - \boldsymbol{f}(t), \boldsymbol{u}_\rho(t) - \boldsymbol{u}(t))_V \le \delta_{\rho n} \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \tag{5.17}$$

where

$$\delta_{\rho n} = \max_{r\in[0,n]} \|\boldsymbol{f}_\rho(r) - \boldsymbol{f}(r)\|_V \tag{5.18}$$

and, using assumption (4.1) it follows that

$$(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q \ge m_{\mathcal{A}} \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V^2. \tag{5.19}$$

We combine now inequalities (5.12)–(5.14), (5.17) and (5.19) to deduce that

$$\|\boldsymbol{u}_\rho(t)-\boldsymbol{u}(t)\|_V \le \frac{G(\rho)}{m_{\mathcal{A}}}\big(c_0^2\,\|\boldsymbol{u}(t)\|_V + c_0\beta\,\mathrm{meas}\,(\Gamma_3)^{\frac12}\big) \tag{5.20}$$
$$+\frac{\theta_{\rho n}}{m_{\mathcal{A}}}\int_0^t \|\boldsymbol{u}_\rho(s)-\boldsymbol{u}(s)\|_V\,ds + \frac{\omega_{\rho n}}{m_{\mathcal{A}}}\int_0^t \|\boldsymbol{u}(s)\|_V\,ds + \frac{\delta_{\rho n}}{m_{\mathcal{A}}}\,.$$

Denote

$$\xi_{n,u} = max\Big\{\frac{1}{m_{\mathcal{A}}}\big(c_0^2\max_{r\in[0,n]}\|\boldsymbol{u}(r)\|_V + c_0\beta\,\mathrm{meas}\,(\Gamma_3)^{\frac12}\big),\,\frac{1}{m_{\mathcal{A}}}\int_0^n \|\boldsymbol{u}(s)\|_V\,ds,\,\frac{1}{m_{\mathcal{A}}}\Big\}.$$

Then, (5.20) yields

$$\|\boldsymbol{u}_\rho(t)-\boldsymbol{u}(t)\|_V \le \big(G(\rho)+\omega_{\rho n}+\delta_{\rho n}\big)\xi_{n,u} + \frac{\theta_{\rho n}}{m_{\mathcal{A}}}\int_0^t \|\boldsymbol{u}_\rho(s)-\boldsymbol{u}(s)\|_V\,ds$$

and, using the Gronwall inequality we obtain

$$\|\boldsymbol{u}_\rho(t)-\boldsymbol{u}(t)\|_V \le \big(G(\rho)+\omega_{\rho n}+\delta_{\rho n}\big)\xi_{n,u}\,e^{\frac{\theta_{\rho n}}{m_{\mathcal{A}}}t}. \tag{5.21}$$

We use the assumptions (5.2), (5.3) and the equivalence (2.4) to see that the sequence $(\theta_{\rho n})_\rho$ defined by (5.15) is bounded. Therefore, there exists $\zeta_n > 0$ which depends on n and is independent of ρ such that

$$0 \le \theta_{\rho n} \le \zeta_n \quad \text{for all} \quad \rho > 0. \tag{5.22}$$

We pas to the upper bound as $t \in [0, n]$ in (5.21) and use (5.22) to obtain

$$\max_{t\in[0,n]}\|\boldsymbol{u}_\rho(t)-\boldsymbol{u}(t)\|_V \le \big(G(\rho)+\omega_{\rho n}+\delta_{\rho n}\big)\xi_{n,u}\,e^{\frac{n\zeta_n}{m_{\mathcal{A}}}} \quad \text{for all} \quad \rho > 0. \tag{5.23}$$

We use now assumption (5.2)–(5.5) and definitions (5.16), (5.18) to see that

$$\omega_{\rho n} \to 0 \quad \text{and} \quad \delta_{\rho n} \to 0 \quad \text{as } \rho \to 0. \tag{5.24}$$

We combine now the convergences (5.24) and (5.6)(b) with inequality (5.23) to obtain that

$$\max_{t\in[0,n]}\|\boldsymbol{u}_\rho(t)-\boldsymbol{u}(t)\|_V \to 0 \quad \text{as } \rho \to 0. \tag{5.25}$$

Since the convergence (5.25) holds for each $n \in \mathbb{N}^*$, we deduce from (2.4) that (5.7) holds, which concludes the proof. $\square$

Note that the convergence result in Theorem 5.1 can be easily extended to the corresponding stress functions. Indeed, let $\boldsymbol{\sigma}$ be the function defined by (3.1) and, for all $\rho > 0$, denote by $\boldsymbol{\sigma}_\rho$ the function given by

$$\boldsymbol{\sigma}_\rho(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) + \int_0^t \mathcal{B}_\rho(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(s))\,ds \tag{5.26}$$

for all $t \in \mathbb{R}_+$. Then, it follows that $\boldsymbol{\sigma}_\rho \in C(\mathbb{R}_+; Q_1)$ and, moreover, (5.1) yields

$$\mathrm{Div}\, \boldsymbol{\sigma}_\rho(t) + \boldsymbol{f}_{\rho 0}(t) = \mathbf{0} \qquad \forall\, t \in \mathbb{R}_+. \tag{5.27}$$

We combine now equalities (3.1), (4.20), (5.26) and (5.27), then we use the convergences (5.2), (5.4) and (5.7) to see that

$$\boldsymbol{\sigma}_\rho \to \boldsymbol{\sigma} \quad \text{in} \quad C(\mathbb{R}_+; Q_1) \quad \text{as } \rho \to 0. \tag{5.28}$$

In addition to the mathematical interest in the convergence result (5.7), (5.28), it is of importance from mechanical point of view, since it states that the weak solution of problem (3.1)–(3.5) depends continuously on the relaxation operator, the normal compliance function, the surface memory function and the densities of body forces and surface tractions.

