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PERTURBED-STEFFENSEN-AITKEN PROJECTION METHODS FOR SOLVING EQUATIONS WITH NONDIFFERENTIABLE OPERATORS

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ABSTRACT In this study we use perturbed-Steffensen- Aitken methods to approximate a locally unique solution of an operator equation in a Banach space. Using projection operators we reduce the problem to solving a linear system of algebraic equations of finite order. Since iterates can rarely be computed exactly we control the residuals to guarantee convergence of the method. Sufficient convergence conditions as well as an error analysis are given for our method.

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Key Words and Phrases Steffensen-Aitken methods, Banach space, projection operator, residuals.

I. INTRODUCTION In this study we are concerned with the problem of approximating a locally unique fixed point x^* of the nonlinear equation.

$$T(x) = x, \tag{1}$$

where T is a continuous operator defined on a convex subset D of a Banach space E with values in E. The differentiability of T is not assumed. Let T_1 be another nonlinear continuous operator from E into E, and let P be a projection operator $(P^2 = P)$ on E.

We introduce the perturbed-Steffensen-Aitken method

$$x_{n+1} = T(x_n) + PA_n(x_{n+1} - x_n) - z_n. \quad A_n = [g_1(x_n), g_2(x_n)] \quad (n \geq 0), \tag{2}$$

where: $[x, y]$ denotes a divided difference of order one of T_1 at the points x, y satisfying

$$[x, y](y - x) = T_1(y) - T_1(x) \quad \text{for all} \quad x, y \in D \quad \text{with} \quad x \neq y \tag{3}$$

and

$$[x, x] = F'(x) \quad (x \in D) \tag{4}$$

if T_1 is Frechet-differentiable $D; g_1, g_2 : D \to E$ are continuous operators; the residual points $\{x_n\}(n \geq 0)$ are chosen in such a way that iteration $\{x_n\}(n \geq 0)$ generated by (2) converges to x^*. The important of studying perturbed Steffensen-Aitken methods comes from the fact that many commonly used variants can be considered procedures of this type. Indeed the above approximation characterizes any iterative process in which corrections are taken as approximate solutions of the Steffensen-Aitken equations. Moreover we note that if for example an equation on the real line is solved $x_n - T(x_n) \geq 0 (n \geq 0)$ and $I - PA_n$ overestimates the derivative, $x_n - (I - PA_n)^{-1}(x_n - T(x_n))$ is always *larger* than the corresponding Steffensen-Aitken iterate. In such cases, a positive $z_n (n \geq 0)$ correction term is appropriate.

For: $P = I (I$ is the identity operator on $E), T(x) = T_1(x)(x \in D), g_1(x) = g_2(x)(x \in D)$, and $z_n = 0(n \geq 0)$ we obtain the ordinary Newton method [1], [2]; $P = I, T_1(x) = T(x)(x \in D), g_1(x) = x(x \in D)$, and $z_n = 0(n \geq 0)$ we obtain Steffensen method [4], [5]; $P = I, T_1(x) = T(x)(x \in D), g_2(x) = g_1(x - T(x))(x \in D)$, and $z_n = 0(n \geq 0)$ we obtain Steffensen-Aitken method [4], [5].

It is easy to see that the solution of (2) reduces to solving certain operator equations in the space E_p. If moreover E_p is a finite dimensional space of dimension N, we obtain a system of linear algebraic equations of at most order N.

We provide sufficient convergence conditions as well as an error analysis for the Steffensen-Aitken method generated by (2).

II. CONVERGENCE ANALYSIS

We state the following semilocal convergence theorem.

Theorem *Let $T, T_1, g_1, g_2 : D \to E$ be continuous operators defined on a convex subset D of a Banach space E with values in E, and P be a projection operator on E. Moreover, assume:*

(a) there exists $x_0 \in D$ such that $B_0 = I - PA_0$ is invertible;

(b) there exist nonnegative numbers a_i, R, $\quad i = 0, 1, 2, \cdots, 9$ such that

$$\|B_0^{-1}P([x,y] - [v,w])\| \leq a_0(\|x - v\| + \|y - w\|), \tag{5}$$

$$\|B_0^{-1}(x_0 - T(x_0))\| \leq a_1, \tag{6}$$

$$\|B_0^{-1}P([x,y] - [g_1(x), g_2(x)])\| \leq a_2(\|x - g_1(x)\| + \|y - g_2(x)\|), \tag{7}$$

$$\|B_0^{-1}(QT_1(x) - QT_1(y))\| \leq a_3\|x - y\|, \quad Q = 1 - P, \tag{8}$$

$$\|B_0^{-1}(F(x) - F(y))\| \leq a_4\|x - y\|, \quad F(x) = T(x) - T_1(x), \tag{9}$$

