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CONTINUATION RESULTS FOR MAPPINGS OF CONTRACTIVE TYPE

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Abstract. We survey fixed point results for mappings of contractive type in metric spaces, quasi-uniform and uniform spaces and, more general, in syntopogenous spaces. The results involve usual contractions, non-expansive mappings, Caristi type mappings and generalized contractions. The focus is on the continuation theory for such types of mappings. We stress on our own results which have been obtained since 1980. For the reader convenience, some of the results which appeared in less accessible publications are here presented together with their proofs.
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1 The discrete continuation principle

Let us first recall Banach contraction principle.

Proposition 1 *Let (X,d) be a complete metric space and $T:X\to X$ a contraction, i.e.*

$$d\left(T\left(x\right),T\left(y\right)\right)\leq l\,d\left(x,y\right)$$

for all $x,y\in X$ and some fixed $l\in[0,1)$. Then T has a unique fixed point x^* and for every $x_0\in X$ and $k\in\mathbf{N}$, one has

$$d\left(T^k\left(x_0\right),x^*\right)\leq\frac{l^k}{1-l}d\left(x_0,T\left(x_0\right)\right).$$

A continuation version of the Banach contraction principle was given by Gatica-Kirk [10] in case of Banach spaces and later by Granas [14] for general complete metric spaces. Extensions for set-valued maps are due to Frigon-Granas [8] and for weakly contractive maps, to Frigon [6]. In [24] we stated and proved the following computational version of Granas' continuation principle for contraction maps on complete metric spaces.

Theorem 1 *[24] Let (X, d) be a complete metric space and U be an open set of X. Let $H : \overline{U} \times [0,1] \to X$, $H_\lambda = H(.,\lambda)$, and assume that the following conditions are satisfied:*

(a1) *there is $l \in [0,1)$ such that*

$$d(H(x,\lambda), H(y,\lambda)) \le l\, d(x,y)$$

for all $x, y \in \overline{U}$ and $\lambda \in [0,1]$;

(a2) *$H(x,\lambda) \ne x$ for all $x \in \partial U$ and $\lambda \in [0,1]$;*

(a3) *H is continuous in λ, uniformly for $x \in \overline{U}$, i.e. for each $\varepsilon > 0$ and $\lambda \in [0,1]$, there is $\rho > 0$ such that $d(H(x,\lambda), H(x,\mu)) < \varepsilon$ whenever $x \in \overline{U}$ and $|\lambda - \mu| < \rho$.*

In addition suppose that H_0 has a fixed point. Then, for each $\lambda \in [0,1]$, there exists a unique fixed point $x(\lambda)$ of H_λ. Moreover, $x(\lambda)$ depends continuously on λ and there exists $0 < r \le \infty$, integers $m, n_1, n_2, ..., n_{m-1}$ and numbers $0 < \lambda_1 < \lambda_2 < ... < \lambda_{m-1} < \lambda_m = 1$ such that for any $x_0 \in X$ satisfying $d(x_0, x(0)) \le r$, the sequences $(x_{j,k})_{k\ge 0}$, $j = 1, 2, ..., m$,

$$\begin{aligned} &x_{1,0} = x_0 \\ &x_{j,k+1} = H_{\lambda_j}(x_{j,k}), \quad k = 0,1,... \\ &x_{j+1,0} = x_{j,n_j}, \quad j = 1,2,...,m-1 \end{aligned}$$

are well defined and satisfy

$$d(x_{j,k}, x(\lambda_j)) \le \frac{l^k}{1-l} d(x_{j,0}, H_{\lambda_j}(x_{j,0})) \quad (k \in \mathbf{N}).$$

To make applicable the above iterative method we have to know how to obtain r, m, $n_1, ..., n_{m-1}$ and the partition $0 < \lambda_1 < ... < \lambda_{m-1} < 1$. Thus, we may take

$$r \le \inf\{d(x(\lambda), y) : \ y \in \partial U, \ \lambda \in [0,1]\}.$$

Next we consider $h > 0$ such that

$$d(H(x,\lambda), H(x,\mu)) \le (1-l)\, r$$

for all $x \in \overline{U}$ and $\lambda, \mu \in [0,1]$ with $|\lambda - \mu| \le h$. Such a h exists because of (a3). Now we choose any partition $0 = \lambda_0 < \lambda_1 < ... < \lambda_{m-1} < \lambda_m = 1$ of $[0,1]$ such that $\lambda_{j+1} - \lambda_j \le h$ for $j = 0, 1, ..., m-1$. Suppose we know the unique fixed point $x(0)$ of H_0 and we wish to obtain an approximation $\overline{x}_1$ of $x(1)$ with $d(\overline{x}_1, x(1)) \le \varepsilon$. Then we take any point x_0 such that $d(x_0, x(0)) \le r$ and we apply the following

