Mixed convection boundary layer flow from a vertical truncated cone in a nanofluid

F.O. Pătrulescu[a], T. Groşan[b*] I. Pop[b]

[a]"Tiberiu Popoviciu" Institute of Numerical Analysis of Romanian Academy, Cluj-Napoca, Romania
[b] Babeş -Bolyai University, Department of Mathematics, Cluj-Napoca, Romania

Abstract

Purpose – In this paper we investigate the steady mixed convection boundary layer flow from a vertical frustum of a cone in water-based nanofluids. The problem is formulated to incorporate three kinds of nanoparticles: Copper (Cu), Alumina (Al_2O_3) and Titanium oxide (TiO_2). The working fluid is chosen as water with the Prandtl number of $\mathrm{Pr} = 6.2$. The mathematical model used for the nanofluid incorporates the particle volume fraction parameter, the effective viscosity and the effective thermal diffusivity. The entire regime of the mixed convection includes the mixed convection parameter λ, which is positive for the assisting flow (heated surface of the frustum cone) and negative for the opposing flow (cooled surface of the frustum cone), respectively.

Design/methodology/approach – The transformed nonlinear partial differential equations are solved numerically for some values of the governing parameters. The derivatives with respect to ξ were discretized using the first order upwind finite differences and the resulting ordinary differential equations with respect to η were solved using *bvp4c* routine from Matlab. The absolute error tolerance in *bvp4c* was 1e-9.

Findings – The features of the flow and heat transfer characteristics for different values of the governing parameters are analysed and discussed. The effects of the particle volume fraction parameter ϕ, the mixed convection parameter λ and the dimensionless coordinate ξ on the flow and heat transfer characteristics are determined only for the Cu nanoparticles. It is found that dual solutions exist for the case of opposing flows. The range of the mixed convection parameter for which the solution exists increases in the presence of the nanofluids.

Originality/value – The paper models the mixed convection from a vertical truncated cone using the boundary layer approximation. Multiple (dual) solutions for the flow reversals are obtained and the range of existence of the solutions was found.Particular cases for $\xi = 0$(full cone), $\xi >> 1$ and $\lambda >> 1$ (free convection limit) were studied.To the authors best knowledge this problem has not been studied before and the results are new and original.

Keywords Truncated cone, Boundary layer, Mixed convection, Nanofluids, Numerical results, Dual solutions

Paper type Research paper

***Corresponding author**
Dr. T. Grosan can be contacted at: tgrosan@math.ubbcluj.ro

Nomenclature

C_f	skin friction coefficient

C_p	specific heat at constant pressure
$f(\eta)$	similarity variable
g	acceleration due to gravity
Gr_x	local Grashof number
k	thermal conductivity of the nanofluid
Nu_x	local Nusselt number
Pr	Prandtl number
q_w	heat flux from the surface of the cone
r	radial distance
Re_x	local Reynolds number
T	temperature of the nanofluid
T_w	wall temperature
T_∞	temperature of the ambient nanofluid
u, v	velocity components along x and y directions, respectively
U_∞	characteristic velocity
x, y	Cartesian coordinates measured along the surface of the truncated cone and normal to it, respectively
x_0	distance measured from the leading edge of the truncated cone

Greek letters

α	thermal diffusivity
β	thermal expansion coefficient
φ	particle volume fraction
η	similarity variable
λ	mixed convection parameter
λ_{cr}	critical value of λ
μ	dynamic viscosity
ν	kinematic viscosity
$\theta(\eta)$	dimensionless temperature
ρ	density
τ_w	skin friction or shear stress
ξ	dimensionless variable
ψ	stream function

Subscripts

nf	nanofluid
f	base fluid
s	solid particle

1. Introduction

Convective heat transfer is present in many engineering processes such as thermal powerplants, heat exchangers, environmental comfort, energy storage systems and electronic cooling. Fluids such as oil,water and ethylene glycol mixture are poor heat transfer fluids, sincethe thermal conductivity of these fluids, which is generally small, play important role on theheat transfer coefficient between the heat transfer medium andthe heat transfer surface. The termnanofluid refersto a liquid containing (e.g. water, ethylene glycol, engine oils, etc.)a suspension of solid particles (nanoparticles) in conventional heat transfer fluids. It seems that Choi (1995) is the first who used the term nanofluids to refer to the fluid with suspended nanoparticles. Nanoparticles can be metallic particles such as those of Cu, Ag, Au, etc. or metallic oxides or non-metallic oxide particles: CuO, Al_2O_3, TiO, SiO , etc. having dimensions in the range 1 to 100nm. Significant features of nanofluids over base fluids include enhanced thermal conductivity, greater viscosity, and enhanced value of critical heat flux. Of these the most talked about is the enhanced thermal conductivity, a phenomenon which was first reported by Masuda et al. (1993), Ding et al. (2007) reported the use of nanofluids in a wide variety of industries ranging from transportation, heating, ventilation, and air conditioning(HVAC), and energy production and supply to electronics, textiles, paper production, etc. It is worth to mention that there are several patents which use nanofluids in heat transfer problemssuch as heat pumps and solar energy collectors (see,Laforgia et al. 2013 and Olson, 2013).Choi et al. (2001) showed that the addition of a small amount (less than 1% by volume) of nanoparticles to conventional heat transfer liquids increased the thermal conductivity of the fluid up to approximately two times. Therefore, the effective thermal conductivityof nanofluids is expected to enhance heat transfer compared withconventional heat transfer liquids (Masuda et al., 1993). All of these industries deal with heat transfer in some or the other way, and thus have a strong need for improved heat transfer mediums. This could possiblybe nanofluids, because of some potential benefits over normal fluids− large surface area provided by nanoparticles for heat exchange, reduced pumping power due to enhanced heat transfer, minimal clogging, innovation of miniaturized systems leading to savings of energy and cost. Choi's (1995) results have been supported by other researchersfrom time to time. Eastman et al. (2001) reported an increase of 40% in the effectivethermal conductivity of ethylene-glycol with 0.3% volume of copper nanoparticles of10nm diameter. Further 10-30% increase of the effective thermal conductivity in alumina/water nanofluids with 1-4% of alumina was reported by Das et al. (2003). Buongiorno and Hu(2005)suggested the possibility of using nanofluids inadvanced nuclear

systems.Another recent application of the nanofluid flow is in thedelivery of nano-drug as suggested by Kleinstreuer et al. (2008).