6 A second convergence result

In this section we provide a second convergence result in the study of Problem $\mathcal{P}$, based on the penalization of the unilateral constraint. For simplicity we assume that the function p does not depend on $\boldsymbol{x} \in \Gamma_3$, i.e. we consider the homogeneous case. Note that in this case assumption (4.3) can be written as follows:

$$\left\{\begin{array}{l} \text{(a) } p : \mathbb{R} \to \mathbb{R}_+. \\ \text{(b) There exists } L_p > 0 \text{ such that} \\ \qquad |p(r_1) - p(r_2)| \le L_p |r_1 - r_2| \quad \forall r_1,\, r_2 \in \mathbb{R}. \\ \text{(c) } (p_(r_1) - p_(r_2))(r_1 - r_2) \ge 0 \quad \forall r_1,\, r_2 \in \mathbb{R}. \\ \text{(d) } p(r) = 0 \ \text{ for all } r \le 0. \end{array}\right. \tag{6.1}$$

Let q be a function which satisfies

$$\left\{\begin{array}{l} \text{(a) } q:\ [g, +\infty] \to \mathbb{R}_+. \\ \text{(b) There exists } L_q > 0 \text{ such that} \\ \qquad |q(r_1) - q(r_2)| \le L_q |r_1 - r_2| \quad \forall\, r_1,\, r_2 \ge g. \\ \text{(c) } (q(r_1) - q(r_2))(r_1 - r_2) > 0 \quad \forall\, r_1,\, r_2 \ge g,\, r_1 \ne r_2. \\ \text{(d) } q(g) = 0. \end{array}\right. \tag{6.2}$$

Also, let $\mu > 0$ and consider the function $p_\mu : \mathbb{R} \to \mathbb{R}$ defined by

$$p_\mu(r) = \left\{\begin{array}{cl} p(r) & \text{if} \quad r \le g, \\ \frac{1}{\mu}\, q(r) + p(g) & \text{if} \quad r > g. \end{array}\right. \tag{6.3}$$

Using assumptions (6.1) and (6.2) it follows that the function p_μ satisfies condition (6.1), i.e.

$$
\begin{cases}
\text{(a) } p_\mu : \mathbb{R} \to \mathbb{R}_+. \\
\text{(b) There exists } L_\mu > 0 \text{ such that} \\
\qquad |p_\mu(r_1) - p_\mu(r_2)| \le L_\mu |r_1 - r_2| \quad \forall r_1,\, r_2 \in \mathbb{R}. \\
\text{(c) } (p_\mu(r_1) - p_\mu(r_2))(r_1 - r_2) \ge 0 \quad \forall r_1,\, r_2 \in \mathbb{R}. \\
\text{(d) } p_\mu(r) = 0 \ \text{ for all } r < 0.
\end{cases}
\tag{6.4}
$$

With these preliminaries, we consider the following contact problem.

Problem $\mathcal{P}_\mu$. *Find a displacement field* $\boldsymbol{u}_\mu : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma}_\mu : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ *such that, for all* $t \in \mathbb{R}_+$,

$$\boldsymbol{\sigma}_\mu(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)) + \int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(s)))\, ds \qquad \text{in} \quad \Omega, \tag{6.5}$$

$$\mathrm{Div}\, \boldsymbol{\sigma}_\mu(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \qquad \text{in} \quad \Omega, \tag{6.6}$$

$$\boldsymbol{u}_\mu(t) = \boldsymbol{0} \qquad \text{on} \quad \Gamma_1, \tag{6.7}$$

$$\boldsymbol{\sigma}_\mu(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \qquad \text{on} \quad \Gamma_2, \tag{6.8}$$

$$-\sigma_{\mu\nu}(t) = p_\mu(u_{\mu\nu}(t)) + \int_0^t b(t-s)\, u_{\mu\nu}^+(s)\, ds \qquad \text{on} \quad \Gamma_3, \tag{6.9}$$

$$\boldsymbol{\sigma}_{\mu\tau}(t) = \boldsymbol{0} \qquad \text{on} \quad \Gamma_3. \tag{6.10}$$

Note that here and below $u_{\mu\nu}$ is the normal component of the displacement field $\boldsymbol{u}_\mu$ and $\sigma_{\mu\nu}$, $\boldsymbol{\sigma}_{\mu\tau}$ represent the normal and tangential components of the stress tensor $\boldsymbol{\sigma}_\mu$, respectively. The equations and boundary conditions in problem (6.5)–(6.10) have a similar interpretations as those in problem (3.1)–(3.6). The difference arises in the fact that here we replace the contact condition with normal compliance, memory term and unilateral constraint (3.5) with the contact condition with normal compliance and memory term (6.9). In this condition μ represents a penalization parameter which may be interpreted as a deformability coefficient of the foundation, and then $\frac{1}{\mu}$ is the surface stiffness coefficient. Indeed, when μ is smaller the reaction force of the foundation to penetration is larger and so the same force will result in a smaller penetration, which means that the foundation is less deformable. When μ is larger the reaction force of the foundation to penetration is smaller, and so the foundation is less stiff and more deformable.

Assume now that (4.1), (4.2), (4.4), (4.5), (6.1) and (6.2) hold. Then using arguments similar to those used in the study of Problem $\mathcal{P}$ we obtain the following variational formulation of Problem $\mathcal{P}_\mu$.