Iterative procedure:

Set $n_0 := 0$ and $x_{0,n_0} := x_0$;

For $j := 1$ to $m-1$ do
$x_{j,0} := x_{j-1,n_{j-1}}$
$k := 0$
While $l^k (1-l)^{-1} d(x_{j,0}, H_{\lambda_j}(x_{j,0})) > r$
$x_{j,k+1} := H_{\lambda_j}(x_{j,k})$
$k := k+1$
$n_j := k$
Set $k := 0$
While $l^k (1-l)^{-1} d(x_{m,0}, H_1(x_{m,0})) > \varepsilon$
$x_{m,k+1} := H_1(x_{m,k})$
$k := k+1$
Finally take $\bar{x}_1 = x_{m,k}$.

This result has a more general version for spaces endowed with two metrics.

Theorem 2 *[24] Let (X, δ) be a complete metric space and d another metric on X. Let $D \subset X$ be δ-closed and U a d-open set of X with $U \subset D$. Let $H : D\times[0,1] \to X$ and assume that the following conditions are satisfied:*

(i) *there is $l \in [0,1)$ such that*

$$d(H(x,\lambda), H(y,\lambda)) \le l\, d(x, y)$$

for all $x, y \in D$ and $\lambda \in [0,1]$;

(ii) *$H(x,\lambda) \ne x$ for all $x \in D \setminus U$ and $\lambda \in [0,1]$;*

(iii) *H is uniformly (d, δ)-continuous;*

(iv) *H is (δ, δ)-continuous;*

(v) *$H(x,\lambda)$ is d-continuous in λ, uniformly for $x \in U$, i.e. for each $\varepsilon > 0$ and $\lambda \in [0,1]$, there is $\rho > 0$ such that $d(H(x,\lambda), H(x,\mu)) < \varepsilon$ whenever $x \in U$ and $|\lambda - \mu| < \rho$.*

In addition suppose that H_0 has a fixed point. Then, for each $\lambda \in [0,1]$, there exists a unique fixed point $x(\lambda)$ of $H_\lambda := H(\,.\,,\lambda)$. Moreover, $x(\lambda)$ depends d-continuously on λ and there exists $0 < r \le \infty$, integers $m, n_1, n_2, ..., n_{m-1}$ and numbers $0 < \lambda_1 < \lambda_2 < ... < \lambda_{m-1} < \lambda_m = 1$ such that for any $x_0 \in X$ satisfying $d(x_0, x(0)) \le r$, the sequences $(x_{j,k})_{k\ge 0}$, $j = 1, 2, ..., m$,

$$\begin{array}{ll} x_{1,0} = x_0 & \\ x_{j,k+1} = H_{\lambda_j}(x_{j,k}), & k = 0, 1, ... \\ x_{j+1,0} = x_{j,n_j}, & j = 1, 2, ..., m-1 \end{array}$$

are well defined and satisfy

$$d(x_{j,k}, x(\lambda_j)) \le \frac{l^k}{1-l} d(x_{j,0}, H_{\lambda_j}(x_{j,0})) \quad (k \in \mathbf{N}),$$

$$\delta(x_{j,k}, x(\lambda_j)) \to 0 \quad \text{as } k \to \infty.$$

The following application of Theorem 1 to evolution equations in Hilbert spaces was presented in [25]. Consider the initial-value problem for a nonlinear evolution equation in a Hilbert space E

$$\begin{cases} u'(t) + Au(t) = f(t, u(t)), & 0 \leq t \leq T \\ u(0) = 0 \end{cases} \tag{1}$$

where $A : D(A) \subset E \to E$ is a linear maximal monotone map. Let $\{S(t)\}_{t \geq 0}$ be the continuous semigroup of linear contractions generated by A. We seek generalized solution (mild solution) of (1), that is a function $u \in C([0,T]; E)$ with

$$u(t) = \int_0^t S(t-s) f(s, u(s)) \, ds, \quad 0 \leq t \leq T.$$

Theorem 3 *[25] Let E be a Hilbert space, $f : [0,T] \times E \to E$ and $A : D(A) \subset E \to E$. Assume that the following conditions are satisfied:*

(i) A is a maximal monotone linear map;

(ii) f is a continuous map, and for each $r > 0$, there exists $L_r \geq 0$ such that

$$|f(t,x) - f(t,y)| \leq L_r |x - y| \tag{2}$$

for all $t \in [0,T]$ and $x, y \in E$ satisfying $|x|, |y| \leq r$;

(iii) there exists a nondecreasing continuous function $\psi : [0, \infty) \to (0, \infty)$ such that

$$|f(t,x)| \leq \psi(|x|) \tag{3}$$

for all $t \in [0,T]$, $x \in E$, and

$$T < \int_0^\infty \frac{1}{\psi(\tau)} d\tau. \tag{4}$$

Then (1) has a unique generalized solution which can be approximated by iterations.