A comprehensive survey of convectivetransport in nanofluids wasmade by Buongiorno(2006),Choi (2009), Kakaç and Pramuanjaroenkij(2009), Das and Choi (2009),Fan and Wang (2011), etc.We have alsotomentionthe valuable published books by Das et al. (2007) and Schaefer (2009).Buongiorno(2006) haspointed out that the nanoparticle absolute velocity can be viewed asthe sum of the base fluid velocity and a relative velocity (that hecalls the slip velocity). He considered in turn seven slip mechanisms:inertia, Brownian diffusion, thermophoresis, diffusiophoresis,Magnuseffect, fluid drainage, and gravity settling (Nield and Kuznetsov, 2009).Numerous models and methods have beenproposed by different authors to study convective flows of nanofluidsand we mention here the papers by Khanafer et al. (2003), Tiwari andDas (2007),Oztop and Abu-Nada (2008),Rohni et al. (2009), Mansur et al. (2009), Nield and Kuznetsov(2010), Kuznetsov andNield (2010), Tham et al. (2012),Aminossadati and Ghasemi (2012),etc.

The problem of viscous boundary-layer flowon a full and truncated cone is a classical problem, and it has been considered by manyresearchers, for example Chiou and Na (1980), Kumari et al. (1989), Chamkha (2001), Molla et al. (2009), Mahdy et al. (2010), Chamkhaand Rashad (2012), Chamkha et al. (2012), etc. Engineering applications of this particular geometry (truncated cone) are in the field of heat exchangers, cooling of electronic devices, etc. (seeShinmura, 1996; Hamilton et al. 1999; McCutcheon et al., 2005 and Nakamura et al., 2010).

Differentfrom the previous investigations, following the nanofluid equationsmodel proposed by Tiwari and Das (2007), we consider in thispaper the development of the steady mixed convection boundary-layer flowon a vertical impermeablefrustum of a cone in a nanofluid. The problem is formulated so that we can consider three different types of naoparticles, namely Cu (cooper), Al_2O_3 (alumina) and TiO_2(titania), and water as a base fluid. However, in order to save space, we have considered here only the case of Cu nanoparticles. The motivation for this study is that nanotechnologyhas been widely used in industry because materials withsizes of nanometers possess unique physical properties. The dimensionless governing partial differential equations have been solved numerically and the results for the particular case of the full cone have been compared with results reported by Kumari et al. (1989). The results are in very good agreement.

2. Basic equations

Consider the steady mixed boundary layer flow near a vertical truncated cone (with half angle ϕ) as shown in Fig. 1 in a water-based nanofluid. The origin O of the coordinate system is placed at the vertex of the full cone, where x is the co-ordinatemeasured from the origin O along the surface of the full cone and y is the coordinate normal to the surface of the truncated cone, respectively. The distance of the leading edge of the truncated cone measured from the origin O is denoted as x_0. The surface of the truncated cone is held at a constant temperature T_w, while the temperature of the ambient nanofluid is T_∞ with $T_w > T_\infty$ for a heated truncated cone (assisting flow) and $T_w < T_\infty$ for a cooled truncated cone (opposing flow), respectively.The velocity far away from the truncated cone in the vertical direction is U_∞ and the viscous dissipation is assumed to be negligible.

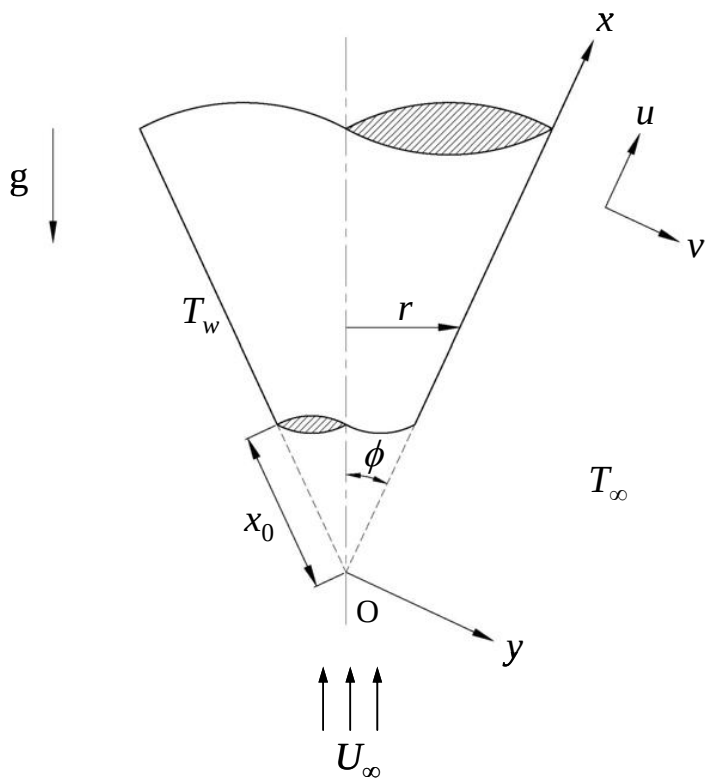

Figure 1.Physical model and coordinate system.

Under the above assumptions along with the assumptions of Boussinesq and boundary layer approximations, the basic equations governing the steady mixed convection boundary layer flow of a

nanofluidnear a vertical frustum of a cone can be written as in Chamkha and Rashad (2012) or Pop and Ingham (2001) combined with the mathematical nanofluid model given by Tiwari and Dass(2007):

$$\frac{\partial}{\partial x}(ru)+\frac{\partial}{\partial y}(rv)=0 \quad (1)$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\mu_{\mathrm{nf}}}{\rho_{\mathrm{nf}}}\frac{\partial^2 u}{\partial y^2}+\frac{(\rho\beta)_{\mathrm{nf}}}{\rho_{\mathrm{nf}}}g\,(T-T_\infty)\sin\phi \quad (2)$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha_{nf}\frac{\partial^2 T}{\partial y^2} \quad (3)$$

along with the boundary conditions

$$\begin{aligned} &v=0, \quad u=0, \quad T=T_w \quad \text{at} \quad y=0 \\ &u=U_\infty, \quad T=T_\infty \quad \text{as} \quad y\to\infty \end{aligned} \quad (4)$$

where u and v are the velocity components along x and y axes, respectively, T is the temperature of the nanofluid, g is the gravity acceleration, U_∞ is the constant velocity of the outer (inviscid) flow, $r=x\sin\phi$, ρ_{nf} is the effective density, μ_{nf} is the effective dynamic viscosity, $(\rho\beta)_{nf}$ is the thermal expansion coefficient and α_{nf} is the thermal diffusivity of the nanofluid, which are given as (see Khanafer et al., 2003, and Oztop and Abu-Nada, 2008),

$$\begin{aligned} &\rho_{nf}=(1-\varphi)\rho_f+\varphi\rho_s, \qquad \mu_{nf}=\frac{\mu_f}{(1-\varphi)^{2.5}} \\ &(\rho\beta)_{nf}=(1-\varphi)(\rho\beta)_f+\varphi(\rho\beta)_s, \qquad \alpha_{nf}=\frac{k_{nf}}{(\rho C_p)_{nf}} \end{aligned} \quad (5)$$