Problem $\mathcal{P}_\mu^V$**.** *Find a displacement field* $\boldsymbol{u}_\mu : \mathbb{R}_+ \to V$ *such that the equality below holds, for all* $t \in \mathbb{R}_+$:

$$(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)),\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q+\Big(\int_0^t\mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(s))\,ds,\boldsymbol{\varepsilon}(\boldsymbol{v})\Big)_Q \tag{6.11}$$
$$+(p_\mu(u_{\mu\nu}(t)),v_\nu)_{L^2(\Gamma_3)}+\Big(\int_0^t b(t-s)u_{\mu\nu}^+(s)\,ds,v_\nu)\Big)_{L^2(\Gamma_3)}$$
$$=(\boldsymbol{f}_0(t),\boldsymbol{v})_{L^2(\Omega)^d}+(\boldsymbol{f}_2(t),\boldsymbol{v})_{L^2(\Gamma_2)^d}\quad\forall\,\boldsymbol{v}\in V.$$

Our main result in this section, which states unique solvability of Problem $\mathcal{P}_\mu^V$ and describes the behavior of its solution as $\mu\to 0$, is the following.

Theorem 6.1 *Assume that* (4.1), (4.2), (4.4), (4.5), (6.1) *and* (6.2) *hold. Then:*

1) *For each $\mu>0$ Problem $\mathcal{P}_\mu^V$ has a unique solution which satisfies $\boldsymbol{u}_\mu\in C(\mathbb{R}_+;V)$.*

2) *The solution $\boldsymbol{u}_\mu$ of the Problem $\mathcal{P}_\mu^V$ converges to the solution $\boldsymbol{u}$ of the Problem $\mathcal{P}^V$, that is*

$$\|\boldsymbol{u}_\mu(t)-\boldsymbol{u}(t)\|_V\to 0 \tag{6.12}$$

as $\mu\to 0$, for all $t\in\mathbb{R}_+$.

The proof of Theorem 6.1 is carried out in several steps that we present in what follows. In the rest of this section we suppose that the assumption of Theorem 6.1 hold and we denote by c a positive generic constant that may depend on time but does not depend on μ, and whose value may change from line to line. We use notation (4.12), (4.13) and (4.14). Moreover, condition (6.4) allows us to consider the operator $P_\mu:V\to V$ defined by

$$(P_\mu\boldsymbol{u},\boldsymbol{v})_V=\int_{\Gamma_3}p_\mu(u_\nu)v_\nu\,da\qquad\forall\,\boldsymbol{u},\,\boldsymbol{v}\in V. \tag{6.13}$$

Then, it is easy to see that Problem $\mathcal{P}_\mu^V$ is equivalent to the problem of finding a function $\boldsymbol{u}_\mu:\mathbb{R}_+\to V$ such that, for all $t\in\mathbb{R}_+$,

$$(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)),\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q+(P_\mu\boldsymbol{u}_\mu(t),\boldsymbol{v})_V+(\mathcal{S}\boldsymbol{u}_\mu(t),\boldsymbol{v})_V \tag{6.14}$$
$$=(\boldsymbol{f}(t),\boldsymbol{v})_V\quad\forall\,\boldsymbol{v}\in V.$$

For this reason, we start by proving the unique solvability of this variational equation.

Lemma 6.2 *There exists a unique solution $\boldsymbol{u}_\mu\in C(\mathbb{R}_+;V)$ which satisfies* (6.14), *for all $t\in\mathbb{R}_+$.*

Proof. We use Theorem 2.1 with $K = X = V$. Let $A_\mu : V \to V$ be the operator defined by

$$(A_\mu \boldsymbol{u}, \boldsymbol{v})_V = (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (P_\mu \boldsymbol{u}, \boldsymbol{v})_V \quad \forall\, \boldsymbol{u},\ \boldsymbol{v} \in V, \tag{6.15}$$

We use (4.1), (6.4) and (6.13) to see that A_μ is a strongly monotone and Lipschitz continuous operator i.e. it verifies condition (2.6). Therefore, it follows from Theorem 2.1 that there exists a unique function $\boldsymbol{u}_\mu \in C(\mathbb{R}_+; V)$ which satisfies the inequality

$$\begin{aligned}(A_\mu \boldsymbol{u}_\mu(t), \boldsymbol{v} - \boldsymbol{u}_\mu(t))_V + (\mathcal{S}\boldsymbol{u}_\mu(t), \boldsymbol{v} - \boldsymbol{u}_\mu(t))_V \\ \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}_\mu(t))_V \quad \forall\, \boldsymbol{v} \in V,\end{aligned}$$

for all $t \in \mathbb{R}_+$. We replace $\boldsymbol{v}$ with $\boldsymbol{u}_\mu(t) \pm \boldsymbol{v}$ to see that the previous inequality is equivalent to the variational equation

$$(A\boldsymbol{u}_\mu(t), \boldsymbol{v})_V + (\mathcal{S}\boldsymbol{u}_\mu(t), \boldsymbol{v})_V = (\boldsymbol{f}(t), \boldsymbol{v})_V \quad \forall\, \boldsymbol{v} \in V,$$

for all $t \in \mathbb{R}_+$. Therefore, using (6.15) we deduce that that there exists a unique function $\boldsymbol{u}_\mu \in C(\mathbb{R}_+; V)$ which satisfies the inequality (6.14) for all $t \in \mathbb{R}_+$, which concludes the proof. □

In the second step we consider the auxiliary problem of finding a displacement field $\tilde{\boldsymbol{u}}_\mu : \mathbb{R}_+ \to V$ such that, for all $t \in \mathbb{R}_+$,

$$\begin{aligned}(\mathcal{A}\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_\mu(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (P_\mu \tilde{\boldsymbol{u}}_\mu(t), \boldsymbol{v})_V + (\mathcal{S}\boldsymbol{u}(t), \boldsymbol{v})_V \\ = (\boldsymbol{f}(t), \boldsymbol{v})_V \quad \forall\, \boldsymbol{v} \in V.\end{aligned} \tag{6.16}$$

Note that the difference between problems (6.14) and (6.16) arises in the fact that in (6.16) the operator $\mathcal{S}$ is applied to a known function. We have the following existence and uniqueness result.