$$\|x - g_1(x)\| \leq a_5\|B^{-1}(x)(x - T(x) - z(x))\|, \quad B(x) = I - PA(x),$$
$$\text{for some continuous function } \; z : D \to E, \tag{10}$$

$$\|x - g_2(x)\| \leq a_6\|B^{-1}(x)(x - T(x) - z(x))\|, \tag{11}$$

$$\|B_0^{-1}(z_n - z_{n-1})\| \leq a_7\|x_n - x_{n-1}\| \quad (n \geq 1), \tag{12}$$

$$\|g_1(x) - g_1(y)\| \leq a_8\|x - y\|, \quad a_8 \in [0,1), \tag{13}$$

and

$$\|g_2(x) - g_2(y)\| \leq a_9\|x - y\|, \quad a_9 \in [0,1), \tag{14}$$

for all $x, y, v, w \in U(x_0, R) = \{x \in E | \|x - x_0\| \leq R\} \subseteq D$;

(c) the sequence $\{z_n\} (n \geq 0)$ *is null;*

(d) there exists a minumum nonnegative number r^* *satisfying*

$$G(r^*) \leq r^* \quad \text{and} \quad r^* \leq R \tag{15}$$

where

$$G(r) = a_1 + \frac{a_2(1 + a_8 + a_9)r + (a_3 + a_4 + a_7)}{[a - a_0(a_8 + a_9)r][1 - a_2(a_5 + a_6)r\beta(r)]} r; \tag{16}$$

and

$$\beta(r) = [1 - a_0(a_8 + a_9)r]^{-1} \tag{17}$$

(e) the numbers r^*, R *also satisfy*

$$r^* < \frac{1}{a_2(a_5 + a_6) + a_0(a_8 + a_9)} \tag{18}$$

$$r^* \geq \frac{\|g_1(x_0) - x_0\|}{1 - a_8} \tag{19}$$

$$r^* \geq \frac{\|g_2(x_0) - x_0\|}{1 - a_9} \tag{19}$$

$$b = \alpha(r, R) < 1. \tag{21}$$

where

$$\alpha(s,t) = \frac{a_2(1 + a_8 + a_9)(s + t) + a_3 + a_4}{[1 - a_0(a_8 + a_9)s][1 - a_2(a_5 - a_6)(s + t)\beta(s)]}, \quad s, t \in [0, R] \tag{22}$$

and

$$\lim_{n\to\infty} q_n = 0 \tag{23}$$

where

$$q_n = \sum_{m=0}^{n} b^{n-m} c_m, \quad c_m = \|z_n\|, \quad B_n = I - PA_n \quad (n \geq 0) \tag{24}$$

Then

(i) the scalar sequence $\{t_n\}$ $(n \geq 0)$ *generated by*

$$t_0 = 0, \quad t_1 = a_1 \geq \|x_1 - x_0\|, \tag{25}$$

$$t_{n+1} = t_n + \frac{a_2(1 + a_8 + a_9)(t_n - t_{n-1}) + a_3 + a_4 + a_7}{[1 - a_0(a_8 + a_9)t_n][1 - a_2(a_5 + a_6)(t_n - t_{n-1}\beta_n]}(t_n - t_{n-1}) \quad (n \geq 1) \tag{26}$$

is monotonically increasing, bounded above by r^* *and* $\lim_{n\to\infty} t_n = r^*$, *with* $\beta_n = [1 - a_0(a_8 + a_9)t_n]^{-1}$ $(n \geq 0)$.

(ii) The perturbed-Steffensen-Aitken method generated by (2) is well defined, remains in $U(x_0, r^*)$ *for all* $n \geq 0$, *converges to a unique fixed point* x^* *of* T *in* $U(x_0, R)$.

Moreover the following error bounds hold:

$$\|x_{n+1} - x_n\| \leq \frac{a_2(1 + a_8 + a_9)\|x_n - x_{n-1}\| + a_3 + a_4 + a_7}{[1 - a_0(a_8 + a_9)\|x_n - x_0\|][1 - a_2(a_5 + a_6)\|x_n - x_{n-1}\|\beta_n]} \|x_n - x_{n-1}\| (n \geq 1) \tag{27}$$