Here, φ is the solid volume fraction, μ_f is the dynamic viscosity of the basic fluid, ρ_f and ρ_s are the densities of pure fluid and nanoparticles, respectively, $(\rho C_p)_{\mathrm{nf}}$ is the heat capacity of the nanofluidand k_{nf} is the thermal conductivity of the nanofluidgiven by

$$(\rho C_p)_{nf}=(1-\varphi)(\rho C_p)_f+\varphi(\rho C_p)_s, \quad \frac{k_{nf}}{k_f}=\frac{k_s+2k_f-2\varphi(k_f-k_s)}{k_s+2k_f+2\varphi(k_f-k_s)} \quad (6)$$

where $(\rho C_p)_f$ and $(\rho C_p)_s$ are the specific heat parameters of the base fluid and nanoparticles and k_f and k_s are the thermal conductivities of the base fluid and nanoparticles. The viscosity of the nanofluid μ_{nf} can be approximated as the viscosity of a base fluid μ_f containing dilute suspension of fine spherical particles and is given byBrinkman (1952). The effective thermal conductivity of the nanofluid

k_{nf} is approximated by the Maxwell-Garnett's model, which is found to be appropriate for studying heat transfer enhancement using nanofluids (Khanafer et al., 2003;Maïga et al., 2004, etc.).

Further, we introduce the following non-similarity variables

$$\xi = \bar{x}/x_0 = (x-x_0)/x_0, \quad \eta = \mathrm{Re}_x^{1/2}(y/\bar{x}), \quad \psi = \mathrm{Re}_x^{1/2}\nu_f \; r\, f(\xi,\eta)$$
$$\theta(\xi,\eta) = (T-T_\infty)/(T_w-T_\infty) \tag{7}$$

where $\bar{x} = x - x_0$, ν_f is the kinematic viscosity of the based fluid, $\mathrm{Re}_x = U_\infty \bar{x}/\nu_f$ is the local Reynold number and ψ is the stream function, which is defined as

$$ru = \frac{\partial \psi}{\partial y}, \quad rv = -\frac{\partial \psi}{\partial x} \tag{8}$$

Substituting (7) and (8) into Eqs. (2) and (3), we obtain the following partial differential equations of parabolic type

$$\frac{1}{(1-\varphi)^{2.5}\left[(1-\varphi)+\varphi\rho_s/\rho_f\right]}\frac{\partial^3 f}{\partial \eta^3} + \left(\frac{1}{2}+\frac{\xi}{1+\xi}\right) f \frac{\partial^2 f}{\partial \eta^2} + \frac{(1-\varphi)+\phi\,(\rho\beta)_s/(\rho\beta)_f}{(1-\varphi)+\varphi\rho_s/\rho_f}\lambda\,\theta = \xi\left(\frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \xi \partial \eta} - \frac{\partial f}{\partial \xi}\frac{\partial^2 f}{\partial \eta^2}\right) \tag{9}$$

$$\frac{1}{\mathrm{Pr}}\frac{k_{nf}/k_f}{(1-\varphi)+\varphi(\rho C_p)_s/(\rho C_p)_f}\frac{\partial^2 \theta}{\partial \eta^2} + \left(\frac{1}{2}+\frac{\xi}{1+\xi}\right) f \frac{\partial \theta}{\partial \eta} = \xi\left(\frac{\partial f}{\partial \eta}\frac{\partial \theta}{\partial \xi} - \frac{\partial f}{\partial \xi}\frac{\partial \theta}{\partial \eta}\right) \tag{10}$$

and the boundary conditions (6) become

$$f = 0, \quad \frac{\partial f}{\partial \eta} = 0, \quad \theta = 1 \quad \text{at} \quad \eta = 0$$
$$\frac{\partial f}{\partial \eta} = 1, \quad \theta = 0 \quad \text{as} \quad \eta \to \infty \tag{11}$$

where λ is the constant mixed convection parameter, which is given by

$$\lambda = \frac{Gr}{\mathrm{Re}_x^{\,2}} \tag{12}$$

with $Gr_x = g\beta_f (T_w - T_\infty) x^3 \sin\phi/\nu_f^{\,2}$ being the local Grashof number. It should be stated that $\lambda > 0$ ($T_w > T_\infty$) corresponds to a heated truncated cone (assisting flow), $\lambda < 0$ ($T_w < T_\infty$) corresponds to a

cooled truncated cone (opposing flow) and $\lambda = 0$ ($T_w = T_\infty$) corresponds to the forced convection flow along the truncated cone, respectively. Using (7) and (8), the velocity components u and v are given by

$$u = U_\infty \frac{\partial f}{\partial \eta}, \quad v = -\frac{\nu_f \, \mathrm{Re}_x^{1/2}}{x}\left[\left(\frac{1}{2} + \frac{\xi}{1+\xi}\right) f + \xi \frac{\partial f}{\partial \xi} - \frac{\eta}{2}\frac{\partial f}{\partial \eta}\right] \quad (13)$$

The quantities of physical interest are also the skin friction coefficient C_f and the local Nusselt number Nu_x which are defined as

$$C_f = \frac{2\tau_w}{\rho_f U_\infty^2}, \quad Nu_x = \frac{x\, q_w}{k_f (T_w - T_\infty)} \qquad (14)$$

where τ_w is the wall skin friction and q_w is the heat flux from the surface of the truncated cone, which are given by

$$\tau_w = \mu_{nf}\left(\frac{\partial u}{\partial y}\right)_{y=0}, \quad q_w = -k_{nf}\left(\frac{\partial T}{\partial y}\right)_{y=0} \qquad (15)$$

Using relations (7) in (15) and (14), we obtain

$$\mathrm{Re}_x^{1/2} C_f = \frac{2}{(1-\varphi)^{2.5}} \frac{\partial^2 f}{\partial \eta^2}(\xi,0), \quad \mathrm{Re}_x^{-1/2} Nu_x = -\frac{k_{nf}}{k_f}\frac{\partial \theta}{\partial \eta}(\xi,0) \quad (16)$$

The following two particular cases are also of interest:

i) Full cone ($\xi = 0$)

In this case, Eqs. (9) and (10) reduce to the following ordinary differential equations

$$\frac{1}{(1-\varphi)^{2.5}\left[(1-\varphi) + \varphi \rho_s / \rho_f\right]} f''' + \frac{1}{2} f f'' + \frac{(1-\varphi) + \varphi(\rho\beta)_s/(\rho\beta)_f}{(1-\varphi) + \varphi \rho_s / \rho_f} \lambda \theta = 0 \quad (17)$$

$$\frac{1}{\mathrm{Pr}} \frac{1}{(1-\varphi) + \varphi(\rho C_p)_s/(\rho C_p)_f} \theta'' + \frac{1}{2} f \theta' = 0 \quad (18)$$

subject to

$$f(0) = 0, \quad f'(0) = 0, \quad \theta(0) = 1, \quad f'(\infty) = 1, \quad \theta(\infty) = 0 \quad (19)$$

where primes denote differentiation with respect to η.

