

Chapter 1

Continuous fields

In physics the notion of continuum has no rigorous definition. The current definitions are based on the fact that the continuum is characterized by three separate space scales: the macroscopic scale of the global dimensions of the continuum, the constituent microscopic particles (usually molecules) scale, and an intermediary scale of the Lagrangian particles of the continuum, abbreviated continuum particle. For example, in [41] *the continuum particle* is defined as "an infinitesimal volume element, small enough in comparison with the volume of the body, but large in comparison with the distances between the molecules such that it should contain a significant number of molecules." In the fluid mechanics the name of *fluid particle* is used. Hence by the motion of a fluid particle we understand not the motion of a single molecule, but that of a volume containing many molecules.

From the thermodynamics point of view the continuum is a thermodynamic system in *incomplete equilibrium* [40]: "The relaxation time of an isolated system increases with the system dimensions. Due to this fact, the different small parts of the system reach the equilibrium faster than the entire system. This means that the different small parts of the system are described by their own equilibrium distribution functions, but with different values of the thermodynamic parameters. It is said in this case that the system is in incomplete equilibrium. During the time the incomplete equilibrium is slowly transformed into complete equilibrium, the thermodynamic parameters for each small part vary slowly as time passes and become finally the same for the entire isolated system." If the system is not completely isolated then the system has no more a global equilibrium state. However, if the interchanges of energy and mass with the environment are small enough, then its evolution consists in a sequence of incomplete equilibrium states. Each of these states can be described by means of a *continuous field* obtained by assigning to each point in the space the equilibrium thermodynamic parameters characteristic

to the continuum particle centered at that point.

Another name for the incomplete equilibrium is *local thermodynamic equilibrium*. Sometimes one talks about the local equilibrium principle [22]. We say that a macroscopic body obeys the local equilibrium principle if at any moment and in any point of the body there is a vicinity of macroscopic scale of that point which should behave as a near equilibrium system to which the linear thermodynamics of the irreversible process can be applied. A mathematical expression of the local thermodynamic equilibrium is formulated in [31] and [52].

Unlike statistical physics, in continuum mechanics one does not try to justify the concept of continuous field, but the existence of continuous fields is postulated. One assumes that they satisfy some generalizations of the conservation laws from mechanics and thermodynamics called *balance equations*. [59], [60] and [15]. In the case of mechanical and thermic phenomena, the balance equations are those well known for mass (the continuity equation), momentum (the second law of dynamics), energy (the first law of thermodynamics), and entropy (the second law of thermodynamics). The balance equations do not form a closed system of equations since the number of continuous fields included is larger than the number of balance equations. Therefore among the continuous fields there must be some additional equations called *constitutive relations* dependent on the type of the matter of the macroscopic continuous body, i.e., on its microscopic structure. The constitutive functions defining these constitutive relations must obey some restrictive conditions such as the principle of the independence on the reference frame, the entropy principle, and the thermodynamic stability [46].

Let Ψ be an additive physical quantity associated to a macroscopic body that can be modelled by a continuum, i.e., it is in incomplete equilibrium. Hence Ψ is a measurable function with respect to the Lebesgue measure of the three-dimensional Euclidian space. If in addition Ψ vanishes on every null measure set (it is absolutely continuous), then the Radon-Nicodym theorem [34] assures the existence of a function of space and time ψ named the volume density of Ψ , such that for any volume V , the integral $\int_V \psi d\mathbf{r}$ represent the amount of the physical quantity Ψ contained in V . This means that the mass distribution can be described by an ensemble of particles with their number approaching infinity and the mass of each particle converges to zero. So the mass of a finite volume remains finite.

At a regular point of a continuous medium (without shocks or other discontinuities) the local form of the balance equation is [46]

$$\partial_t \psi + \nabla \cdot (\psi \mathbf{u} + \mathbf{\Phi}) - (p + s) = 0, \quad (1.1)$$

where ∂_t is the time derivative, $\mathbf{\Phi}$ is the flux density of Ψ , \mathbf{u} is the mean

velocity, p is the production density of Ψ due to interior processes, and s is the supply density of Ψ controlled from the exterior of V . When $p = 0$, the balance equation becomes a conservation equation.

If the equation (1.1) is integrated over an arbitrary volume of the continuum, we obtain the integral form of the balance equation which has a direct physical interpretation. Thus, the amount of the physical quantity Ψ within an arbitrary volume can vary in time because of three processes. The first process is the flux of Ψ through the volume boundary, due either to the mean macroscopic motion, or to the microscopic fluctuations. The second process is the production of Ψ by phenomena taking place inside the volume. The third process is the generation of Ψ because of outside causes. So the balance equations represent a very general frame for the continuous physical phenomena. Therefore the Poincaré's statements [55] on the energy conservation can be also extended on the balance equations: "as we can not provide a general definition of the energy, the energy conservation principle means simply that there exists *something* that remains constant. Well, whatever would be the new notions that the future experiences will provide us on this world, we are sure in advance that it will be something that remains constant and which we may call *energy*." Similarly, whatever discoveries in the future, we can be sure that they will be included in one of the terms of the balance equation (1.1), such that any additive physical quantity would satisfy an equation of the same type.