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n \quad (n \geq 0) \tag{28}$$

and

$$\|x_n - x^*\| \leq r^* - t_n \quad (n \geq 0), \tag{29}$$

where $\bar{\beta}_n = [1 - a_0(a_8 - a_9)\|x_n - x_0\|]^{-1} \quad (n \geq 0)$

Proof (i). By (15) and (25) we get $0 \leq t_0 \leq t_1 \leq r^*$. Let us assume $0 \leq t_{k-1} \leq t_k \leq r^*$ for $k = 1, 2, \cdots, n$. It follows from (18) and (26) that $0 \leq t_k \leq t_{k+1}$. Hence, the sequence $\{t_n\}(n \geq 0)$ is monotonically increasing. Moreover using (26) we get in turn

$$\begin{aligned} t_{k+1} &\leq t_k + \frac{a_2(1 + a_8 + a_9)r^* + a_3 + a_4 + a_7}{[1 - a_0(a_8 + a_9)r^*][1 - a_2(a_5 + a_6)r^*\beta(r^*)]}(t_k - t_{k-1}) \\ &\leq \cdots \leq a_1 + \frac{a_2(1 + a_8 + a_9)r^* a_3 + a_4 + a_7}{[1 - a_0(a_8 + a_9)r^*][a - a_2(a_5 + a_6)r^*\beta(r^*)]}(t_k - t_0) \\ &\leq G(r^*) \leq r^* \quad \text{(by (15))} \end{aligned}$$

That is the sequence $\{t_n\}(n \geq 0)$ is also bounded above by r^*. Since r^* is the minimum nonnegative number satisfying $G(r^*) \leq r^*$, it follows that $\lim_{n\to\infty} t_n = r^*$.

(ii) By hypothesis (15) and the choice of a_1 it follows that $x_1 \in U(x_0, r^*)$. From (19) and (20) we get $g_1(x_0)$, $g_2(x_0) \in U(x_0, r^*)$. Let us assume $x_{k+1}, g_1(x_k)$, $g_2(x_k) \in U(x_0, r^*)$ for $k = 0, 1, \cdots, n-1$. Then from (13), (14), (19) and (20) we get

$$\|g_1(x_k) - x_0\| \leq \|g_1(x_k) - g_1(x_0)\| + \|g_1(x_0) - x_0\| \leq a_8\|x_k - x_0\| + \|g_1(x_0) - x_0\|$$

$$\leq a_8 r^* + \|g_1(x_0) - x_0\| \leq r^*$$

and

$$\begin{aligned} \|g_2(x_k) - x_0\| &\leq \|g_2(x_k) - g_2(x_0)\| + \|g_2(x_0) - x_0\| \leq a_9\|x_k - x_0\| + \|g_2(x_0) - x_0\| \\ &\leq a_9 r^* + \|g_2(x_0) - x_0\| \leq r^* \end{aligned}$$

Hence $g_1(x_n)$, $g_2(x_n) \in U(x_0, r^*)$. Using (5), (13), (14) and (17) we obtain

$$\|B_0^{-1}(B_k - B_0)\| \leq a_0(\|g_1(x_0) - g_1(x_k)\| + \|g_2(x_0) - g_2(x_k)\|)$$

$$\leq a_0(a_8 + a_9)\|x_0 - x_k\| \leq a_0(a_8 + a_9)r^* < 1$$

It follows from the Banach lemma on invertible operators [3] that B_k is invertible and

$$\|B_k^{-1}B_0\| \leq \frac{1}{1 - a_0(a_8 + a_9)\|x_k - x_0\|} = \bar{\beta}_k \tag{30}$$

Using (2) we obtain the approximation

$$x_{k+1} - x_k = B_k^{-1}(T(x_k) - x_k - z_k) = (B_k^{-1}B_0)B_0^{-1}$$
$$\{(PT_1(x_k) - PT_1(x_{k-1}) - P[g_1(x_{k-1}),\ g_2(x_{k-1})]\ (x_k - x_{k-1})$$
$$+(QT_1(x_k) - QT_1(x_{k-1}) + (F(x_k) - F(x_{k-1})) + (z_{k-1} - z_k)\} \tag{31}$$

From (7), we get

$$\|B_0^{-1}[PT_1(x_k) - PT_1(x_{k-1}) - PA_{k-1}(x_k - x_{k-1})]\| \leq \|B_0^{-1}P([x_{k-1}, x_k] - A_{k-1})(x_k - x_{k-1})\|$$

$$\leq a_2(\|x_{k-1} - g_1(x_{k-1})\| + \|x_k - g_2(x_{k-1})\|)\|x_k - x_{k-1}\| \tag{32}$$

and since by (10), (11), (13), (14)

$$\begin{aligned}
\|x_{k-1} - g_1(x_{k-1})\| &\leq \|x_{k-1} - x_k\| + \|g_1(x_k) - g_1(x_{k-1})\| + \|x_k - g_1(x_k)\| \\
&\leq \|x_k - x_{k-1}\| + a_8\|x_k - x_{k-1}\| + a_5\|B_k^{-1}(x_k - T(x_k) - z_k\| \\
\|x_k - g_2(x_{k-1})\| &\leq x_k - g_2(x_k)\| + \|g_2(x_k) - g_2(x_{k-1})\| \\
&\leq a_6\|B_k^{-1}(x_k - T(x_k) - z_k)\| + a_9\|x_k - x_{k-1}\|
\end{aligned}$$