Lemma 6.3 *There exists a unique solution $\tilde{\boldsymbol{u}}_\mu \in C(\mathbb{R}_+; V)$ which satisfies* (6.16), *for all $t \in \mathbb{R}_+$.*

Proof. Besides the operator $A_\mu : V \to V$ defined by (6.15) we define the function $\tilde{\boldsymbol{f}} : \mathbb{R}_+ \to V$ by equality

$$(\tilde{\boldsymbol{f}}(t), \boldsymbol{v})_V = (\boldsymbol{f}(t), \boldsymbol{v})_V - (\mathcal{S}\boldsymbol{u}(t), \boldsymbol{v})_V \quad \forall\, \boldsymbol{v} \in V,\ t \in \mathbb{R}_+ \tag{6.17}$$

and we note that assumptions on $\boldsymbol{f}_0$, $\boldsymbol{f}_2$, $\mathcal{B}$ and b yield

$$\tilde{\boldsymbol{f}} \in C(\mathbb{R}_+; V). \tag{6.18}$$

Let $t \in \mathbb{R}_+$. Based on (6.15) and (6.17), it is easy to see that (6.16) is equivalent to equality

$$A_\mu \tilde{\boldsymbol{u}}_\mu(t) = \tilde{\boldsymbol{f}}(t). \tag{6.19}$$

Recall that A_μ is a strongly monotone and Lipschitz continuous operator. Therefore, by standard arguments we deduce the existence of a unique function $\tilde{\boldsymbol{u}}_\mu \in C(\mathbb{R}_+; V)$ such that (6.19) holds for all $t \in \mathbb{R}_+$, which concludes the proof. $\Box$

We proceed with the following convergence result.

Lemma 6.4 *As* $\mu \to 0$,

$$\tilde{\boldsymbol{u}}_\mu(t) \rightharpoonup \boldsymbol{u}(t) \quad \text{in } V,$$

for all $t \in \mathbb{R}_+$.

Proof. Let $t \in \mathbb{R}_+$. We take $\boldsymbol{v} = \tilde{\boldsymbol{u}}_\mu(t)$ in (6.16) to obtain

$$\begin{aligned}(\mathcal{A}\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_\mu(t)), \boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_\mu(t)))_Q + (P_\mu \tilde{\boldsymbol{u}}_\mu(t), \tilde{\boldsymbol{u}}_\mu(t))_V \\ +(\mathcal{S}\boldsymbol{u}(t), \tilde{\boldsymbol{u}}_\mu(t))_V = (\boldsymbol{f}(t), \tilde{\boldsymbol{u}}_\mu(t))_V.\end{aligned} \tag{6.20}$$

On the other hand, the properties (6.4) of the function p_μ yield

$$(P_\mu \tilde{\boldsymbol{u}}_\mu(t), \tilde{\boldsymbol{u}}_\mu(t))_V \geq 0. \tag{6.21}$$

We combine (6.20), (6.21) and use (4.1) to obtain that

$$\|\tilde{\boldsymbol{u}}_\mu(t)\|_V \leq c\,(\|\boldsymbol{f}(t)\|_V + \|\mathcal{S}\boldsymbol{u}(t)\|_V + \|\mathcal{A}\boldsymbol{0}\|_Q). \tag{6.22}$$

This inequality shows that the sequence $\{\tilde{\boldsymbol{u}}_\mu(t)\}_\mu \subset V$ is bounded. Hence, there exists a subsequence of the sequence $\{\tilde{\boldsymbol{u}}_\mu(t)\}_\mu$, still denoted $\{\tilde{\boldsymbol{u}}_\mu(t)\}_\mu$, and an element $\tilde{\boldsymbol{u}}(t) \in V$ such that

$$\tilde{\boldsymbol{u}}_\mu(t) \rightharpoonup \tilde{\boldsymbol{u}}(t) \quad \text{in } V. \tag{6.23}$$

It follows from (6.20) that

$$(P_\mu \tilde{\boldsymbol{u}}_\mu(t), \tilde{\boldsymbol{u}}_\mu(t))_V = (\boldsymbol{f}(t), \tilde{\boldsymbol{u}}_\mu(t))_V - (\mathcal{A}\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_\mu(t)), \boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_\mu(t)))_Q - (\mathcal{S}\boldsymbol{u}(t), \tilde{\boldsymbol{u}}_\mu(t))_V$$

and, since $\{\tilde{\boldsymbol{u}}_\mu(t)\}_\mu$ is a bounded sequence in V, using (4.1) we deduce that

$$(P_\mu \tilde{\boldsymbol{u}}_\mu(t), \tilde{\boldsymbol{u}}_\mu(t))_V \leq c.$$

This implies that

$$\int_{\Gamma_3} p_\mu(\tilde{u}_{\mu\nu}(t))\tilde{u}_{\mu\nu}(t)\, da \leq c$$

and, since p_μ and g are positive, it follows that

$$\int_{\Gamma_3} p_\mu(\tilde{u}_{\mu\nu}(t))(\tilde{u}_{\mu\nu}(t) - g)\, da \leq c. \tag{6.24}$$

