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Dedicated to Costică Mustăţa on his 60th anniversary

ON THE APPROXIMATION OF SOLUTIONS TO NONLINEAR OPERATORS BETWEEN METRIC SPACES

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Abstract. A Gauss-Seidel-type method for the solution of linear systems, based on the decomposition of the system matrix into four matrices blocks, has been proposed by R. Varga in [3]. The convergence of this method was studied in [1] and [2].

In this paper we shall extend the ideas contained in the above quoted works to the case of nonlinear system equations.

In the paper [3], R. Varga proposes a Gauss-Seidel type method for solving linear systems, which is based on decomposing the matrix of the system in four submatrix blocks. The convergence of this method has been proved in [1] and [2].

We shall extend these idees to the case of nonlinear systems.

Let (X_i, ρ_i), $i = 1, 2$, two complete metric spaces, and $X = X_1 \times X_2$, $F : X \to X_1$, $G : X \to X_2$ two mappings. We are interested in studying the existence and uniqueness of the solution of the system

$$\begin{cases} u = F(u, v) \\ v = G(u, v), \qquad (u, v \in X). \end{cases} \tag{1}$$

In this sense, we shall consider the sequences $(u_n)_{n\geq 0}, (v_n)_{n\geq 0}$ generated by the Gauss-Seidel method, i.e.,

$$\begin{cases} u_{n+1} = F(u_n, v_n) \\ v_{n+1} = G(u_{n+1}, v_n), \qquad n = 0, 1, 2, \ldots, (u_0, v_0) \in X. \end{cases} \tag{2}$$

Let $D_i \subset X_i, i=1,2$ and $D=D_1 \times D_2$. We shall assume that F and G verify Lipschitz-type conditions on D, i.e., there exist $\alpha, \beta, a, b \geq 0$ such that

$$\left\{ \begin{array}{l} \rho_1 \left(F\left(x_1,y_1\right), F\left(x_2,y_2\right)\right) \leq \alpha \rho_1\left(x_1,x_2\right) + \beta \rho_2\left(y_1,y_2\right) \\ \rho_2\left(G\left(x_1,y_1\right), G\left(x_2,y_2\right)\right) \leq \alpha \rho_1\left(x_1,x_2\right) + b \rho_2\left(y_1,y_2\right) \end{array} \right. \tag{3}$$

for all $(x_i, y_i) \subseteq D, i=1,2$.

For the study of the convergence of (2) we consider two sequences of real numbers $(f_n)_{n\geq 0}, (g_n)_{n\geq 0}$ with nonnegative terms, obeying the following system of difference inequalities:

$$\left\{ \begin{array}{l} f_n \leq \alpha f_{n-1} + \beta g_{n-1} \\ g_n \leq \alpha f_n + b g_{n-1}, \qquad n=1,2,\ldots, \end{array} \right. \tag{4}$$

where α, β, a, b are given in (3).

We associate to (4) the following system in the unknowns h, k:

$$\left\{ \begin{array}{l} \alpha + \beta h = hk \\ ah + b = hk \end{array} \right. \tag{5}$$

It was shown in [1] that if α, β, a, b obey

$$\left\{ \begin{array}{c} \alpha + b + a\beta < 2 \\ (1-\alpha)(1-b) - a\beta > 0 \\ \alpha > 0, b > 0, \end{array} \right. \tag{6}$$

then the system (5) has two real solutions (h_i, k_i), $i=1,2$ such that $0 < h_i,\ k_i < 1$, $i=1,2$, and one of these solutions has both the components positive. Denote by (h_1, k_1) this solution, i.e., $h_1 > 0, k_1 > 0$, so that the elements of the sequences $(f_n)_{n\geq 0}$ and $(g_n)_{n\geq 0}$ obey

$$\left\{ \begin{array}{l} f_n \leq C h_1^{n-1} k_1^{n-1} \\ g_n \leq C h_1^{n} k_1^{n-1}, \qquad n=1,2,\ldots \end{array} \right. \tag{7}$$

where $C = \max\left\{\alpha f_0 + \beta g_0, \left(a f_1 + b g_0\right)/h_1\right\}.$

Let $p_1 = h_1 k_1$ and $d_1 > 0$ be a positive number such that the sets

$$\begin{array}{l} S_1 = \left\{x \in X_1 : \rho_1\left(x, u_0\right) \leq \frac{d_1}{1-p_1}\right\}; \\ S_2 = \left\{x \in X_2 : \rho_2\left(x, v_0\right) \leq \frac{d_1 h_1}{1-p_1}\right\}, \end{array} \tag{8}$$

verify $S_i \subseteq D_i,\ i=1,2$.

Denoting $f_n = \rho_1\left(u_n, u_{n-1}\right), g_n = \rho_2\left(v_n, v_{n-1}\right), n=1,2,\ldots$ and taking into account the above relations we obtain the following result.

Theorem 1. *If the mappings F and G verify conditions* (3) *on the set D, $S_1 \times S_2 \subseteq D$, the numbers α, β, a, b verify* (6) *and $u_1 = F(u_0, v_0)$, $v_1 = G(u_1, v_0)$ are such that $\rho_1(u_1, u_0) \leq d_1$, $\rho_2(u_1, v_0) \leq d_1 h_1$, then the following statements hold:*

a) *the sequences $(u_n)_{n\geq 0}, (v_n)_{n\geq 0}$ converge, and denoting $\lim u_n = \bar{u}$, $\lim v_n = \bar{v}$, then $(\bar{u}, \bar{v})$ is the unique solution of* (1) *in the set $S = S_1 \times S_2$;*

b) *the following inequalities are true*

$$\left\{ \begin{array}{l} \rho_1(\bar{u}, u_n) \leq \frac{d_1 p_1^n}{1-p_1} \\ \rho_2(\bar{v}, v_n) \leq \frac{d_1 h_1 p_1^n}{1-p_1}, \qquad n = 0, 1, \ldots \end{array} \right. \tag{9}$$

This theorem is proved using (3) and inequalities (4). We shall apply this Theorem to the study of a Gauss-Seidel type method for solving nonlinear operator equations.