(32) gives

$$\|B_0^{-1}[PT_1(x_k) - PT_1(x_{k-1}) - PA_{k-1}(x_k - x_{k-1})\| \leq a_2(1 + a_8 + a_9)\|x_k - x_{k-1}\|^2$$

$$+ a_2(a_5 + a_6)\|B_k^{-1}(x_k - T(x_k) - z_k)\|\|x_k - x_{k-1}\| \tag{33}$$

Moreover from (8), (9) and (12) we obtain respectively

$$\|B_0^{-1}(QT_1(x_k) - QT_1(x_{k-1})\| \leq a_3\|x_k - x_{k-1}\| \quad (k \geq 1) \tag{34}$$

$$\|B_0^{-1}(F(x_k) - F(x_{k-1}))\| \leq a_4\|x_k - x_{k-1}\| \quad (k \geq 1) \tag{35}$$

and

$$\|B_0^{-1}(z_k - z_{k-1})\| \leq a_7\|x_k - x_{k-1}\| \quad (k \geq 1) \tag{36}$$

Furthermore (31) because of (30), (33)-(36) finally gives (27) for $n = k$.

Estimate (28) is true for $n = 0$ by (25). Assume (28) is true for $k = 0, 1, 2, \cdots, n-1$. Then from (26), (27) and the induction hypothesis it follows that (28) is true for $k = n$. By (28) and part (i) it follows that iteration $\{x_n\}(n \geq 0)$ is Cauchy in a Banach space E and as such it converges to some $x^* \in U(x_0, r^*)$ (since $U(x_0, r^*)$ is a closed set). Using hypothesis (c) and letting $n \to \infty$ in (2) we get $x^* = T(x^*)$. That is x^* is a fixed point of T. Estimate (29) follows immediately from (28) using standard majorization techniques [2], [3].

Finally to show uniqueness let us assume $y^* \in U(x_0, R)$ is a fixed point of equation (1). As in (31) we start from the approximation.

$$\begin{aligned} x_{n+1} - y^* &= (B_n^{-1} B_0) B_0^{-1} \{[PT_1(x_n) - PT_1(y^*) - PA_n(x_n - y^*)] \\ &\quad + [QT_1(x_n) - QT_1(y^*)] + [F(x_n) - F(y^*)] - \quad z_n\} \end{aligned}$$

and using (5), (7)-(11), (13), (14), (21), (22) and (24) we get

$$\|x_{n+1} - y^*\| \leq b\|x_n - y^*\| + c_n \leq \cdots \leq b^{n+1}\|x_0 - y^*\| + q_n \quad (n \geq 0) \tag{37}$$

By letting $n \to \infty$ as using (21) and (23) we get $\lim_{n\to\infty} x_n = y^*$. It follows from the uniqueness of the limit that $x^* = y^*$.

That completes the proof of the Theorem.

Remarks

(1) Conditions (19) and (20) guarantee $g_1(x), g_2(x) \in U(x_0, r^*)$ for $x \in U(x_0, r^*)$. Hence condition (7) can be dropped and we can set $a_2 = a_0$. However it is hoped that $a_2 \leq a_0$.

(2) It can easily be seen that the first inequality in (15) can be replaced by the system of inequalities (17), (18) and

$$f(r^*) \leq 0$$

where

$$f(r) = d_2 r^2 + d_1 r + d_0$$

with

$$b_1 = a_2(1 + a_8 + a_9), \quad b_2 = a_3 + a_4 + a_7, \quad b_3 = a_0(a_8 + a_9) + a_2(a_5 + a_6)$$

$$\begin{aligned} d_2 &= b_1 + b_3 \\ d_1 &= b_1 - 1 - b_3 a_1 \end{aligned}$$

and $d_0 = a_1$.

(3) Condition (23) is satisfied if and only if $z_n = 0 (n \geq 0)$

(4) It can easily be seen from (10) and (11) that conditions (19) and (20) will be satisfied if $a_5 + a_8 \leq 1$ and $a_6 + a_9 \leq 1$ for $r^* \neq 0$. Indeed from (10) we have $\|x_0 - g_1(x_0)\| \leq a_5\|x_1 - x_0\| \leq a_5 r^*$. Hence (19) will be certainly satisfied if $a_5 r^* \leq (1 - a_8) r^*$. That is if $a_5 + a_8 \leq 1$. We argue similarly for (20).

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