Proof. Consider the family of equations

$$u(t) = \lambda \int_0^t S(t-s) f(s, u(s)) \, ds, \quad 0 \leq t \leq T, \tag{5}$$

for $\lambda \in [0,1]$. According to (4), there exists $R > 0$ with

$$T < \int_0^R \frac{1}{\psi(\tau)} d\tau. \tag{6}$$

Suppose that $u \in C([0,T]; E)$ is any solution of (5), for some $\lambda \in [0,1]$. We have $|u(t)| < R$ for all $t \in [0,T]$. Indeed, from (5), we obtain

$$|u(t)| \leq \lambda \int_0^t |S(t-s) f(s, u(s))| \, ds.$$

Since $S(t)$ is a contractive linear map and $\lambda \in [0,1]$, by (3), we deduce that

$$|u(t)| \leq \int_0^t |f(s,u(s))|\, ds \leq \int_0^t \psi(|u(s)|)\, ds, \quad 0 \leq t \leq T. \tag{7}$$

Denote

$$\varphi(t) = \int_0^t \psi(|u(s)|)\, ds.$$

Then, using the monotonicity of ψ and the inequality $|u(t)| \leq \varphi(t)$, which is exactly (7), we obtain

$$\varphi'(t) = \psi(|u(t)|) \leq \psi(\varphi(t)), \quad 0 \leq t \leq T.$$

Dividing by $\psi(\varphi(t))$ and integrating from 0 to any $t \in (0,T]$, we get

$$\int_0^{\varphi(t)} \frac{1}{\psi(\tau)} d\tau = \int_0^t \frac{\varphi'(s)}{\psi(\varphi(s))} ds \leq t \leq T.$$

This together with (6) guarantees that $\varphi(t) < R$ for all $t \in [0,T]$. Consequently, $|u(t)| < R$ on $[0,T]$ as we claimed. Now we choose any $\theta > L_R$ if $L_R \geq 1$, and $\theta = 0$ when $L_R < 1$, where L_R is the Lipschitz constant in (2), and we consider on $C([0,T];E)$ the norm

$$|u|_\theta = \max\left\{e^{-t\theta}|u(t)| : t \in [0,T]\right\}.$$

We apply Theorem 1 to: $X = C([0,T];E)$ endowed with the norm $|.|_\theta$ (thus d is the metric induced by $|.|_\theta$),

$$U = \{u \in C([0,T];E) : |u(t)| < R \text{ for all } t \in [0,T]\}$$

and $H : \overline{U} \times [0,1] \to C([0,T];E)$ given by

$$H(u,\lambda)(t) = \lambda \int_0^t S(t-s) f(s,u(s))\, ds, \quad t \in [0,T].$$

It is easy to show that all the assumptions of Theorem 1 are satisfied with $l = \theta^{-1}L_R$. Thus (1) has a unique generalized solution. Finally, the iterative procedure of Theorem 1 can be used in order to approximate the mild solution of (1). In this case we may take

$$r = (R - R_0)\, e^{-\theta T},$$

where R_0 is such that

$$T = \int_0^{R_0} \frac{1}{\psi(\tau)} d\tau,$$

and

$$h = \frac{(1 - L_R/\theta)\, r}{T\psi(R)}.$$

Here the approximation sequences are given by

$$u_{j,k+1}(t) = \lambda_j \int_0^t S(t-s) f(s, u_{j,k}(s))\, ds, \quad k \in \mathbf{N}$$

and we may start with $u_{1,0} = 0.$ ■

Applications of Theorem 2 have been given in [24] to boundary value problems on bounded sets in Banach spaces and in [26] to abstract integral equations. Thus, in [26] we proved the following existence and uniqueness result for the equation

$$u(t) = \int_0^T f(t, s, u(s))\, ds, \quad t \in [0,T] \tag{8}$$

in a Banach space $(E, |.|)$. We denote by B the closed ball $\{x \in E : |x| \le R\}$, by K a closed convex set of continuous functions from $[0,T]$ into E, and by K_R the set $\{u \in K : \|u\|_\infty \le R\}$. Here $\|.\|_\infty$ is the max norm on the space of continuous functions from $[0,T]$ into E.