We consider now the measurable subsets of Γ_3 defined by

$$\Gamma_{31} = \{\, \boldsymbol{x} \in \Gamma_3 \,:\, \tilde{u}_{\mu\nu}(t)(\boldsymbol{x}) \leq g \,\}, \qquad \Gamma_{32} = \{\, \boldsymbol{x} \in \Gamma_3 \,:\, \tilde{u}_{\mu\nu}(t)(\boldsymbol{x}) > g \,\}. \tag{6.25}$$

Clearly, both Γ_{31} and Γ_{32} depend on t and μ but, for simplicity, we do not indicate explicitly this dependence. We use (6.24) to write

$$\int_{\Gamma_{31}} p_\mu(\tilde{u}_{\mu\nu}(t))(\tilde{u}_{\mu\nu}(t) - g)\, da + \int_{\Gamma_{32}} p_\mu(\tilde{u}_{\mu\nu}(t))(\tilde{u}_{\mu\nu}(t) - g)\, da \le c$$

and, since

$$\int_{\Gamma_{31}} p_\mu(\tilde{u}_{\mu\nu}(t))\tilde{u}_{\mu\nu}(t)\, da \ge 0,$$

we obtain

$$\int_{\Gamma_{32}} p_\mu(\tilde{u}_{\mu\nu}(t))(\tilde{u}_{\mu\nu}(t) - g)\, da \le \int_{\Gamma_{31}} p_\mu(\tilde{u}_{\mu\nu}(t)) g\, da + c.$$

Thus, taking into account that $p_\mu(r) = p(r)$ for $r \le g$, by the monotonicity of the function p we can write

$$\int_{\Gamma_{32}} p_\mu(\tilde{u}_{\mu\nu}(t))(\tilde{u}_{\mu\nu}(t) - g)\, da \le \int_{\Gamma_{31}} p(\tilde{u}_{\mu\nu}) g\, da + c \le \int_{\Gamma_3} p(g) g\, da + c.$$

Therefore, we deduce that

$$\int_{\Gamma_{32}} p_\mu(\tilde{u}_{\mu\nu}(t))(\tilde{u}_{\mu\nu}(t) - g)\, da \le c. \tag{6.26}$$

We use now the definitions (6.3) and (6.25) to see that

$$p_\mu(\tilde{u}_{\mu\nu}(t)) = \frac{1}{\mu} q(\tilde{u}_{\mu\nu}(t)) + p(g), \quad p(g)(\tilde{u}_{\mu\nu}(t) - g) \ge 0 \quad \text{a.e. on } \Gamma_{32}.$$

Consequently, the inequality (6.26) yields

$$\int_{\Gamma_{32}} q(\tilde{u}_{\mu\nu}(t))(\tilde{u}_{\mu\nu}(t) - g)\, da \le c\mu. \tag{6.27}$$

Next, we consider the function defined by

$$\tilde{p} : \mathbb{R} \to \mathbb{R}_+ \quad \tilde{p}(r) = \begin{cases} 0 & \text{if} \quad r \le g, \\ q(r) & \text{if} \quad r > g \end{cases}$$

and we note that by (6.2) it follows that $\tilde{p}$ is a continuous increasing function and, moreover,

$$\tilde{p}(r) = 0 \qquad \text{iff} \qquad r \le g. \tag{6.28}$$

We use (6.27), equality $q(\tilde{u}_{\mu\nu}(t)) = \tilde{p}(\tilde{u}_{\mu\nu}(t))$ a.e. on Γ_{32} and (6.25) to deduce that

$$\int_{\Gamma_3} \tilde{p}(\tilde{u}_{\mu\nu}(t))(\tilde{u}_{\mu\nu}(t) - g)^+ \le c\mu,$$

where, recall, $(\tilde{u}_{\mu\nu}(t)-g)^+$ denotes the positive part of $\tilde{u}_{\mu\nu}(t)-g$. Therefore, passing to the limit as $\mu\to 0$, using (6.23) as well as compactness of the trace operator we find that

$$\int_{\Gamma_3} \tilde{p}(\tilde{u}_\nu(t))(\tilde{u}_\nu(t)-g)^+\,da \le 0.$$

Since the integrand $\tilde{p}(\tilde{u}_\nu(t))(\tilde{u}_\nu(t)-g)^+$ is positive a.e. on Γ_3, the last inequality yields

$$\tilde{p}(\tilde{u}_\nu(t))(\tilde{u}_\nu(t)-g)^+ = 0 \quad \text{a.e. on } \Gamma_3$$

and, using (6.28) and definition (4.6) we conclude that

$$\tilde{\boldsymbol{u}}(t)\in U. \tag{6.29}$$

Next, we test in (6.16) with $\boldsymbol{v}-\tilde{\boldsymbol{u}}_\mu(t)$, where $\boldsymbol{v}\in U$, to obtain

$$\begin{aligned}(\mathcal{A}\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_\mu(t)),\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_\mu(t)))_V+(P_\mu\tilde{\boldsymbol{u}}_\mu(t),\boldsymbol{v}-\tilde{\boldsymbol{u}}_\mu(t))_V\\ +(\mathcal{S}\boldsymbol{u}(t),\boldsymbol{v}-\tilde{\boldsymbol{u}}_\mu(t))_V=(\boldsymbol{f}(t),\boldsymbol{v}-\tilde{\boldsymbol{u}}_\mu(t))_V.\end{aligned} \tag{6.30}$$