ii) **Case of ξ large** ($\xi >> 1$)

In this case Eqs. (9) and (10) reduce to

$$\frac{1}{(1-\varphi)^{2.5}\left[(1-\varphi)+\varphi\rho_s/\rho_f\right]}\, f''' + \frac{3}{2} f\, f'' + \frac{(1-\varphi)+\varphi(\rho\beta)_s/(\rho\beta)_f}{(1-\varphi)+\varphi\rho_s/\rho_f}\lambda\,\theta = 0 \quad (20)$$

$$\frac{1}{\Pr}\frac{1}{(1-\varphi)+\varphi(\rho C_p)_s/(\rho C_p)_f}\,\theta'' + \frac{3}{2} f\,\theta' = 0 \quad (21)$$

subjected to the boundary conditions (19). Also, the skin friction coefficient C_f and the local Nusselt number Nu_x become

$$\frac{1}{2}\mathrm{Re}_x^{1/2} C_f = \frac{1}{(1-\varphi)^{2.5}} f''(0,0), \quad \mathrm{Re}_x^{-1/2} Nu_x = -\frac{k_{nf}}{k_f}\theta'(0,0) \quad (22)$$

and

$$\frac{1}{2}\mathrm{Re}_x^{1/2} C_f = \frac{2}{(1-\varphi)^{2.5}} f''(\xi >> 1,\, 0), \quad \mathrm{Re}_x^{-1/2} Nu_x = -\frac{k_{nf}}{k_f}\theta'(\xi >> 1,\, 0) \quad (23)$$

3. Aiding case. The free convection limit $(\lambda >> 1)$

In this case, we limit the analysis to the case when ξ >> 1, the other case ξ = 0 (full cone) being similar. To discuss the behaviour of the solution of the boundary value problems (17-19) for $\lambda >> 1$, we introduce the following new variables

$$f(\eta) = \lambda^{1/4} F(z), \quad \theta(\eta) = \theta(z), \quad z = \lambda^{1/4}\eta \quad (24)$$

Substituting (24) into Eqs. (17) and (18), we obtain

$$\frac{1}{(1-\varphi)^{2.5}\left[(1-\varphi)+\varphi\rho_s/\rho_f\right]}\, F''' + \frac{3}{2} F\, F'' + \frac{(1-\varphi)+\varphi(\rho\beta)_s/(\rho\beta)_f}{(1-\varphi)+\varphi\rho_s/\rho_f}\,\theta = 0 \quad (25)$$

$$\frac{1}{\Pr}\frac{1}{(1-\varphi)+\varphi(\rho C_p)_s/(\rho C_p)_f}\,\theta'' + \frac{3}{2} F\,\theta' = 0 \quad (26)$$

and the boundary conditions (19) become

$$F(0) = 0, \quad F'(0) = 0, \quad \theta(0) = 1, \quad F'(\infty) = \lambda^{-1/2}, \quad \theta(\infty) = 0 \qquad (27)$$

whereprimes denote now the differentiation with respect to z .The boundary conditions (27) suggest an expansion for F and θ of the following form

$$F = F_0(z) + \lambda^{-1/2} F_1(z) + \mathrm{O}(\lambda^{-1}), \quad \theta = \theta_0(z) + \lambda^{-1/2}\theta_1(z) + \mathrm{O}(\lambda^{-1}) \quad (28)$$

where (F_0, θ_0) and (F_1, θ_1) are given by

$$\frac{1}{(1-\varphi)^{2.5}\left[(1-\varphi)+\varphi\rho_s/\rho_f\right]}F_0'''+\frac{3}{2}F_0F_0''+\frac{(1-\varphi)+\varphi(\rho\beta)_s/(\rho\beta)_f}{(1-\varphi)+\varphi\rho_s/\rho_f}\theta_0=0$$

$$\frac{1}{\Pr}\frac{1}{(1-\varphi)+\varphi(\rho C_p)_s/(\rho C_p)_f}\theta_0''+\frac{3}{2}F_0\theta_0'=0 \qquad (29)$$

$$F_0(0)=0,\quad F_0'(0)=0,\quad \theta_0(0)=1,\quad F_0'(\infty)=0,\quad \theta_0(\infty)=0$$

$$\frac{1}{(1-\varphi)^{2.5}\left[(1-\varphi)+\varphi\rho_s/\rho_f\right]}F_1'''+\frac{3}{2}(F_0F_1''+F_0''F_1)$$

$$+\frac{(1-\varphi)+\varphi(\rho\beta)_s/(\rho\beta)_f}{(1-\varphi)+\varphi\rho_s/\rho_f}\theta_1=0 \qquad (30)$$

$$\frac{1}{\Pr}\frac{1}{(1-\varphi)+\varphi(\rho C_p)_s/(\rho C_p)_f}\theta_1''+\frac{3}{2}(F_0\theta_1'+F_1\theta_0')=0$$

$$F_1(0)=0,\quad F_1'(0)=0,\quad \theta_1(0)=0,\quad F_1'(\infty)=1,\quad \theta_1(\infty)=0$$

Thus, from (22), we have

$$\frac{1}{2}\mathrm{Re}_x^{1/2}C_f=\frac{1}{(1-\varphi)^{2.5}}\lambda^{3/4}\left[F_0''(0)+\lambda^{-1/2}F_1''(0)+\mathrm{O}(\lambda^{-1})\right]$$

$$\mathrm{Re}_x^{-1/2}Nu_x=-\frac{k_{nf}}{k_f}\lambda^{1/4}\left[\theta_0'(0)+\lambda^{-1/2}\theta_1'(0)+\mathrm{O}(\lambda^{-1})\right] \qquad (31)$$

for $\lambda >> 1$. It should be mentioned that Eqs. (29) describe the problem of free convection from a full vertical cone in a nanofluid.

4. Results and discussion

As we have mentioned before, the problem is formulated so that we can consider different types of nanoparticles (e.g. Cu, Al_2O_3, TiO_2, etc.) and water as a base fluid. However, in order to save space, we have considered here only the case of Cu nanoparticles.The thermophysical properties of the base fluidand nanoparticles are listed in Table 1.Following Aminossadati and Ghasemi(2012), we considered the range of nanoparticle fraction parameter ϕ as $0 \le \phi \le 0.05$ and the Prandtl number of the base fluid (water) is kept constant at $\Pr = 6.7$. Itcan be stated that the present study reduces to that of a viscous (Newtonian) fluid when $\varphi = 0$.

