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BILATERAL APPROXIMATIONS OF SOLUTIONS OF EQUATIONS BY ORDER THREE STEFFENSEN-TYPE METHODS

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Dedicated to Professor Ştefan Cobzaş at his 60^{th} *anniversary*

Abstract. The convergence of method of Steffensen-type which is obtained from the Lagrange polynomial of inverse interpolation with controlled nodes - is studied in this paper. Conditions are given sequences which bilaterally approximates the solution of an equation.

1. Introduction

In order to approximate the solutions of scalar equations it is suitable to use iteration methods which lead to monotone sequences. Suppose that such a method generates two such sequences, i.e., an increasing sequence $(u_n)_{n\geq 0}$ and a decreasing sequence $(v_n)_{n\geq 0}$. If both converge to the solution $\bar{x}$ of a given equation, then at each step one obtains the following error control:

$$\max\{\bar{x}-u_n, v_n-\bar{x}\} \leq v_n - u_n$$

Such methods can be generated, for example, by combining simultaneously both Newton and chord methods [1], [2], [3], [10].

Conditions for Steffensen and Aitken-Steffensen methods which lead to monotone sequences which bilaterally approximate the root of a given equation, were studied in [6], [10].

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It is known that both the Steffensen and Aitken-Steffensen methods are obtained from the chord method in which the interpolation nodes are controlled.

In this paper we shall consider a Steffensen-type method, obtained from the inverse interpolation polynomial of second degree, using three controlled interpolation nodes.

More exactly, consider the following equation

$$f(x) = 0, \tag{1}$$

where $f : [a, b] \longrightarrow \mathbb{R}$, $a, b \in \mathbb{R}$, $a < b$.

Denote by $F = f([a, b])$ the range of f for $x \in [a, b]$.

Suppose that $f : [a, b] \longrightarrow F$ is a bijection, that is, there exists $f^{-1} : F \longrightarrow [a, b]$

Let $a_1, a_2, a_3 \in [a, b]$, $a_i \neq a_j$, for $i \neq j$, $i; j = \overline{1, 3}$, three distinct interpolation nodes and let $b_1 = f(a_1)$, $b_2 = f(a_2), b_3 = f(a_3)$. The inverse interpolation Lagrange polynomial for f^{-1} on the nodes b_1, b_2, $b_3 \in F$ is given by the following relation:

$$\begin{aligned} L(b_1, b_2, b_3; f^{-1} \mid y) =& a_1 + [b_1, b_2; f^{-1}](y - b_1) \\ &+ [b_1, b_2, b_3; f^{-1}](y - b_1)(y - b_2) \end{aligned} \tag{2}$$

with the remainder given by:

$$f^{-1}(y) - L(b_1, b_2, b_3; f^{-1} \mid y) = [y, b_1, b_2, b_3; f^{-1}](y - b_1)(y - b_2)(y - b_3). \tag{3}$$

It is known that (2) is symmetric with respect to nodes order. Thus, if (i_1, i_2, i_3) is a permutation of $(1, 2, 3)$, then the following relations are satisfied:

$$L(b_1, b_2, b_3; f^{-1} \mid y) = a_{i_1} + [b_{i_1}, b_{i_2}; f^{-1}](y - b_{i_1,}) + [b_{i_1}, b_{i_2,} b_{i_3}; f^{-1}](y - b_{i_1})(y - b_{i_2}) \tag{4}$$

Apart from (2), these relations lead to five more representations for Lagrange's polynomial.

In order to obtain a Steffensen-type method, and to approximate the solutions of equations (1), we shall consider one additional equation:

$$x - g(x) = 0, \quad g : [a, b] \longrightarrow [a, b],$$

which we shall assume is equivalent with (1).

