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A CONVERGENCY THEOREM CONCERNING THE CHORD METHOD

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Let X be a Banach space, and let $f : X \to X$ be a mapping to solve the equation:

$$f(x) = 0, \tag{1}$$

the chord method is well known, consisting of approximating the solution of (1) by elements of the sequence $(x_n)_{n\geq 0}$ generated by the following relations:

$$x_{n+1} = x_n - [x_{n-1}, x_n; f]^{-1} f(x_n), \quad n = 1, 2, \dots, x_0, x_1 \in X, \tag{2}$$

where $[x, y; f] \in \mathcal{L}(X)$ stands for the divided difference of f on $x, y \in X$. It is clear that to generate the elements of the sequence $(x_n)_{n\geq 0}$ by means of (2) we must ensure ourselves that at every iteration step the linear mapping $[x_{n-1}, x_n; f]$ is invertible. The mathematical literature dealing with the convergency of the chord method contains results which state by hypothesis that the mapping $[x, y; f]$ admits a bounded inverse for every $x, y \in D$, where D is a subset of X.

In this note we intend to establish convergency conditions for the method (2), supposing the existence of the inverse mapping only for the divided difference $[x_0, x_1; f]$.

Let $r > 0$ be a real number, and write $S(x_0, r) = \{x \in X : \|x - x_0\| \leq r\}$.

Theorem. If the mapping $f : X \to X$, the real number $r > 0$ and the element $x_1 \in X$ fulfil the conditions:

(i) the mapping $[x_0, x_1; f]$ admits a bounded inverse mapping, and $\| [x_0, x_1; f]^{-1} \| \leq B < +\infty$;

(ii) the bilinear mapping $[x, y, z; f]$ (the second order divided difference of f on x, y, z) is bounded for every $x, y, z \in S(x_0, r)$, that is, $\|[x, y, z; f]\| \leq L < \infty$;

(iii) $3BLr < 1$;

(iv) $\rho_0 = \alpha \|f(x_0)\| < 1$, $\rho_1 = \alpha \|f(x_1)\| \leq \rho_0^{t_1}$, where $\alpha = LB^2/(1 - 3BLr)^2$ and $t_1 = (1 + \sqrt{5})/2$;

(v) $B\rho_0/[\alpha(1 - \rho_0^{t_1 - 1})(1 - 3BLr)] \leq r$,

then the following properties hold:

(j) $x_n \in S(x_0, r)$ for every $n = 0, 1, \dots$;

(jj) the mapping $[x_{i-1}, x_i; f]$ admits bounded inverse for every $i = 1, 2, \dots$;

(jjj) equation (1) has at least one solution $x^* \in S(x_0, r)$;

(jv) the sequence $(x_n)_{n\geq 0}$ is convergent, and $\lim x_n = x^*$;

(v) $\|x^* - x_n\| \leq \frac{B\rho_0^{t_1^n}}{\left[\alpha(1-3BLr)\left(1-\rho_0^{t_1^n(t_1-1)}\right)\right]}$.

Proof. We shall firstly show that for every $x, y \in S(x_0, r)$ the following inequality holds:

$$\|\,[x_0, x_1; f]^{-1}\left([x_0, x_1; f] - [x, y; f]\right)\| \leq 3BLr < 1. \tag{3}$$

Taking into account hypothesis (ii) and the definition of the second order divided difference [2], it results:

$$\begin{aligned}\|[x_0, x_1; f] - [x, y, f]\| &\leq \|[x_0, x_1; f] - [x_1, x; f]\| + \|[x_1, x; f] - [x, y; f]\| \\ &\leq L\|x - x_0\| + L\|y - x_1\| < 3Lr.\end{aligned}$$

From the above inequality and hypothesis (i) there follows (3).

Using Banach's lemma on inverse mapping continuousness, it results from (3) that there exists $[x, y; f]^{-1}$, and:

$$\|\,[x, y; f]^{-1}\| \leq B/(1 - 3BLr).$$

Suppose now that the following properties hold:

(a) $x_i \in S$, $i = \overline{0, k}$;

(b) $\rho_i = \alpha\|f(x_i)\| \leq \rho_0^{t_1^i}, i = \overline{0, k}$;

and prove that they hold for $i = k + 1$, too.

