

ON AN APPROXIMATION FORMULA

ION PĂVĂLOIU

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1. INTRODUCTION

This Note contains some remarks concerning an approximation formula for functions, which is a generalization of some interpolation formulae given in [2] and [4]. In particular, we shall show that only one of the formulae of this type, mentioned in [4], has a maximal degree of exactness. Some particular cases of such formulae were also mentioned in [4, p.163].

Denote by I_x the closed interval determined by two distinct points x_0, x in \mathbb{R} . For a $(2n + 1)$ -times derivable function $f : I_x \rightarrow \mathbb{R}$ and $n \in \mathbb{N}$, consider the class G of functions given by

$$(1.1) \quad G = \left\{ g : g(t) = f(x_0) + (t - x_0) \sum_{i=1}^n a_i f'(x_0 + b_i(t - x_0)), \right. \\ \left. a_i, b_i \in \mathbb{R}, \quad i = \overline{1, n}, t \in I_x \right\}.$$

Consider the following problem: Find a function $\bar{g} \in G$ such that

$$(1.2) \quad f^{(i)}(x_0) = \bar{g}^{(i)}(x_0), \quad i = \overline{1, m}.$$

In [4] this problem was solved in some particular cases. We shall show that, for $m = 2n$, this problem has a unique solution and we shall give a representation for the remainder.

2. DETERMINATION OF THE APPROXIMATING FUNCTION

For $m = 2n$, we are looking for a function \bar{g} in G verifying conditions (1.2) and having a maximal degree of approximation.

It is easily seen that conditions (1.2) lead to the following system, having the real numbers $a_i, b_i, i = \overline{1, n}$, as unknowns:

$$(2.1) \quad \sum_{i=1}^n a_i b_i^k = \frac{1}{(k+1)}, \quad k = 0, 1, \dots, 2n-1.$$

Consider now a continuous function $\varphi : [0, 1] \rightarrow \mathbb{R}$ and let

$$(2.2) \quad \int_0^1 \varphi(t) dt = \sum_{i=1}^n a_i \varphi(b_i) + R[\varphi]$$

be a quadrature formula, having $\{b_i\}_1^n$ as knots and $\{a_i\}_1^n$ as coefficients. Asking that $R[\varphi_k] = 0$ for $\varphi_k(t) = t^k, k = \overline{0, 2n-1}$, (2.2) becomes the classical Gauss quadrature formula. On the other hand, the conditions $R[\varphi_k] = 0$, for $\varphi_k(t) = t^k, k = \overline{0, 2n-1}$, lead again to the system (2.1), implying that b_i must be the roots of the Legendre polynomial w_n of degree n , i.e., the roots of the equation

$$(2.3) \quad w_n(i) := \frac{n!}{(2n)! dt^n} [t^n (t-l)^n] = 0.$$

The coefficients a_i are given by the following formula

$$(2.4) \quad a_i = \frac{(n!)^4}{[(2n)!]^2 b_i (1-b_i) [w_n'(b_i)]^2}, \quad i = \overline{1, n},$$

(see [1, p.261]).

Now, it is clear that the following theorem holds:

THEOREM 2.1. *If $f : I_x \rightarrow \mathbb{R}$ is a $(2n+1)$ -times derivable function on I_x , then there exists only one function $\bar{g} \in G$ verifying conditions (1.2) for $m = 2n$. The parameters $\{a_i\}_{i=1}^n$ are given by formula (2.4), where $\{b_i\}_{i=1}^n$ are the roots of equation (2.3).*

3. DETERMINATION OF THE REMAINDER

Consider the approximation formula

$$(3.1) \quad f(x) = \bar{g}(x) + r[f],$$

where $\bar{g} \in G$ is a function verifying (2.1) and $r[f]$ is the remainder.

In the conditions of Theorem 2.1, it follows that

$$(3.2) \quad f'(x_0 + b_i(x - x_0)) = \sum_{j=1}^{2n} \frac{f^{(j)}(x_0)}{(j-1)!} b_i^{j-1} (x - x_0)^{j-1} + r_i(x),$$

where

$$(3.3) \quad r_i(x) = \frac{f^{(2n+1)}(\theta_i)}{(2n)!} b_i^{2n} (x - x_0)^{2n},$$

and θ_i is a number contained in the open interval determined by x_0 and $x_0 + b_i(x - x_0)$, $1 \leq i \leq n$.