The non-linear partial differential equations (9) and (10) with the boundary conditions (11) have beensolved numerically for several values of the governing parameters. The derivatives with respect to ξ were discretized using the first order upwind finitedifferences and the resulting ordinary differential equations with respect to ηwere solved using *bvp4c*routine from Matlab. For marching in ξ direction a step $\Delta\xi = 0.01$ has been used, while the absolute error tolerancesin *bvp4c* was 1e-9. Also the sets (17-19), (20,21) and (29,30) of ordinary differential equations were solved numerically using the same Matlab routine (*bvp4c*).We believe this is a better numerical method than the popular Keller box method used for solving such problems, especially in obtaining the multiple (dual) solutions. Therefore, in order to validate the present numerical method, wehave compared our results with those reported byKumari et al. (1989)fortwo values of Prwhen $\xi = 0$(full cone) $\lambda = 0$(forced convection)and $\phi = 0$(Newtonian fluid)as shown in Table 2. The comparisons arefound to be in very good agreement. However, in order to compare the results, we have to take in Kumari et al. (1989), $\xi = 1$and to use the transformation$\eta = \sqrt{3}\bar{\eta}$, $f(\eta) = \frac{1}{\sqrt{3}}\,\mathrm{F}(\bar{\eta})$and $-\theta(\eta) = \Theta(\bar{\eta})$.

Figures 2and 5 display the variation of the reduced skin friction coefficient $f''(0)$ and the reduced Nusselt number $-\theta'(0)$ for $\xi = 0$ (full cone) and $\xi >> 1$when $\lambda < 0$ (opposing flow) and $\phi = 0$ (regular fluid), 0.03 and 0.05.It was found that there are regions of uniquesolutions for $\lambda > 0$(assisting flow), dual (upper and lower branch) solutions exist for $\lambda < 0$(opposing flow). This happens because the forced flow and the flow due the buoyancy are in opposite directions. The solution for each value of ϕ exists up to a critical value of λ ($\lambda_c < 0$say) (opposing flow). Beyond this value we are unable to get the solution using the boundary layer approximations and the full Navier-Stokes equation has to be used. The upper curve for aparticular value of ϕends at λ_c (< 0) and the lower branchsolution continues further and terminates at a certain value of $\lambda < 0$. Itshould be remarked that the computations have been performed increasing λuntil the point where the solution does not converge($\lambda = \lambda_c < 0$), and the calculations were terminated at that point. In both cases of $\xi = 0$ and $\xi >> 1$, the values of $\lambda_c$decrease with the increases of the nanoparticle concentration parameterϕ. The values of $\lambda_c < 0$ are given in Table 3 for $\phi = 0, 0.03, 0.05$, and $\xi = 0, 5$ and $\xi >> 1$.

Figures 3and 6 present thevelocity and temperature (upper and lower branch) profilesfor $\xi = 0$(full cone) and $\xi >> 1$ when $\lambda = -0.2$ and $\varphi = 0, 0.03$and 0.05, which support the existence of the dual nature of the solutions presented inFigs.2 and 5.The velocity and thermalboundary layer thicknessesincrease

with the increases of the nanoparticle volume fraction parameter ϕ.This is due to the increase of the viscosity and thermal conductivity of the nanofluid with the increasing of the nanoparticles volume fraction, see Eqs. (5) and (6).A reverse flow is noticed from Figs. 3a and 6a in the case of the lower branch solution.It is worth mentioning that the lower branch solutions are unstable and the boundary layer thicknesses of these solutions are higher as for the upper branch solutions.This can be shown by performing a stability analysis, but this is out of the scope of the present paper. Such an analysis has been done by Weidman et al. (2006), Rosca and Pop (2013) and Trîmbițaș et al. (2013).

Furthermore, Figs. 4 ($\xi = 0$) and 7 ($\xi >> 1$) illustrate thevelocity and temperature profilesfor $\lambda > 0$ (assisting flow) and $\phi = 0, 0.03, 0.05$. It easily to notice thatnear the surface of the truncated cone there is an overshoot of the velocity profile for large values of $\lambda\ (> 0)$. This is in agreement with the physical phenomenon that for large values of $\lambda\ (> 0)$ the convective heat transfer dominates the conductive heat transferand buoyancy is much larger than convective flow. In addition, it is seen that the velocity profiles increase with λ, while the temperature profiles decrease with $\lambda\ (> 0)$, which is in agreement with Fig. 8.

Figure 8 displays the asymptotic behaviour of the reduced skin friction coefficient $f''(0)$ and reducedNusselt number $-\theta'(0)$ for $\lambda >> 1$ when $\xi >> 1$ and $\phi = 0, 0.03$ and 0.05. It is further observed that there is a very good agreement between the numerical solution of Eqs. (20) and (21) and the asymptotic solution of Eqs. (29) and (30). Reduced skin friction and Nusselt number decrease with ϕ and increase with λ.The values of the coefficients $F_0''(0)$, $F_1''(0)$, $\theta_0'(0)$ and $\theta_1'(0)$ involved in Eq. (31) are given in Table 4.

Finally, Fig. 9 presents the variation of velocity profiles and temperature profiles with ξ between the two limit cases: full cone ($\xi = 0$) and ξ large ($\xi >> 1$). One can see that both velocity and temperature profiles decrease with the increases of ξ and as a consequence the skin friction coefficient is lower and Nusselt number is higher for larger values of ξ (truncated cone), which is important in some practical applications.

5. Conclusions

The steady mixed convection boundary layer flow of a nanofluidover a vertical frustum of a cone has been theoretically studied.A frequently used nanofluid model based on formulas for thermal

conductivity and dynamic viscosity is considered.The working fluid ischosen as water with the Prandtl number of $\mathrm{Pr} = 6.7$. The effects of theparticle volume fraction parameter ϕ, the mixed convection parameter λ and the dimensionless coordinate ξ on the flow and heat transfer characteristics are determined for the Cu nanoparticles. From this investigation, some conclusions were summarized as follows:

a. The reduced skin friction coefficient $f''(\xi,0)$ increases due to increase in the parameter λ while it decreases due toincrease in particle volume fraction parameter ϕ when $\lambda > 0$.

b. The reduced local Nusselt number $-\theta'(\xi,0)$ is also influenced by the particle volume fraction parameter ϕ and the mixed convection parameter λ. Thus, it increases with the increase of λ and decreases with the increase of the volume fraction parameter ϕ for positive λ. The increasing volume fraction of Cu nanoparticle increases the thermal conductivity which results in increasing of the thermal boundary layer.

c. The nanofluidtemperature increases due to increasing in theparticle volume fraction parameter ϕ, while it decreases due to increase in the mixed convection parameter λ when assisting flow is considered.

d. Dual solutions were found in the opposing flow case in the region $\lambda \in [\lambda_c, 0]$. When $\lambda < \lambda_c$ it is not possible to find a solution if the boundary approximation model is used.

e. It was shown that the numerical method based on *bvp4c* from Matlab works very efficient for this kind of problems.

