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ON SOME FIXED POINT THEOREMS OF DEIMLING

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1. INTRODUCTION

Let E be a real Banach space, $K \subset E$ a cone, i.e. K is closed convex such that $\lambda K \subset K$ for all $\lambda \geq 0$ and $K \cap (-K) = \{0\}$, and let $f: K_r = \{x \in K;\ |x| \leq r\} \to E$ $(r > 0)$ be compact or at least α-condensing. The following results on the fixed points of f in K_r or in a shell $K_{\rho,r} = \{x \in K;\ \rho \leq |x| \leq r\}$ $(0 < \rho < r)$ were established by Deimling [1] under the assumption that f is weakly inward on the conical boundary of K_r, i.e.

$$x \in \partial K, \qquad |x| \leq r, \qquad x^* \in K^*, \qquad x^*(x) = 0 \qquad \text{imply } x^*(f(x)) \geq 0. \tag{1.1}$$

THEOREM 1.1 [1]. If $f: K_r \to E$ is α-condensing and satisfies (1.1) and

$$f(x) \neq \lambda x \qquad \text{on } |x| = r \qquad \text{for all } \lambda > 1 \tag{1.2}$$

then f has a fixed point in K_r.

THEOREM 1.2 [1]. If $f: K_r \to E$ is α-condensing and satisfies (1.1), (1.2) and

$$x - f(x) \neq \lambda e \qquad \text{on } |x| = \rho \qquad \text{for all } \lambda > 0 \tag{1.3}$$

for some $\rho \in (0, r)$ and $e \in K\backslash\{0\}$, then f has a fixed point in $K_{\rho,r}$.

THEOREM 1.3 [1]. If K_1 is not compact, $f: K_r \to E$ is compact and satisfies (1.1), (1.2) and

$$f(x) \neq \lambda x \qquad \text{on } |x| = \rho \qquad \text{for } \lambda \in (0, 1) \qquad \text{and} \qquad \inf_{|x| = \rho} |f(x)| > 0 \tag{1.4}$$

for some $\rho \in (0, r)$, then f has a fixed point in $K_{\rho,r}$.

Clearly, condition (1.1) is satisfied if f maps K_r into K, and in this case it is known (see [2, Section 20]) that such results are consequences of the properties of the topological index. As we shall see, in this case these results can be derived as well from the topological transversality theorem of Granas (see [3]) together with the Schauder–Sadovskii fixed point theorem, without using index theory.

The purpose of this note is to show that under assumption (1.1) such results can still be derived in this way, but this time from the generalized topological transversality theorem given in [4] together with a Schauder–Sadovskii-type theorem for weakly inward maps. Briefly, the existence of a fixed point for f will be a consequence of the "essentiality" of $I - f$ (I being the identity map) in a certain class of maps. By this method, we shall give new proofs for theorems 1.1, 1.2 and 1.3 which are closer to the proofs (based on index theory) of the corresponding classical results than those in [1]. Moreover, by our method we show that such results as theorems 1.2 and 1.3 remain true if we replace the balls $B_\rho(0)$ and $B_r(0)$ by any open bounded U_1, U such that $\bar{U}_1 \subset U$ and $0 \in U_1$.

2. PRELIMINARIES

(1) *The Sadovskii-type fixed point theorem for weakly inward maps*

We denote by α the Kuratowski measure of noncompactness. A map $f: D \to E$, $D \subset E$, is called α-condensing if it is continuous bounded and $\alpha(f(M)) < \alpha(M)$ for each bounded $M \subset D$ with $\alpha(M) > 0$.

A first tool in our proofs will be the following theorem.

THEOREM 2.1 [2, theorem 18.3]. Let E be a Banach space, $D \subset E$ closed bounded convex, $f: D \to E$ α-condensing and weakly inward, i.e.

$$f(x) \in \overline{J_D(x)} \qquad \text{for all } x \in D, \tag{2.1}$$

where $J_D(x) = \{x + \lambda(y - x);\ \lambda \geq 0,\ y \in D\}$. Then f has a fixed point.