Since $\boldsymbol{v}\in U$ we have $p_\mu(v_\nu)=p(v_\nu)$ a.e. on Γ_3. Thus, taking into account the monotonicity of the function p_μ yields

$$p(v_\nu)(v_\nu-\tilde{u}_{\mu\nu}(t))\ge p_\mu(\tilde{u}_{\mu\nu}(t))(v_\nu-\tilde{u}_{\mu\nu}(t)) \quad \text{a.e. on } \Gamma_3$$

and, therefore, we obtain

$$(P\boldsymbol{v},\boldsymbol{v}-\tilde{\boldsymbol{u}}_\mu(t))_V\ge(P_\mu\tilde{\boldsymbol{u}}_\mu(t),\boldsymbol{v}-\tilde{\boldsymbol{u}}_\mu(t))_V. \tag{6.31}$$

Then, using (6.31) and (6.30) we find that

$$\begin{aligned}(\mathcal{A}\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_\mu(t)),\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_\mu(t)))_Q+(P\boldsymbol{v},\boldsymbol{v}-\tilde{\boldsymbol{u}}_\mu(t))_V\\ +(\mathcal{S}\boldsymbol{u}(t),\boldsymbol{v}-\tilde{\boldsymbol{u}}_\mu(t))_V\ge(\boldsymbol{f}(t),\boldsymbol{v}-\tilde{\boldsymbol{u}}_\mu(t))_V \quad \forall\,\boldsymbol{v}\in U.\end{aligned} \tag{6.32}$$

We take $\boldsymbol{v}=\tilde{\boldsymbol{u}}(t)$ in (6.32) to obtain

$$\begin{aligned}(\mathcal{A}\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_\mu(t)),\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_\mu(t))-\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}(t)))_Q\le(P\tilde{\boldsymbol{u}}(t),\tilde{\boldsymbol{u}}(t)-\tilde{\boldsymbol{u}}_\mu(t))_V\\ +(\mathcal{S}\boldsymbol{u}(t),\tilde{\boldsymbol{u}}(t)-\tilde{\boldsymbol{u}}_\mu(t))_V+(\boldsymbol{f}(t),\tilde{\boldsymbol{u}}_\mu(t)-\tilde{\boldsymbol{u}}(t))_V\end{aligned} \tag{6.33}$$

and, passing to the upper limit as $\mu\to 0$, by (6.23) we find that

$$\limsup_{\mu\to 0}\,(\mathcal{A}\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_\mu(t)),\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_\mu(t))-\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}(t)))_Q\le 0.$$

Therefore, by a pseudomonotonicity argument is follows that

$$\begin{aligned}\liminf_{\mu\to 0}\,(\mathcal{A}\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_\mu(t)),\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_\mu(t))-\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q\\ \ge(\mathcal{A}\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}(t)),\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}(t))-\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q \quad \forall\,\boldsymbol{v}\in V.\end{aligned} \tag{6.34}$$

We use now (6.32) to see that

$$\begin{aligned}(P\boldsymbol{v},\boldsymbol{v}-\tilde{\boldsymbol{u}}_\mu(t))_V+(\mathcal{S}\boldsymbol{u}(t),\boldsymbol{v}-\tilde{\boldsymbol{u}}_\mu(t))_V+(\boldsymbol{f}(t),\tilde{\boldsymbol{u}}_\mu(t)-\boldsymbol{v})_V\\ \geq(\mathcal{A}\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_\mu(t)),\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_\mu(t))-\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q\quad\forall\,\boldsymbol{v}\in U,\end{aligned}$$

then we pass to the lower limit in this inequality and use (6.34) and (6.23) to find that

$$\begin{aligned}(\mathcal{A}\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}(t)),\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}(t)))_Q+(P\boldsymbol{v},\boldsymbol{v}-\tilde{\boldsymbol{u}}(t))_V\\ +(\mathcal{S}\boldsymbol{u}(t),\boldsymbol{v}-\tilde{\boldsymbol{u}}(t))_V\geq(\boldsymbol{f}(t),\boldsymbol{v}-\tilde{\boldsymbol{u}}(t))_V\quad\forall\,\boldsymbol{v}\in U.\end{aligned}\tag{6.35}$$

Next, we take $\boldsymbol{v}=\boldsymbol{u}(t)$ in (6.35) and $\boldsymbol{v}=\tilde{\boldsymbol{u}}(t)$ in (4.15). Then, adding the resulting inequalities we obtain

$$(\mathcal{A}\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}(t))-\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}(t)-\boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q\leq 0.$$

This inequality combined with (4.1) implies that

$$\tilde{\boldsymbol{u}}(t)=\boldsymbol{u}(t).$$

It follows from here that the whole sequence $\{\tilde{\boldsymbol{u}}_\mu(t)\}_\mu$ is weakly convergent to the element $\boldsymbol{u}(t)\in V$, which concludes the proof. □

We proceed with the following strong convergence result.

Lemma 6.5 *As* $\mu\to 0$,

$$\|\tilde{\boldsymbol{u}}_\mu(t)-\boldsymbol{u}(t)\|_V\to 0,$$

for all $t\in\mathbb{R}_+$.