Indeed, to prove that $x_{n+1} \in S$ we estimate the difference:

$$\begin{aligned}\|x_{k+1} - x_0\| &\leq \sum_{i=0}^{k}\|x_{i+1} - x_i\| \leq \tfrac{B\alpha^{-1}}{1-3BLr}\sum_{i=0}^{k}\alpha\|f(x_i)\| \\ &\leq B\rho_0\left[\alpha(1 - \rho_0^{t_1-1})(1 - 3BLr)\right]^{-1} \leq r\end{aligned}$$

To prove (b) for $i = k + 1$ we use Newton's identity:

$$f(z) = f(x) + [x, y; f](z - x) + [x, y, z; f](z - x)(z - y) \tag{4}$$

and the obvious identity:

$$x - [x, y; f]^{-1} f(x) = y - [x, y; f]^{-1} f(y). \tag{5}$$

Applying (4) and taking into account (2) and (5), we deduce:

$$\begin{aligned}\|f(x_{k+1})\| &= \|f(x_{k+1}) - f(x_k) - [x_{k-1}, x_k; f](x_{k+1} - x_k)\| \\ &\leq \|[x_{k-1}, x_k, x_{k+1}; f]\| \cdot \|x_{k+1} - x_k\| \cdot \|x_{k+1} - x_{k-1}\| \\ &\leq LB^2\|f(x_k)\| \cdot \|f(x_{k-1})\| \cdot (1 - 3BLr)^{-2} \\ &\leq LB^2(1 - 3BLr)^{-2} \cdot \alpha^{-2}\rho_k\rho_{k-1},\end{aligned}$$

and writing $\rho_{k+1} = \alpha\|f(x_{k+1})\|$ we obtain:

$$\rho_{k+1} \leq \rho_k\rho_{k-1} < \rho_0^{t_1^k + t_1^{k-1}} = \rho_0^{t_1^{k+1}}$$

that is, the property (b) holds for $i = k + 1$, too.

From (2) one obtains the following inequalities:

$$\|x_{n+1} - x_n\| \leq B\alpha^{-1} (1 - 3BLr)^{-1} \rho_n \leq \frac{B\rho_0^{t_1^n}}{\alpha(1-3BLr)}$$

for every $n = 0, 1, \ldots$

From these relations, for every $m, n \in \mathbb{N}$ we deduce:

(6)
$$\|x_{n+m} - x_n\| \leq \sum_{i=n}^{m+n-1} \frac{B\rho_0^{t_1^n}}{\alpha(1-3BLr)}$$
$$\leq B\rho_0^{t_1^n}\alpha^{-1}(1 - 3BLr)^{-1}\left(1 - \rho_0^{t_1^{n(t_1-1)}}\right)^{-1}$$

from which, taking into account the fact that $t_1 > 1$, there follows that the sequence $(x_n)_{n\geq 0}$ is fundamental.

At limit $(m \to \infty)$, (6) leads to

$$\|x^* - x_n\| < B\rho_0^{t_1^n}\alpha^{-1} (1 - 3BLr)^{-1} \left(1 - \rho_0^{t_1^n(t_1-1)}\right)^{-1}$$

where $x^* = \lim_{n\to\infty} x_n$. For $n = 0$ follows that $x^* \in S(x_0, r)$.

It is obvious that $f(x^*) = 0$. □

Remark. In the conditions of the above proved theorem, it results form (3) that x^* is the unique solution of equation (1) in the sphere $S(x_0, r)$.

Indeed, supposing that x^* and y^* are two solutions of equation (1) in $S(x_0, r)$, $x^* \neq y^*$, and using the identities:

$$x^* = x^* - [x_0, x_1; f]^{-1} f(x^*)$$
$$y^* = y^* - [x_0, x_1; f]^{-1} f(y^*)$$

we deduce

$$x^* - y^* = \left(I - [x_0, x_1; f]^{-1} [x^*, y^*; f]\right)(x^* - y^*)$$

from which, taking into account (3) it follows that:

$$\|x^* - y^*\| \leq 3BLr \|x^* - y^*\|$$

but, since $3BLr < 1$, it results that the relation $x^* \neq y^*$ is impossible. □

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