Recall that in the case where D is a cone, condition (2.1) becomes

$$x \in \partial K, \qquad x^* \in K^*, \qquad x^*(x) = 0 \qquad \text{imply } x^*(f(x)) \geq 0,$$

where $K^* = \{x^* \in E^*;\ x^*(x) \geq 0 \text{ on } K\}$ is the dual cone of K.

Clearly, condition (2.1) holds if $f(D) \subset D$. In this case theorem 2.1 is just the Sadovskii fixed point theorem.

(2) *Generalized topological transversality*

Let X be a normal topological space, A a proper closed subset of X, Y a set and B a proper subset of Y. Consider a nonvoid class of maps

$$\mathcal{A} = \mathcal{A}_A^B(X, Y) \subset \{F: X \to Y;\ F^{-1}(B) \cap A = \emptyset\}$$

whose elements are called admissible maps and let

$$d: \{F^{-1}(B);\ F \in \mathcal{A}_A^B(X, Y)\} \cup \{\emptyset\} \to \Lambda$$

be any map with values in a nonempty set Λ. Denote $\theta = d(\emptyset)$. An admissible map F is said to be d-essential if

$$d(F^{-1}(B)) = d(F'^{-1}(B)) \neq \theta$$

for any admissible map F' having the same restriction to A as F, i.e. $F|_A = F'|_A$. Otherwise, F is said to be d-inessential. Also consider an equivalence relation $\sim$ on $\mathcal{A}$ such that the following two conditions hold:

(A) if $F|_A = F'|_A$ then $F \sim F'$;

(H) if $F \sim F'$ then there exists $H: [0, 1] \times X \to Y$ such that $\operatorname{cl}(\bigcup\{H(t, \cdot)^{-1}(B);\ t \in [0, 1]\}) \cap A = \emptyset$, $H(1, \cdot) = F$, $H(0, \cdot) = F'$ and $H(\eta(\cdot), \cdot) \in \mathcal{A}$ for any continuous $\eta: X \to [0, 1]$ satisfying $\eta(x) = 1$ for all $x \in A$.

A second tool in our proofs will be the following theorem.

THEOREM 2.2 [4]. If F and F' are two admissible maps such that $F \sim F'$, then F and F' are both d-essential or both d-inessential, and in the first case one has

$$d(F^{-1}(B)) = d(F'^{-1}(B)) \neq \theta. \tag{2.2}$$

When we deal only with the existence of solutions to the inclusion $F(x) \in B$ and we do not have to "measure" the set $F^{-1}(B)$ of all solutions, it is sufficient to take as d the simplest indicator function

$$d(\varnothing) = 0 \quad \text{and} \quad d(M) = 1 \quad \text{for } M \neq \varnothing, \tag{2.3}$$

taking $\Lambda = \{0, 1\}$ and $\theta = 0$. In the case where d is given by (2.3), we shortly speak about essentiality instead of d-essentiality (see [5]). This will be the case throughout the paper except in remarks 3.5 and 3.6.

As an example, let $C \subset E$ be closed convex, $U \subset C$ be bounded open in C, $0 \in C$ and $\bar{U}$ and ∂U denote the closure and the boundary of U in C. Then, if we set: $X = \bar{U}$, $A = \partial U$, $Y = E$, $B = \{0\}$,

$$\mathcal{A} = \mathcal{A}^0_{\partial U}(\bar{U}, E) = \{F = I - g;\ g: \bar{U} \to C \text{ is } \alpha\text{-condensing and } x \neq g(x) \text{ for } x \in \partial U\} \tag{2.4}$$

and $F = I - g \sim F' = I - g'$ if and only if

there exists h: $[0, 1] \times \bar{U} \to C$ α-condensing such that
$h(0, \cdot) = g'$, $h(1, \cdot) = g$ and $x \neq h(t, x)$ for $t \in [0, 1]$ and $x \in \partial U$;

then theorem 2.2 (with d given by (2.3)) reduces to the transversality theorem of Granas (see [3]) adapted for α-condensing maps. Also, the map I is essential in the class (2.4) as follows by the Sadovskii fixed point theorem.