If equation (1) has one root $\bar{x} \in [a, b]$, then obviously $\bar{x} = f^{-1}(0)$, and from (3) one obtains

$$\bar{x} = L(b_1, b_2, b_3; f^{-1} \mid 0) - [0, b_1, b_2, b_3; f^{-1}]b_1 b_2 b_3,$$

and if we neglect the remainder, we obtain:

$$\bar{x} \simeq L(b_1, b_2, b_3; f^{-1} \mid 0). \tag{5}$$

For divided differences of first and second order of f^{-1}, one knows that [5], [7], [8], [10]:

$$[b_1, b_2; f^{-1}] = \frac{1}{[a_1, a_2; f]} \tag{6}$$

and

$$[b_1, b_2; b_3; f^{-1}] = -\frac{[a_1, a_2, a_3; f]}{[a_1, a_2; f][a_1, a_3; f][a_2, a_3; f]}. \tag{7}$$

Relations (2), (5), (6) and (7) lead to the following approximation of $\bar{x}$:

$$\bar{x} \simeq a_1 - \frac{f(a_1)}{[a_1, a_2; f]} - \frac{[a_1, a_2, a_3; f]f(a_1)f(a_2)}{[a_1, a_2; f][a_2, a_3; f][a_1, a_3; f]} \tag{8}$$

or the equivalent formal representations from (4). Supposing that f has third degree derivatives at each point from $[a, b]$, then function f^{-1} has third degree derivatives at each point of F.

The following relation is satisfied for the third order derivative of f^{-1} [4], [10], [11], [12]:

$$[f^{-1}(y)]''' = \frac{3[f''(x)]^2 - f'(x)f'''(x)}{[f'(x)]^5} \tag{9}$$

where $y = f\,(x)$.

Denote $x_n \in [a, b]$ an approximation to the root $\bar{x}$ of (1).

We consider the following nodes in (8):

$$a_1 = x_n, a_2 = g(x_n), a_3 = g(g(x_n)).$$

Taking into account all six approximative representations of $\bar{x}$, obtained by permutations of set $(1,2,3)$ one obtains the following representations for the Steffensen-type method.

If

$$D(x_n) = \frac{[x_n, g(x_n), g(g(x_n)); f]}{[x_n, g(x_n); f][x_n, g(g(x_n)); f][g(x_n), g(g(x_n)); f]}$$

then the above considerations lead us to the following:

$$x_{n+1} = x_n - \frac{f(x_n)}{[x_n, g(x_n); f]} - D(x_n) f(x_n) \cdot f(g(x_n)); \tag{10}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{[x_n, g(g(x_n)); f]} - D(x_n) f(x_n) \cdot f(g(g(x_n))); \tag{11}$$

$$x_{n+1} = g(x_n) - \frac{f(g(x_n))}{[x_n, g(x_n); f]} - D(x_n) f(x_n) f(g(x_n)); \tag{12}$$

$$x_{n+1} = g(x_n) - \frac{f(g(x_n))}{[g(x_n), g(g(x_n)); f]} - D(x_n) f(g(x_n)) f(g(g(x_n))); \tag{13}$$

$$x_{n+1} = g(g(x_n)) - \frac{f(g(g(x_n)))}{[x_n, g(g(x_n)); f]} - D(x_n) f(x_n) f(g(g(x_n))); \tag{14}$$

$$x_{n+1} = g(g(x_n)) - \frac{f(g(g(x_n)))}{[g(x_n), g(g(x_n)); f]} - D(x_n) f(g(x_n)) f(g(g(x_n))). \tag{15}$$

From Newton's identity (3) one obtains the error representation:

$$\bar{x} - x_{n+1} = -[0, f(x_n), f(g(x_n)), f(g(g(x_n))); f^{-1}] f(x_n) f(g(x_n)) f(g(g(x_n))). \tag{16}$$

From the mean value formulas for divided differences one obtains for a fixed $x \in [a,b]$ the existence of $\eta \in F$ such that:

$$[0, f(x), f(g(x)), f(g(g(x))); f^{-1}] = \frac{[f^{-1}(\eta)]}{3!}.$$

Since $\eta \in F$ and f is a bijection, using (9) there results the existence of $\xi \in [a,b]$ such that

$$[0, f\,(x)\,, f(g(x)), f(g(g(x))); f^{-1}] = \frac{3[f''(\xi)]^2 - f'(\xi) f'''(\xi)}{6[f'(\xi)]^5}.$$

Denote

$$E(x) = 3[f''(x)]^2 - f'(x)f'''(x) \tag{17}$$

so that we obtain from (16)

$$\bar{x} - x_{n+1} = -\frac{E(\xi_n)}{6[f'(\xi_n)]^5} f(x_n) f(g(x_n)) f(g(g(x_n))). \tag{18}$$

where $\xi_n \in [a,b]$ is assigned to $x = x_n$.The x_{n+1} term is given by each of the relations (10)-(15).