Acknowledgment

The work of the first author was supportedby the Sectorial Operational Programme for Human Resources Development 2007-2013, co-financed by theEuropean Social Fund, under the project number POSDRU/88/1.5/S/60185 withthe title “Modern Doctoral Studies: Internalization andInterdisciplinarity”. The work of the corresponding author was supported from the grant PN-II-RU-TE-2011-3-0013, UEFISCDI, Romania.

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Table 1.Thermophysical properties of fluid and nanoparticles(Oztop and Abu-Nada, 2008)

Physical properties	Fluid phase (water)	Cu	Al_2O_3	TiO_2
C_p (J/kg K)	4179	385	765	686.2
ρ (kg/m^3)	997.1	8933	3970	4250
k (W/mK)	0.613	400	40	8.9538
$\beta \times 10^{-5}$ (1/K)	21	1.67	0.85	0.9

Table 2. Skin friction and heat transfer coefficients for $\xi = 0,\ \lambda = 0$ and $\phi = 0$.

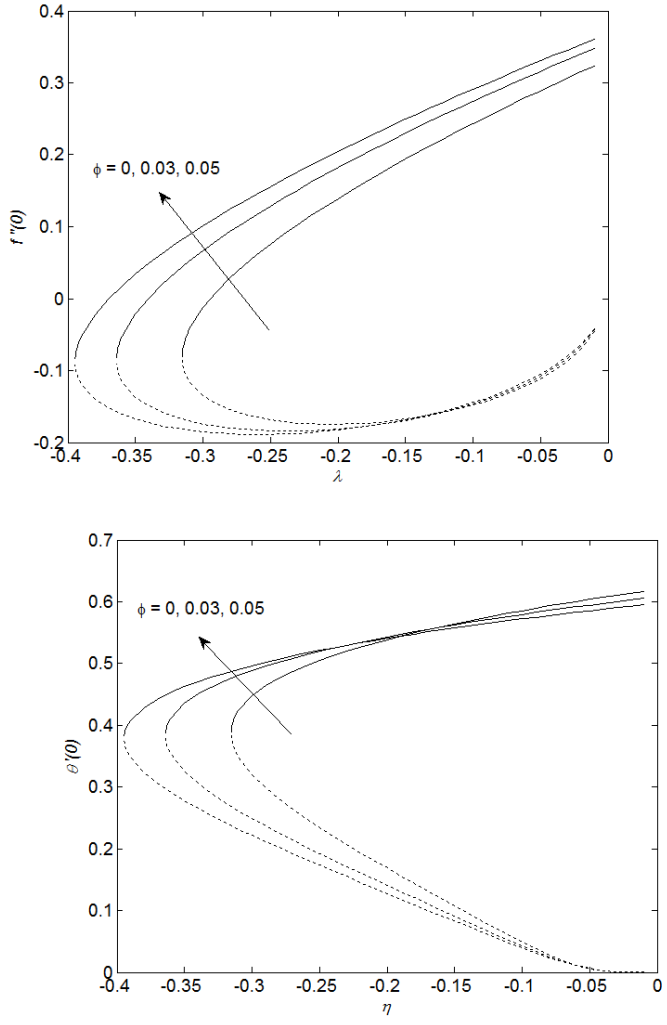
Pr	Kumari et al. (1989)		Present	
	$f''(0)$	$-\theta'(0)$	$f''(0)$	$-\theta'(0)$
0.733	1.1505	0.5158	1.150280	0.515329
6.7	1.1505	1.1025	1.150280	1.102402

Table 3. Values of λ_c.

ϕ	$\xi = 0$	$\xi = 5$	$\xi >> 1$
0	-0.3154	-0.8500	-0.9464
0.03	-0.3640	-1.0010	-1.0921
0.05	-0.3947	-1.1421	-1.1842

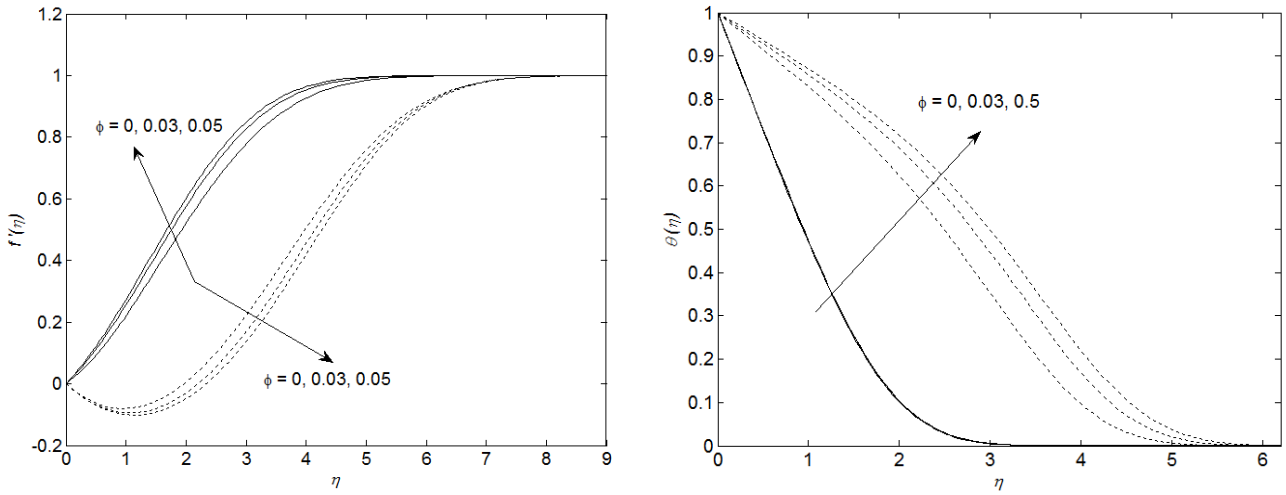
Table 4. Values of $F_0''(0)$, $F_1''(0)$, $\theta_0'(0)$ and $\theta_1'(0)$ for $\xi >> 1$.

ϕ	$F_0''(0)$	$F_1''(0)$	$-\theta_0'(0)$	$-\theta_1'(0)$
0	0.570810	0.122173	0.878720	0.173098
0.03	0.547495	0.129456	0.825799	0.187845
0.05	0.532278	0.131806	0.792171	0.194214



a) b)

Figure 2. Variation of a) reduced skin friction coefficient $f''(0)$ and b) reduced Nusselt number $\theta'(0)$with $\lambda\ (<0)$ for $\xi = 0$and several values of ϕ.



a) b)

Figure 3. Upper (full line) and lower (dot line) profiles: a) velocity $f'(\eta)$ and b) temperature $\theta(\eta)$ for $\xi = 0$, $\lambda = -0.2$ (opposing flow).