3. RESULTS

We start with a Leray–Schauder-type continuation theorem which extends theorem 1.1.

THEOREM 3.1. Let $U \subset E$ be open bounded, $x_0 \in U \cap K = K_U$ and h: $[0, 1] \times \bar{K}_U \to E$ α-condensing such that $h(0, x) = x_0$ for all $x \in \bar{K}_U$ and for each $t \in [0, 1]$ the map $g = h(t, \cdot)$ satisfies

$$x \in \bar{U} \cap \partial K, \qquad x^* \in K^*, \qquad x^*(x) = 0 \qquad \text{imply } x^*(g(x)) \geq 0 \tag{3.1}$$

and

$$g(x) \neq x \qquad \text{for all } x \in K \cap \partial U. \tag{3.2}$$

Then there exists $x \in K_U$ such that $h(1, x) = x$.

Proof. We shall apply theorem 2.2 where: $X = \bar{K}_U$, $A = K \cap \partial U$, $Y = E$, $B = \{0\}$,

$$\begin{aligned}\mathcal{A} &= \mathcal{A}^0_{K\cap\partial U}(\bar{K}_U, E)\\ &= \{F = I - g;\ g: \bar{K}_U \to E \text{ is } \alpha\text{-condensing and satisfies (3.1) and(3.2)}\}\end{aligned} \tag{3.3}$$

and $F = I - g \sim F' = I - g'$ if and only if

$$\begin{aligned}&\text{there is } h\colon [0, 1] \times \bar{K}_U \to E \ \alpha\text{-condensing such that}\\ &h(0, \cdot) = g',\ h(1, \cdot) = g \text{ and } h(t, \cdot) \text{ satisfies (3.1), (3.2) for each } t \in [0, 1].\end{aligned} \tag{3.4}$$

It is easy to see that conditions (A) and (H) are satisfied for $H(t, \cdot) = I - h(t, \cdot)$. Also, $F = I - h(1, \cdot) \sim F' = I - x_0$. Hence in order to prove the existence of a zero for F it is sufficient to prove that $F' = I - x_0$ is essential or, equivalently, that any α-condensing map $g\colon \bar{K}_U \to E$ which satisfies (3.1) and equals x_0 on $K \cap \partial U$, has a fixed point. Indeed, denote $\tilde{g}\colon K_R \to E$, where $R > 0$ is such that $\bar{U} \subset \{x \in E;\ |x| < R\}$, the map defined by $\tilde{g}(x) = g(x)$ for $x \in \bar{K}_U$ and $\tilde{g}(x) = x_0$ otherwise. Clearly, $\tilde{g}$ is α-condensing and weakly inward on K_R. Thus, by theorem 2.1, $\tilde{g}$ has a fixed point in K_R which, obviously, is a fixed point of g. The proof is complete.

Remark 3.1. Theorem 1.1 follows from theorem 3.1 if we take: $U = \{x \in E;\ |x| < r\}$, $x_0 = 0$ and $h(t, \cdot) = tf$. Consequently, we get a new proof of theorem 1.1 which is essentially different from the original one in [1].

Next we shall give a common generalization to theorems 1.2 and 1.3.

THEOREM 3.2. Let U_1, U be open bounded such that $\bar{U}_1 \subset U \subset E$, $x_0 \in U_1 \cap K$, $h\colon [0, 1] \times \bar{K}_U \to E$ be as in theorem 3.1, and let $h_1\colon [0, 1] \times \bar{K}_{U_1} \to E$ α-condensing satisfying (3.1) and (3.2) (with U_1 instead of U). Suppose also that $h_1(1, x) = h(1, x)$ for all $x \in K \cap \partial U_1$ and $h_1(0, x) \neq x$ for all $x \in K_{U_1}$. Then there exists $x \in K \cap (U \backslash \bar{U}_1)$ such that $h(1, x) = x$.