Proof. Let $t\in\mathbb{R}_+$. Using (4.1) we write

$$\begin{aligned}m_{\mathcal{A}}\|\tilde{\boldsymbol{u}}_\mu(t)-\boldsymbol{u}(t)\|_V^2&\leq(\mathcal{A}\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_\mu(t))-\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_\mu(t))-\boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q\\ &=(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\boldsymbol{\varepsilon}(\boldsymbol{u}(t))-\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_\mu(t)))_Q-(\mathcal{A}\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_\mu(t)),\boldsymbol{\varepsilon}(\boldsymbol{u}(t))-\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_\mu(t)))_Q.\end{aligned}$$

Next, we take $\boldsymbol{v}=\boldsymbol{u}(t)$ in (6.32) to obtain

$$\begin{aligned}-(\mathcal{A}\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_\mu(t)),\boldsymbol{\varepsilon}(\boldsymbol{u}(t))-\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_\mu(t)))_Q\leq(P\boldsymbol{u}(t),\boldsymbol{u}(t)-\tilde{\boldsymbol{u}}_\mu(t))_V\\ +(\mathcal{S}\boldsymbol{u}(t),\boldsymbol{u}(t)-\tilde{\boldsymbol{u}}_\mu(t)))_V-(\boldsymbol{f}(t),\boldsymbol{u}(t)-\tilde{\boldsymbol{u}}_\mu(t))_V\end{aligned}$$

and, combining the previous two inequalities, we find that

$$\begin{aligned}m_{\mathcal{A}}\|\tilde{\boldsymbol{u}}_\mu(t)-\boldsymbol{u}(t)\|_V^2\leq(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\boldsymbol{\varepsilon}(\boldsymbol{u}(t))-\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_\mu(t)))_Q+(P\boldsymbol{u}(t),\boldsymbol{u}(t)-\tilde{\boldsymbol{u}}_\mu(t))_V\\ +(\mathcal{S}\boldsymbol{u}(t),\boldsymbol{u}(t)-\tilde{\boldsymbol{u}}(t))_V-(\boldsymbol{f}(t),\boldsymbol{u}(t)-\tilde{\boldsymbol{u}}_\mu(t))_V.\end{aligned}$$

We pass to the upper limit in this inequality and use Lemma 6.4 to conclude that $\tilde{\boldsymbol{u}}_\mu(t)\to\boldsymbol{u}(t)$ in V, as $\mu\to 0$. □

We are now in position to provide the proof of Theorem 6.1.

Proof. 1) Is easy to see that Problem $\mathcal{P}_\mu^V$ is equivalent to the problem of finding a function $\boldsymbol{u}_\mu : \mathbb{R}_+ \to V$ such that, for all $t \in \mathbb{R}_+$, (6.14) holds. Therefore, the existence of a unique solution $\boldsymbol{u}_\mu \in C(\mathbb{R}_+; V)$ to Problem $\mathcal{P}_\mu^V$ is a direct consequence of Lemma 6.2.

2) Let $t \in \mathbb{R}_+$ and let $n \in \mathbb{N}^*$ be such that $t \in [0, n]$. Let also $\mu > 0$. Then, testing with $\boldsymbol{v} = \boldsymbol{u}_\mu(t) - \tilde{\boldsymbol{u}}_\mu(t)$ in (6.16) and (6.14) we have

$$\begin{aligned}(\mathcal{A}\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_\mu(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)) - \boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_\mu(t)))_Q + (P_\mu \tilde{\boldsymbol{u}}_\mu(t), \boldsymbol{u}_\mu(t) - \tilde{\boldsymbol{u}}_\mu(t))_V \\ +(\mathcal{S}\boldsymbol{u}(t), \boldsymbol{u}_\mu(t)) - \tilde{\boldsymbol{u}}_\mu(t))_V = (\boldsymbol{f}(t), \boldsymbol{u}_\mu(t) - \tilde{\boldsymbol{u}}_\mu(t))_V,\end{aligned}$$

$$\begin{aligned}(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)) - \boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_\mu(t)))_V + (P_\mu \boldsymbol{u}_\mu(t), \boldsymbol{u}_\mu(t) - \tilde{\boldsymbol{u}}_\mu(t))_V \\ +(\mathcal{S}\boldsymbol{u}_\mu(t), \boldsymbol{u}_\mu(t)) - \tilde{\boldsymbol{u}}_\mu(t))_V = (\boldsymbol{f}(t), \boldsymbol{u}_\mu(t) - \tilde{\boldsymbol{u}}_\mu(t))_V.\end{aligned}$$

We subtract the previous inequalities and use the monotonicity of the operator P_μ to deduce that

$$\begin{aligned}(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)) - \mathcal{A}\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_\mu(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)) - \boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_\mu(t)))_Q \\ \le (\mathcal{S}\boldsymbol{u}(t) - \mathcal{S}\boldsymbol{u}_\mu(t), \boldsymbol{u}_\mu(t)) - \tilde{\boldsymbol{u}}_\mu(t))_V\end{aligned}$$

and, therefore,

$$\|\boldsymbol{u}_\mu(t) - \tilde{\boldsymbol{u}}_\mu(t)\|_V \le \frac{1}{m_{\mathcal{A}}} \|\mathcal{S}\boldsymbol{u}(t) - \mathcal{S}\boldsymbol{u}_\mu(t)\|_V. \tag{6.36}$$

We combine now (6.36) and with (4.17), (4.18) to find that

$$\|\boldsymbol{u}_\mu(t) - \tilde{\boldsymbol{u}}_\mu(t)\|_V \le \frac{r_n}{m_{\mathcal{A}}} \int_0^t \|\boldsymbol{u}(s) - \boldsymbol{u}_\mu(s)\|_V \, ds.$$

It follows from here that

$$\|\boldsymbol{u}_\mu(t) - \boldsymbol{u}(t)\|_V \le \|\tilde{\boldsymbol{u}}_\mu(t) - \boldsymbol{u}(t)\|_V + \frac{r_n}{m_{\mathcal{A}}} \int_0^t \|\boldsymbol{u}_\mu(s) - \boldsymbol{u}(s)\|_V ds$$

and, using a Gronwall's argument, we obtain

$$\|\boldsymbol{u}_\mu(t) - \boldsymbol{u}(t)\|_V \le \|\tilde{\boldsymbol{u}}_\mu(t) - \boldsymbol{u}(t)\|_V + \frac{r_n}{m_{\mathcal{A}}} \int_0^t e^{\frac{r_n}{m_{\mathcal{A}}}(t-s)} \|\tilde{\boldsymbol{u}}_\mu(s) - \boldsymbol{u}(s)\|_V \, ds. \tag{6.37}$$