2. **The convergence of Steffensen-type method**

In this section we shall see that conditions for the Steffensen-type method of third order given by any relations (10)-(15), lead to sequences which bilaterally approximate the root of (1).

We suppose that g satisfies the following conditions:

a) *there exists* $l \in \mathbb{R}$, $0 < l < 1$ *such that for all* $x \in [a,b]$:

$$|g(x) - g(\bar{x})| \leq l\,|x - \bar{x}|\,, \tag{19}$$

where $\bar{x}$ *is the common root of* (1) *and* $x = g(x)$;

b) *the function* g is *decreasing on* $[a,b]$;

c) *the equations* (1) *and* $x = g(x)$ *ar equivalent.*

The following result holds:

Theorem 1. *If functions* f, g *and element* $x_0 \in [a,b]$ *satisfy the conditions:*

i$_1$. *if* $x_0 \in [a,b]$, *then* $g(x_0) \in [a,b]$;

ii$_2$. f *has third order derivatives on* $[a,b]$;

iii$_3$. $f'(x) > 0$, $f''(x) \geq 0$, *for all* $x \in [a,b]$;

iv$_1$. $E(x) \leq 0$ *for all* $x \in [a,b]$, *where* E *is given by (17);*

v$_1$. *function* g *satisfies* a)-c);

vi$_1$. *equation (1) has a root* $\bar{x} \in [a,b]$.

Then the following properties are true:

j$_1$. *The elements of sequence $(x_n)_{n\geq 0}$ generated by (10), where x_0 satisfies* i$_1$, *remain in $[a,b]$ and for each $n=0,1,\ldots$, the following relations are satisfied:*

$$x_n \leq x_{n+1} \leq \bar{x} \leq g(x_{n+1}) \leq g(x_n) \tag{20}$$

if $f(x_0)<0$, or

$$x_n \geq x_{n+1} \geq \bar{x} \geq g(x_{n+1}) \geq g(x_n) \tag{21}$$

if $f(x_0)>0$.

jj$_1$. $\lim x_n = \lim g(x_n) = \bar{x}$;

jjj$_1$. $\max\{|x_{n+1}-\bar{x}|, |g(x_n)-\bar{x}|\} \leq |x_{n+1}-g(x_n)|$, *for each $n=0,1,\ldots$.*

Proof. Let $x_n \in [a,b]$, $n\geq 0$ for which $g(x_n)\in[a,b]$.

We consider first: $f(x_n)<0$, that is, $x_n<\bar{x}$. Since g is decreasing and using $g(\bar{x})=\bar{x}$ one obtains:

$$g(x_n) > \bar{x}$$

and $g(g(x_n))<\bar{x}$.

Relation (19) implies:

$$|g(g(x_n))-\bar{x}| \leq l\,|g(x_n)-\bar{x}| \leq l^2\,|x_n-\bar{x}|$$

from which one obtains:

$$a \leq x_n < g(g(x_n)) < \bar{x} < g(x_n) \leq b. \tag{22}$$

By use of iii$_1$, (22) and (10) one gets

$$x_{n+1} \geq x_n. \tag{23}$$

The assumptions ii$_1$-iv$_1$and from (22) and (18) we get

$$\bar{x} - x_{n+1} > 0,$$

i.e., $\bar{x} > x_{n+1}$, that is $f(x_{n+1})<0$.

Using (23) and assumption *c*) on g one obtains $g(x_{n+1}) \leq g(x_n)$ and $g(x_{n+1}) > g(\bar{x}) = \bar{x}$. Hence we have shown (20).

We consider now the case $f(x_n) > 0$, that is $x_n > \bar{x}$.

Taking in consideration (11) instead of (10) and by use of

$$g(x_n) < \bar{x}$$

and $g(g(x_n)) > \bar{x}$, one gets:

$$f(g(x_n)) < 0, \quad f(g(g(x_n))) > 0.$$

It is obvious to note relations (21).