Proof. Additionally to the class $\mathcal{A}$ in (3.3), let us consider $\mathcal{A}_1 = \mathcal{A}^0_{K \cap \partial U_1}(\bar{K}_{U_1}, E)$ endowed with the equivalence relation $\sim_1$ defined by (3.4) with U_1 instead of U, and also the following class of maps from $K \cap (\bar{U} \backslash U_1)$ into E

$$\begin{aligned}\mathcal{A}_0 &= \mathcal{A}^0_{K \cap (\partial U \cup \partial U_1)}(K \cap (\bar{U} \backslash U_1), E)\\ &= \{I - g;\ g\colon \bar{K}_U \to E \text{ is } \alpha\text{-condensing, satisfies (3.1) and } g(x) \neq x \text{ for } x \in K \cap (\partial U \cup \partial U_1)\}.\end{aligned} \tag{3.5}$$

Now, by theorem 3.1, the map $F = I - h(1, \cdot)$ is essential in $\mathcal{A}$. On the other hand, by the same theorem, since $F \sim_1 I - h_1(0, \cdot)$ and $h_1(0, x) \neq x$ for all $x \in K_{U_1}$, we have that F is inessential in $\mathcal{A}_1$. Consequently, F is essential in $\mathcal{A}_0$. Therefore, $h(1, \cdot)$ has a fixed point in $K \cap (U \backslash \bar{U}_1)$.

A first consequence of theorem 3.2 is an extension of theorem 1.2.

COROLLARY 3.1. Let U_1, U be open bounded such that $\bar{U}_1 \subset U \subset E$, $x_0 \in U_1 \cap K$ and $f\colon \bar{K}_U \to E$ α-condensing satisfying (3.1),

$$f(x) - x_0 \neq \lambda(x - x_0) \qquad \text{for } x \in K \cap \partial U \qquad \text{and} \qquad \lambda > 1 \tag{3.6}$$

and

$$x - f(x) \neq \lambda e \qquad \text{for } x \in K \cap \partial U_1 \qquad \text{and} \qquad \lambda > 0, \tag{3.7}$$

for some $e \in K \backslash \{0\}$. Then f has a fixed point in $K \cap (\bar{U} \backslash U_1)$.

Proof. Let us suppose $f(x) \neq x$ for $x \in \partial U \cup \partial U_1$. Then we may apply theorem 3.2, where $h(t, \cdot) = (1 - t)x_0 + tf$ and $h_1(t, \cdot) = f + (1 - t)\lambda e$, with $\lambda > 0$ large enough that $f(x) + \lambda e \neq x$ for all $x \in \bar{K}_{U_1}$.

Another consequence of theorem 3.2 is the following extension of theorem 1.3 and of theorem 20.2 in [2].

COROLLARY 3.2. Let U_1, U be open bounded such that $\bar{U}_1 \subset U \subset E$, $x_0 \in U_1 \cap K$ and $f: \bar{K}_U \to E$ compact satisfying (3.1), (3.6) and

$$f(x) \neq \lambda x \qquad \text{for } x \in K \cap \partial U_1, \qquad \lambda \in (0, 1). \tag{3.8}$$

Assume that there exists $e \in K \setminus \{0\}$ such that $-\lambda e \notin \overline{f(K \cap \partial U_1)}$ for all $\lambda \geq 0$. Then f has a fixed point in $K \cap (\bar{U} \setminus U_1)$.