From (3.2) we obtain the equalities

$$(3.4) \quad \begin{aligned} f(x) - f(x_0) - (x - x_0) f'(x_0 + b_i(x - x_0)) = \\ = f(x) - f(x_0) - \sum_{j=1}^{2n} \frac{f^{(j)}(x_0)}{(j-1)!} b_i^{j-1} (x - x_0)^j - r_i(x) (x - x_0), \quad i = \overline{1, n}. \end{aligned}$$

Multiplying equalities (3.4) by a_i , taking into account solution (2.1) and summing up, we obtain

$$(3.5) \quad f(x) - \bar{g}(x) = f(x) - \sum_{j=0}^{2n} \frac{f^{(j)}(x_0)}{j!} (x - x_0)^j - \sum_{i=1}^n a_i r_i(x) (x - x_0).$$

Now, using (3.3) and Lagrange form of the remainder in the Taylor formula, we get

$$(3.6) \quad f(x) - \bar{g}(x) = \left[\frac{f^{(2n+1)}(\eta)}{(2n+1)!} - \sum_{i=1}^n a_i b_i^{2n} \frac{f^{(2n+1)}(\theta_i)}{(2n)!} \right] (x - x_0)^{(2n+1)},$$

where $\eta \in I_x$.

Setting $\varphi(t) = t^{2n}$ in (2.2) and taking into account the form of the remainder term in the Gauss quadrature formula [[1],p.259], we get

$$\sum_{i=1}^n a_i b_i^{2n} + \frac{[n]^4}{[(2n)!]^2 (2n+1)} = \frac{1}{2n+1},$$

implying

$$(3.7) \quad \sum_{i=1}^n a_i b_i^{2n} = \frac{[(2n)!]^2 - [n]^4}{(2n+1)[(2n)!]^2}.$$

Suppose now that the $(2n + 1)$ -order derivative of f is bounded on I_x and let

$$(3.8) \quad M_{2n+1} = \sup_{t \in I_x} \left| f^{(2n+1)}(t) \right|.$$

Taking into account relations (3.6) and (3.7), one obtains the following delimitation for $r[f]$

$$(3.9) \quad |r[f]| \leq \frac{M_{2n+1}}{(2n+1)!} \cdot \frac{2 \cdot [(2n)!]^2 + [n!]^4}{[(2n)!]^2} |x - x_0|^{2n+1}.$$

4. PARTICULAR CASES

a) $n = 1$. In this case, $b_1 = \frac{1}{2}$, $a_1 = 1$ and

$$g(x) = f(x_0) + (x - x_0) f' \left(x_0 + \frac{1}{2} (x - x_0) \right).$$

From (3.9) we get

$$|f(x) - g(x)| \leq \frac{7M_3}{24} |x - x_0|^3,$$

where $M_3 = \sup_{t \in I_x} |f'''(t)|$.

b) $n = 2$. In this case, $b_1 = \frac{3-\sqrt{3}}{6}$, $b_2 = \frac{3+\sqrt{3}}{6}$, $a_1 = a_2 = \frac{1}{2}$ and

$$g(x) = f(x_0) + \frac{1}{2} (x - x_0) \left[f' \left(x_0 + \frac{3-\sqrt{3}}{6} (x - x_0) \right) + f' \left(x_0 + \frac{3+\sqrt{3}}{6} (x - x_0) \right) \right],$$

One also obtains the evaluation

$$|f(x) - g(x)| \leq \frac{71M_5}{4320} |x - x_0|^5,$$

where $M_5 = \sup_{t \in I_x} |f^{(5)}(t)|$.

REMARK. Approximation formula of the type considered in this Note could be useful for the approximate calculation of the values of some functions having rational functions as derivatives.

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Institutul de Calcul "Tiberiu Popoviciu"
Str. G. Bîlaşcu, nr.37
C.P. 68, O.P. 1, 3400 Cluj-Napoca
România