Eventually, (20) and (21) show that sequences $(x_n)_{n\geq 0}$ and $(g(x_n))_{n\geq 0}$ converge. Denote $\lim x_n = a$, then,we obtain $\lim g(x_n) = g(a)$. Using (10) or (11) as $n \to \infty$, it results that $f(a) = 0$ and therefore $a = \bar{x}$, the unique solution of (1) on $[a, b]$. □

Remark 2. *Suppose the assumptions from Theorem 1 are satisfied and excepting* iii_1, *the following assumption holds.*

$f'(x) < 0$ *and* $f''\ (x) < 0$ *for each* $x \in [a, b]$ *and consider instead of (1) the following equation:*

$$h(x) = 0, \tag{24}$$

where h *is given by* $h(x) = -f(x)$.

Note that Theorem 1 holds for (24).

The proof is obvious, since $h'(x) > 0$ *and* $h''(x) > 0$, *for all* $x \in [a, b]$ *and* $E(x) = 3[h''(x)]^2 - h'(x)h''(x) < 0$, *that is* E *remains invariant.*

A result similar to Theorem 1 holds, for the case in which f is decreasing and convex.

Theorem 3. *If functions* f, g *and element* $x_0 \in [a, b]$ *satisfy the following conditions:*

- i_2. *if* $x_0 \in [a, b]$, *then* $g(x_0) \in [a, b]$;
- ii_2. f *has third order derivative on* $[a, b]$;
- iii_2. $f'(x) < 0$ *and* $f''(x) > 0$, *for all* $x \in [a, b]$;
- iv_2. $E(x) \leq 0$, *for all* $x \in [a, b]$;
- v_2. *function* g *satisfies* a)-c);

vi$_2$. *equation* (1) *has one root* $\bar{x} \in [a, b]$.

Then $(x_n)_{n\geq 0}$ *generated by (10) or (11), remains in* $[a, b]$*, and relation* j_1–jjj_1 *from Theorem 1 are satisfied, when* x_0 *satisfies* i_2.

Proof. The assumption iii$_1$ leads to $D(x) < 0$ for all $x \in [a, b]$. Let $x_n \in [a, b]$, $n \geq 0$, an element for which $g(x_n) \in [a, b]$.

If $x_n > \bar{x}$, then $f(x_n) < 0$ and $g(x_n) < \bar{x}$, $g(g(x_n)) > \bar{x}$.

From (19) one gets

$$|g(g(x_n)) - \bar{x}| \leq l^2 \, |x_n - \bar{x}| \, ,$$

that is the following relations hold:

$$a \leq g(x_n) < \bar{x} < g(g(x_n)) < x_n \leq b.$$

From iii$_2$, $f(x_2) < 0$ and using $D(x_n) < 0$, (10) one obtains

$$x_{n+1} < x_n.$$

The assumptions ii$_2$-iv$_2$ and relations (22), and (18) imply

$$\bar{x} - x_{n+1} < 0,$$

that is, $x_{n+1} > \bar{x}$, $f(x_{n+1}) < 0$. Obviously relations (21) hold.

Relations (20) and consequences jj$_1$ and jjj$_1$ are proven analogously to Theorem 1. $\square$

Remark 4. *If* f *is increasing and concave, that is,* $f'(x) > 0$ *and* $f'(x) < 0$, *then obviously* $h = -f$ *is decreasing and convex.*

If we replace in Theorem 3: function f *by function* h, *and if we take into account that function* E *remains invariant by this replacement, then we note that the statements of Theorem 3 remain true.*

3. Determination of the auxiliary function

In the following, by use of function f, we give a method to determine auxiliary function g, which could assure the control of interpolatory nodes.

If f is a convex function, i.e. $f''(x) > 0$, then for function g we consider

$$g(x) = x - \frac{f(x)}{f'(x)}. \tag{25}$$

If f is a concave function, then we can set

$$g(x) = x - \frac{f(x)}{f'(b)}. \tag{26}$$

Obviously in both cases we have $g'(x) < 0$ and thus function g satisfies assumption b).

It is clear that function g given by either (25) or (26), assures the equivalence of (1) and $x = g(x)$, i.e., g satisfies assumption c).

In order that g also satisfies assumption a), it is enough that the following relations hold:

$$\left|1 - \frac{f'(x)}{f(a)}\right| < 1,$$

for all $x \in [a, b]$, if f is convex function, or

$$\left|1 - \frac{f'(x)}{f(b)}\right| < 1,$$

for all $x \in [a, b]$, if f is a concave function.

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