Proof. Let us suppose $f(x) \neq x$ for $x \in \partial U \cup \partial U_1$. Then we may apply theorem 3.2 where $h(t, \cdot) = (1 - t)x_0 + tf$ and $h_1(t, \cdot) = f + \mu I + (1 - 2t)\lambda e$ for $0 \leq t \leq 1/2$ and $h_1(t, \cdot) = f + 2(1 - t)\mu I$ for $1/2 \leq t \leq 1$, with some suitable $\mu \in [0, 1)$ and $\lambda > 0$. First we choose $\lambda > 0$ such that $h_1(0, x) = f(x) + \mu x + \lambda e \neq x$ for all $x \in \bar{K}_{U_1}$ and any $\mu \in [0, 1]$. Next we observe that, since f is compact and μI $(0 \leq \mu < 1)$ is a contraction, we have that h_1 is α-condensing. Clearly, $h_1(t, \cdot)$ satisfies (3.1) (with U_1 instead of U). As regards condition (3.2), observe that for $t \in [1/2, 1]$, by (3.8), we trivially have $h_1(t, x) \neq x$ on $K \cap \partial U_1$, while for $t \in [0, 1/2)$ the condition $h_1(t, x) \neq x$ or equivalently, $-(1 - 2t)\lambda e \neq f(x) - (1 - \mu)x$, is satisfied on $K \cap \partial U_1$ if we choose $\mu \in [0, 1)$ close enough to 1.

Remark 3.2. The conditions

$$\inf_{x \in K \cap \partial U_1} |f(x)| > 0 \text{ and } K_1 \text{ is not compact}$$

are sufficient for the assumption "there exists $e \in K \setminus \{0\}$ such that $-\lambda e \notin \overline{f(K \cap \partial U_1)}$ for all $\lambda \geq 0$" to be satisfied (see [1]). In case $f(K_U) \subset K$, this assumption is equivalent to $\inf_{x \in K \cap \partial U_1} |f(x)| > 0$.

Remark 3.3. The tricks from [1] (see also [6]) can be used to show the inessentiality of $I - f$ in the class $\mathcal{Q}_1 = \mathcal{Q}^0_{K \cap \partial U_1}(\bar{K}_{U_1}, E)$ (where $U = B_r(0)$, $U_1 = B_\rho(0)$) directly, starting from the definition, without using homotopy (continuation) methods. Because of their geometrical rather than topological nature, those tricks do not work for general open bounded sets U_1, U (see [6, remark 2]).

Remark 3.4. In the classical case, i.e. $f(K_U) \subset K$, we may require that all maps in classes (3.3), (3.5) and in the definitions of relations $\sim$ and $\sim_1$, take values into K. Then, the main tools in our proofs are the classical topological transversality theorem of Granas and the fixed point theorem of Sadovskii. Therefore, we get new proofs of the classical results without using index theory and different than those in [1].

Remark 3.5. The proofs based on index theory for the classical results can be found too from our proofs if instead of the notion "essential" we use "d-essential" with d defined by

$$d(F^{-1}(0)) = i(f, \Omega, K) \tag{3.9}$$

where $\Omega \subset K$ is open bounded, $F = I - f$, $f: \bar{\Omega} \to K$ is α-condensing with $f(x) \neq x$ on $\partial\Omega$ and i denotes the fixed point index. Then, the d-essentiality of F means that $i(f, \Omega, K) \neq 0$ $(= \theta)$ and theorem 2.2 expresses the homotopy invariance of the index.

Remark 3.6. In case E is reflexive and the maps are supposed completely continuous, we may also use (3.9), where $i(f, \Omega, K)$ is the fixed point index for weakly inward maps recently defined in [7].

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REFERENCES

1. DEIMLING K., Positive fixed points of weakly inward maps, *Nonlinear Analysis* **12**, 223–226 (1988).
2. DEIMLING K., *Nonlinear Functional Analysis*. Springer, Berlin (1985).
3. DUGUNDJI J. & GRANAS A., *Fixed Point Theory I*. Polish Scientific Publishers, Warsaw (1982).
4. PRECUP R., On the topological transversality principle, *Nonlinear Analysis* **20**, 1–9 (1993).
5. PRECUP R., Generalized topological transversality and existence theorems, *Libertas Mathematica* **11**, 65–79 (1991).
6. DEIMLING K. & CHUAN H. S., Fixed points of weakly inward maps in conical shells, *Nonlinear Analysis* **12**, 227–230 (1988).
7. HU S. & SUN Y., Fixed point index for weakly inward mappings, *J. math. Analysis Applic.* **172**, 266–273 (1993).