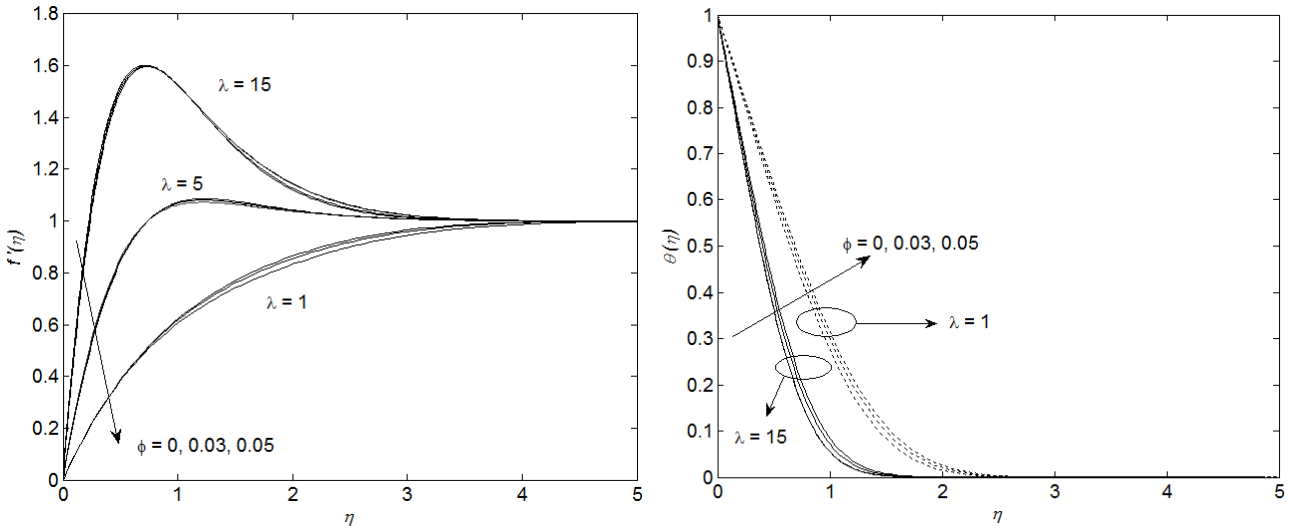


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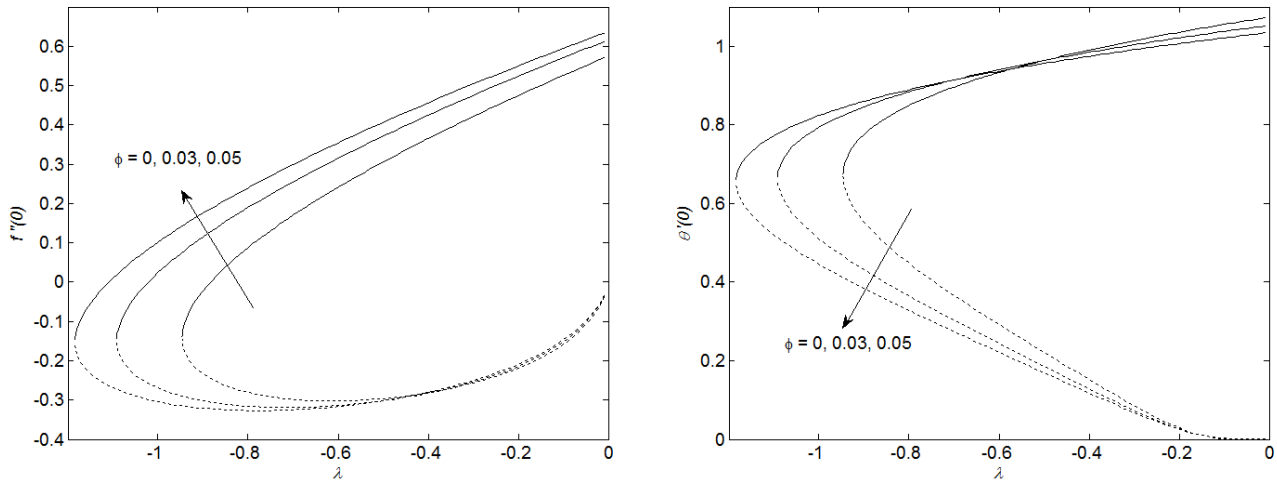
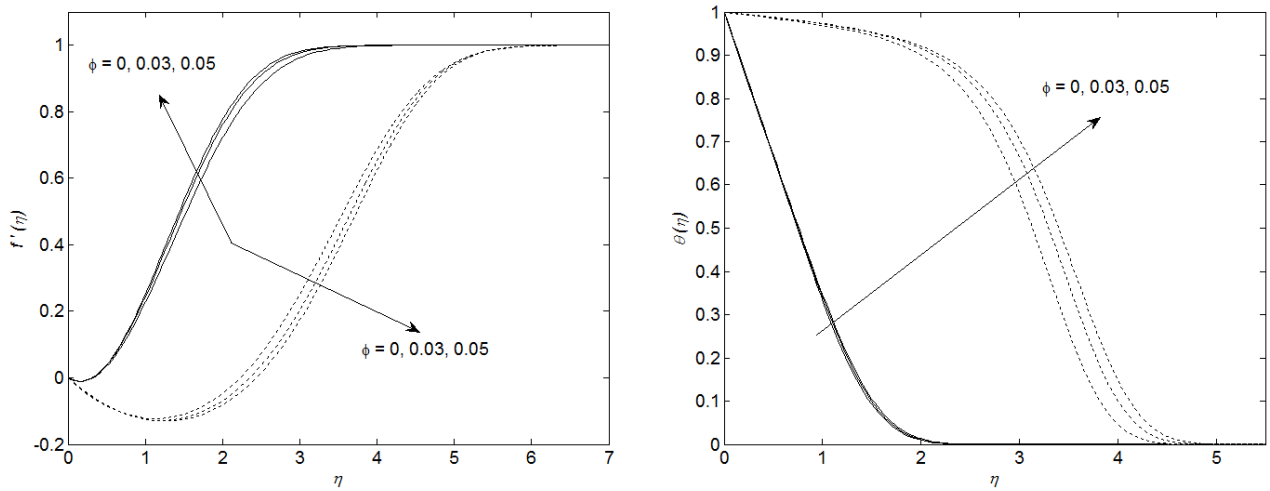
Figure 4. Profiles of a) velocity $f'(\eta)$ and b) temperature $\theta(\eta)$ for $\xi = 0,\ \lambda > 0$ (assisting flow) and several values of ϕ.