The most frequently used balance equations are those for mass, momentum, and energy. In the following we give their most general form, valid for any continuum [46] without their justification on the bases of the dynamics and thermodynamics laws. The balance equation for mass, also known as *the continuity equation* is obtained from (1.1) by identification of ψ with the mass density ρ . Since the mass is conserved we have $p = 0$. Besides $\Phi = 0$ and $s = 0$ and then we have the equation

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0. \quad (1.2)$$

One notices that in this equation we have no continuous field which has to be expressed by a constitutive relation.

The balance equations for momentum are obtained from (1.1) by identifying ψ with the volume density of the component α of the momentum ρu_α . The respective flux is given by the symmetrical stress tensor $(\Phi_\alpha)_\beta = -\sigma_{\alpha\beta}$. The component α of the force which acts on the unity area from a plane surface with the normal \mathbf{n} is equal with $\sum_{\beta=1}^3 n_\beta \sigma_{\alpha\beta}$. The only possibility to generate the momentum is that due to the action of an exterior force. If we assume that they come from a potential field with the intensity \mathbf{g} (for

example the gravitational field), then in (1.1) we have $p = 0$ and $s = -\rho g_\alpha$ and the balance equation of the momentum reads

$$\partial_t(\rho u_\alpha) + \sum_{\beta=1}^3 \partial_\beta(\rho u_\alpha u_\beta) - \sum_{\beta=1}^3 \sigma_{\alpha\beta} = \rho g_\alpha, \quad (1.3)$$

where we denote by ∂_β the derivative with respect to the component β of the position vector. If the stress tensor $\sigma_{\alpha\beta}$ can be expressed by a constitutive relation in terms of ρ and \mathbf{u} , then (1.2) and (1.3) form together a nonlinear system of partial derivative equations. For example, for incompressible viscous fluids $\sigma_{\alpha\beta}$ is expressed in terms of the velocity derivatives by means of a single material constant (the viscosity coefficient) and the Navier-Stokes equation is obtained.

The balance equation for the total energy is obtained from (1.1) by identifying ψ with the volume density of the sum of the kinetic energy and the internal energy $\frac{1}{2}\rho \mathbf{u}^2 + \rho \varepsilon$. The flux Φ is formed in this case by two parts: the heat flux \mathbf{q} and the flux due to the stress tensor $\sum_{\beta=1}^3 u_\beta \sigma_{\alpha\beta}$. The total energy is modified because of the exterior forces such that $p = 0$ and $s = \rho \mathbf{g} \cdot \mathbf{u}$ and the balance equation (1.1) becomes

$$\partial_t \left(\frac{1}{2} \rho \mathbf{u}^2 + \rho \varepsilon \right) + \sum_{\beta=1}^3 \partial_\beta \left[\left(\frac{1}{2} \rho \mathbf{u}^2 + \rho \varepsilon \right) u_\beta - \sum_{\alpha,\beta=1}^3 u_\beta \sigma_{\alpha\beta} + q_\beta \right] = \rho \mathbf{u} \cdot \mathbf{g}. \quad (1.4)$$

Besides the constitutive relation for $\sigma_{\alpha\beta}$ a constitutive relation for the heat flux \mathbf{q} is needed. In the simplest cases this is the Fourier law in which only one parameter (the thermal conductivity) is used. Usually the internal energy is expressed in terms of the temperature by means of the caloric equation of state.

Balance equations can also be written for other physical quantities: entropy, kinetic momentum, internal energy, kinetic energy, etc. They also imply the existence of other constitutive relations than those involved into the equations (1.3) and (1.4). Besides the balance equations presented above, in this work we deal only with the balance equation for mass for the components of a mixture [5], [41], [46]. For mixtures the equation (1.2) holds only for the total density of the mixture. In the general balance equation (1.1) we identify ψ with the concentration $c(\mathbf{r}, t)$ of a component of the mixture defined as the number of particles of the component per unity volume. The concentration can also be defined as mass ratio or particles number ratio, but in this work we always use the definition given above. To simplify we consider that the mixture is in mechanical equilibrium, i.e., $\mathbf{u} = 0$. Then

in the general balance equation (1.1) Φ is the particles flux density denoted by \mathbf{j} which is non vanishing due to the diffusion. The term p can be non vanishing because of the chemical reactions, but in the following we do not consider such processes. Thus we obtain the diffusion equation

$$\partial_t c + \nabla \cdot \mathbf{j} = s, \quad (1.5)$$

where s is the source term . The balance equation (1.5) needs to be completed by a constitutive law that should relate the flux \mathbf{j} to c . The simplest constitutive law is Fick's law, valid for small gradients of the concentration

$$\mathbf{j} = -D \nabla c, \quad (1.6)$$

where D is the diffusion coefficient.