Note that $e^{\frac{r_n}{m_{\mathcal{A}}}(t-s)} \le e^{\frac{r_n}{m_{\mathcal{A}}}t} \le e^{\frac{nr_n}{m_{\mathcal{A}}}}$ for all $s \in [0, t]$ and, therefore, (6.37) yields

$$\|\boldsymbol{u}_\mu(t) - \boldsymbol{u}(t)\|_V \le \|\tilde{\boldsymbol{u}}_\mu(t) - \boldsymbol{u}(t)\|_V + \frac{r_n}{m_{\mathcal{A}}} e^{\frac{nr_n}{m_{\mathcal{A}}}} \int_0^t \|\tilde{\boldsymbol{u}}_\mu(s) - \boldsymbol{u}(s)\|_V \, ds. \tag{6.38}$$

On the other hand, by estimate (6.22), Lemma 6.5 and Lebesgue's convergence theorem it follows that

$$\int_0^t \|\tilde{\boldsymbol{u}}_\mu(s) - \boldsymbol{u}(s)\|_V \, ds \to 0 \qquad \text{as} \quad \mu \to 0. \tag{6.39}$$

We use now (6.38), (6.39) and Lemma 6.5 to obtain the convergence (6.12), which concludes the proof. $\square$

We extend now the convergence result in Theorem 6.1 to the weak solution of the corresponding contact problems $\mathcal{P}$ and $\mathcal{P}_\mu$. Let $n \in \mathbb{N}^*$ be such that $t \in [0, n]$. Then, using (6.5) and (3.1) we obtain

$$\|\boldsymbol{\sigma}_\mu(t) - \boldsymbol{\sigma}(t)\|_Q \leq \|\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)) - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t))\|_Q$$
$$+ \int_0^t \|\mathcal{B}(t-s)(\boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))\|_Q$$

and, using (4.1) and arguments similar to those used to obtain (4.17) it follows that

$$\|\boldsymbol{\sigma}_\mu(t) - \boldsymbol{\sigma}(t)\|_Q \leq c\,\|\boldsymbol{u}_\mu(t) - \boldsymbol{u}(t))\|_V \tag{6.40}$$
$$+d \max_{r\in[0,n]} \|\mathcal{B}(r)\|_{\mathbf{Q}_\infty} \int_0^t \|\boldsymbol{u}_\mu(s) - \boldsymbol{u}(s)\|_V\, ds.$$

Moreover, taking $\boldsymbol{v} = \boldsymbol{u}_\mu(t)$ in (6.14) and using the monotonicity of P_μ and $\mathcal{A}$ we find that
$$\|\boldsymbol{u}_\mu(t)\|_V \leq c\,\big(\|\boldsymbol{f}(t)\|_V + \|\mathcal{S}\boldsymbol{u}_\mu(t)\|_Q.$$
We use now the property (4.17) of the operator $\mathcal{S}$ and the Gronwall argument to see that
$$\|\boldsymbol{u}_\mu(t)\|_V \leq c_n, \tag{6.41}$$
where c_n is a positive constant which depends on n, $\mathcal{B}$ and b. Then, we use the convergence (6.12), estimate (6.41) and Lebesque's theorem, again, and pass to the limit in (6.40). As a result we find that

$$\|\boldsymbol{\sigma}_\mu(t) - \boldsymbol{\sigma}(t)\|_Q \to 0 \qquad \text{as} \quad \mu \to 0. \tag{6.42}$$

Finally, since (4.20) implies that $\mathrm{Div}\,\boldsymbol{\sigma}_\mu(t) = \mathrm{Div}\,\boldsymbol{\sigma}(t) = -\boldsymbol{f}_0(t)$, we conclude that

$$\|\boldsymbol{\sigma}_\mu(t) - \boldsymbol{\sigma}(t)\|_{Q_1} = \|\boldsymbol{\sigma}_\mu(t) - \boldsymbol{\sigma}(t)\|_Q$$

and, therefore, (6.42) yields

$$\|\boldsymbol{\sigma}_\mu(t) - \boldsymbol{\sigma}(t)\|_{Q_1} \to 0 \qquad \text{as} \quad \mu \to 0. \tag{6.43}$$

In addition to the mathematical interest in the convergence result (6.12), (6.43), it is important from the mechanical point of view, since it shows that the weak solution of the viscoelastic contact problem with normal compliance memory term and unilateral constraint may be approached as closely as one wishes by the solution of the viscoplastic contact problem with normal compliance and memory term, with a sufficiently small deformability coefficient.

A brief comparison between the convergence results (5.7), (5.28) on one hand, and the convergence results (6.12), (6.43) on the other hand, show that the convergences (5.7), (5.28) hold in the Fréchet spaces $C(\mathbb{R}_+;V)$ and $C(\mathbb{R}_+;Q_1)$, respectively, and, in contrast, the convergences (6.12), (6.43) hold in the spaces V and Q_1, respectively, at each $t \in \mathbb{R}_+$. This feature arises from the mathematical tools we use on the proof of Theorem 6.1. The extension of (6.12), (6.43) to convergence results on the spaces $C(\mathbb{R}_+;V)$ and $C(\mathbb{R}_+;Q_1)$ remain an open problem which deserves to be investigated in the future.

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