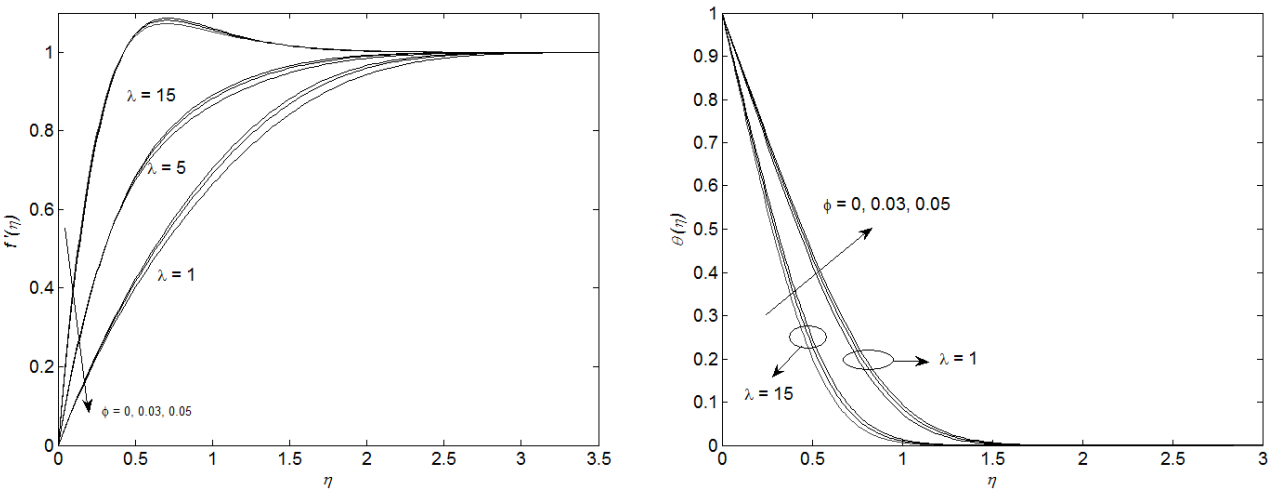
a) b)

Figure 5.Variation of a) reduced skin friction coefficient $f''(0)$ and b) reduced Nusselt number $\theta'(0)$with $\lambda\ (<0)$ for $\xi \gg 1$ and several values of ϕ.



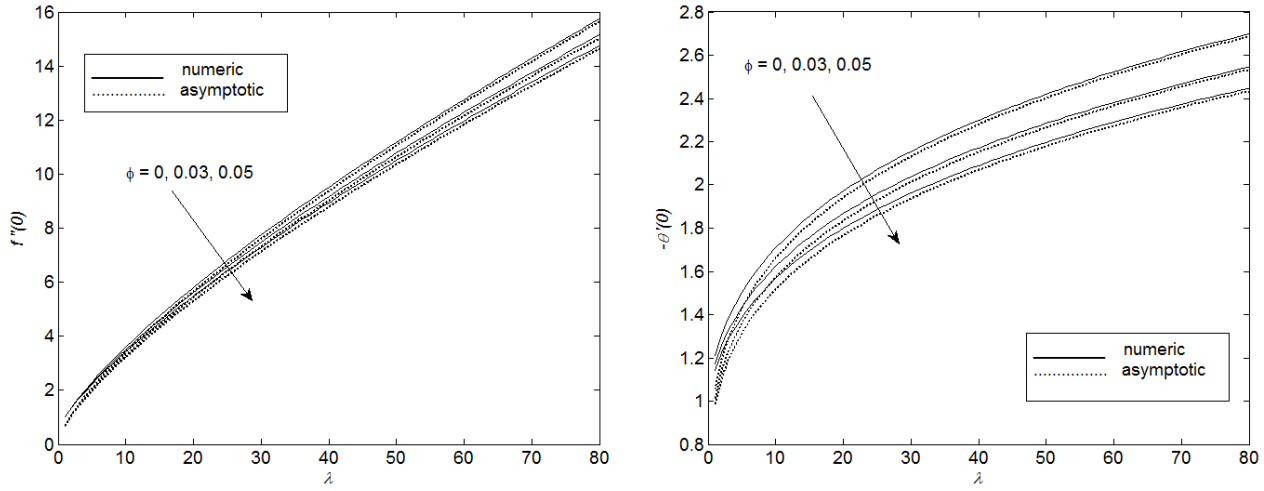
a) b)

Figure 6.Upper (full line) and lower (dot line) profiles: a) velocity$f'(\eta)$ and b) temperature $\theta(\eta)$ for $\xi \gg 1$, λ = -0.2 (opposing flow).



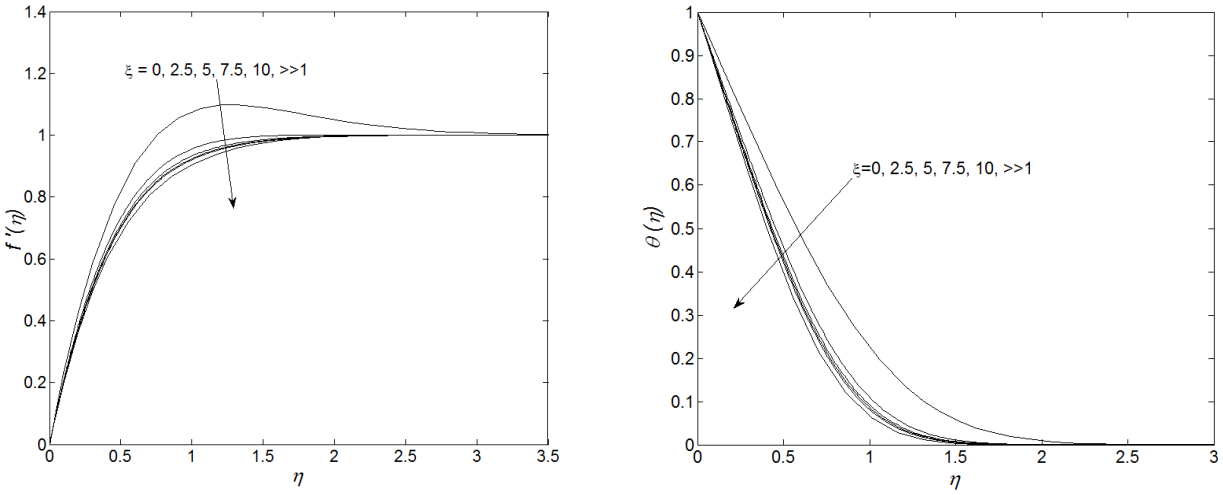
a) b)

Figure 7.Profiles of a) velocity $f^{'}(\eta)$ and b) temperature $\theta(\eta)$for $\xi \gg 1,\ \lambda > 0$ (assisting flow) and several values of ϕ.



a) b)

Figure 8. Variation of a) reduced skin friction $f''(0)$ and b) reduced Nusselt number $-\theta'(0)$ with $\lambda > 0$ (assisting flow) when $\xi >> 1$ and $\phi = 0, 0.03$ and 0.05.



a) b)

Figure 9.Profiles of a) velocity $f^{'}(\eta)$ and b) temperature $\theta(\eta)$for $\phi = 0.05,\ \lambda = 5$ (assisting flow) when $\xi = 0, 2.5, 5, 7.5, 10$ and $\gg 1$.