Let (X, ρ) be a complete metric space and $X^m, X^s, X^{m-s}, 1 \leq s \leq m-1$ the chartesian products.

If $u, v \in X^i$, $i = \{m, s, m-s\}$, we define the metric in such a space in the following way: let $u = (u_1, \ldots, u_i)$, $v = (v_1, \ldots, v_i)$ and put

$$\rho_i(u, v) = \max_{1\leq j\leq i} \{\rho(u_j, v_j)\}, \qquad i \in \{m, s, m-s\}. \tag{10}$$

Consider the mappings $\varphi_k : X^m \to X, k = \overline{1, m}$, and the following system of equations:

$$x_k = \varphi_k(x_1, x_2, \ldots, x_m), \qquad k = \overline{1, m}. \tag{11}$$

and define the mapping $\bar{F} : X^s \times X^{m-s} \to X^s$ resp. $\bar{G} : X^s \times X^{m-s} \to X^{m-s}$ in the following way. If $u = (u_1, \ldots, u_s) \in X^s$ and $v = (v_1, \ldots, v_{m-s}) \in X^{m-s}$ then

(12)

$$\bar{F}(u, v) = (\varphi_1(u_1, u_2, \ldots, u_s, v_1, v_2, \ldots, v_{m-s}), \ldots, \varphi_s(u_1, u_2, \ldots, u_s, v_1, v_2, \ldots, v_{m-s}))$$
$$\bar{G}(u, v) = (\varphi_{s+1}(u_1, u_2, \ldots, u_s, v_1, v_2, \ldots, v_{m-s}), \ldots, \varphi_m(u_1, u_2, \ldots, u_s, v_1, v_2, \ldots, v_{m-s})).$$

For solving of (12) we consider the following iterations.

$$\left\{ \begin{array}{l} u_{n+1} = \bar{F}(u_n, v_n) \\ v_{n+1} = \bar{G}(u_{n+1}, v_n), (u_0, v_0) \in X^s \times X^{m-s}, \qquad n = 0, 1, \ldots \end{array} \right. \tag{13}$$

Assuming that the mappings $\varphi_k, k = \overline{1, m}$ verify the Lipschitz type conditions, i.e., $\exists a_{kl} \geq 0, k, l = \overline{1, m}$ such that $\forall (x_1, \ldots, x_m), (y_1, \ldots, y_m) \in D \leq X^m$ it follows

$$\rho(\varphi_k(x_1, x_2, \ldots, x_m), \varphi_k(y_1, y_2, \ldots, y_m)) \leq \sum_{l=1}^{m} a_{kl} \rho(x_e, y_e), \qquad k = \overline{1, m} \tag{14}$$

Denoting

$$\bar{\alpha} = \max_{1\le k\le s}\left\{\sum_{l=1}^{m} a_{kl}\right\}, \quad \bar{\beta} = \max_{1\le k\le s}\left\{\sum_{l=s+1}^{m} a_{kl}\right\} \tag{15}$$

$$\bar{a} = \max_{s+1\le k\le m}\left\{\sum_{l=1}^{s} a_{kl}\right\}, \quad \bar{b} = \max_{s+1\le k\le m}\left\{\sum_{l=s+1}^{m} a_{kl}\right\}$$

then it can be seen that the mappings $\bar{F}$ and $\bar{G}$ obey

$$\begin{cases} \rho_s\left(\bar{F}(u,v), \bar{F}(x,y)\right) \le \bar{\alpha}\rho_s(u,x) + \bar{\beta}\rho_{n-s}(v,y) \\ \rho_{n-s}\left(\bar{G}(u,v), \bar{G}(x,y)\right) \le \bar{\alpha}\rho_s(u,x) + \bar{b}\rho_{n-s}(v,y) \end{cases}$$

$\forall (u,v), (x,y) \in D = D^s \times D^{m-s}$.

It is clear that if in Theorem 1 we set $X_1 = X^s, X_2 = X^{m-s}, \rho_1 = \rho_s, \rho_2 = \rho_{m-s}, \alpha = \bar{\alpha}, \beta = \bar{\beta}, a = \bar{a}, b = \bar{b}$ then $(u_0, v_0), (u_1, v_1)$ obey $\rho(u_0, u_1) \le d_1, \rho(v_0, v_1) \le d_1 h_1, \bar{S}_1 = D^s, \bar{S}_2 = D^{m-s}$, where

$$\bar{S}_1 = \left\{x \in X^s : \rho_s(x, u_0) \le \tfrac{d_1}{1-p_1}\right\},$$

$$\bar{S}_2 = \left\{x \in X^{m-s} : \rho_{m-s}(x, v_0) \le \tfrac{d_1 h_1}{1-p_1}\right\}$$

verify the relations $S_1 \subseteq D^s, S_2 \subseteq D^{m-s}$, and assuming that the assumptions of Theorem 1 are satisfied, we get the same conclusions regarding the solution of (12).

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