Theorem 4 *[26] Let* $f : [0,T]^2 \times B \to E$. *Suppose*

(h1) *for any* $t \in [0,T]$ *and* $x \in B$, *the map* $f(t, .\,, x)$ *is strongly measurable and* $f(t, .\,, 0) \in L^1([0,T]; E)$;

(h2) *there exists* $\phi : [0,T]^2 \to \mathbf{R}_+$ *and* $q \in [1, \infty]$ *such that*

$$\begin{cases} \text{the map } t \longmapsto \phi(t, .) \text{ (also denoted by } \phi) \text{ belongs to} \\ L^\infty([0,T]; L^q[0,T]) \text{ and} \\ \|\phi\|_{L^p([0,T];\, L^q[0,T])} < 1 \ \ (1/p + 1/q = 1) \end{cases}$$

and

$$|f(t, s, x) - f(t, s, y)| \le \phi(t, s) |x - y|$$

for a.e. $s \in [0,T]$, *all* $x,\ y \in B$ *and each* $t \in [0,T]$;

(h3) *there exists* $w : [0,T] \to \mathbf{R}_+$ *bounded, continuous at* 0 *and with* $w(0) = 0$, *such that*

$$\int_0^T \sup_{|x| \le R} |f(t, s, x) - f(t', s, x)|\, ds \le w(|t - t'|)$$

for all $t,\ t' \in [0,T]$;

(h4) *for each* $\lambda \in (0,1)$, *each possible solution* $u \in K_R$ *of equation*

$$u(t) = \lambda \int_0^T f(t, s, u(s))\, ds, \quad t \in [0,T]$$

is such that $\|u\|_\infty < R$.

Then (8) has a unique solution in K_R.

In this case δ is the metric on K induced by $\|.\|_\infty$ while d is the metric induced by the L^p-norm $\|.\|_p$.

2 Continuation principles for mappings of Caristi type

First we recall the fixed point theorem of Caristi:

Proposition 2 *Let M be a complete metric space, $\varphi : M \to \mathbf{R}_+$ a lower semicontinuous function and $T : M \to M$ a map such that*

$$d(x, T(x)) \leq \varphi(x) - \varphi(T(x))$$

for each $x \in M$. Then T has at least one fixed point.

Theorem 5 *[22] Let M be a complete metric space, $X \subset M$ a non-empty closed set, $\psi : X \times [0,1] \to \mathbf{R}_+$ a lower semicontinuous function and $N : X \times [0,1] \to M$ a map. Let X_λ be the biggest subset invariated by $N_\lambda := N(.,\lambda)$, i.e.*

$$X_\lambda = \cap \left\{ \left(N_\lambda^k\right)^{-1}(X) : \; k = 1, 2, ... \right\}.$$

Suppose

(i) $d(x, N_\lambda(x)) \leq \psi_\lambda(x) - \psi_\lambda(N_\lambda(x))$ for all $x \in X_\lambda$ and $\lambda \in [0,1]$, where $\psi_\lambda = \psi(.,\lambda)$;

(ii) there exists a non-empty closed set $\mathcal{S} \subset \{(x,\lambda) \in X \times [0,1] : \; x \in X_\lambda\}$ such that $(N_1(x), 1) \in \mathcal{S}$ whenever $(x,1) \in \mathcal{S}$ and if $(x_0, \lambda_0) \in \mathcal{S}$ and $\lambda_0 < 1$, then there exists $(x,\lambda) \in \mathcal{S}$ with $\lambda_0 < \lambda$ and $d(x_0, x) \leq \psi_{\lambda_0}(x_0) - \psi_\lambda(x)$.

Then, if N_0 has a fixed point x with $(x, 0) \in \mathcal{S}$, N_1 also has a fixed point.

This theorem immediately yields the following result for continuous mappings N.

Theorem 6 *[22] Let M be a complete metric space, $X \subset M$ a closed set, $\psi : X \times [0,1] \to \mathbf{R}_+$ a lower semicontinuous function and $N : X \times [0,1] \to M$ a continuous map. Suppose*

(1) *$d(x, N_\lambda(x)) \leq \psi_\lambda(x) - \psi_\lambda(N_\lambda(x))$ for all $x \in X_\lambda$ and $\lambda \in [0,1]$;*

(2) *if $N_{\lambda_0}(x_0) = x_0$ and $\lambda_0 < 1$, there exists $\lambda \in]\lambda_0, 1[$ such that $x_0 \in X_\lambda$ and $\psi_\lambda(x_0) \leq \psi_{\lambda_0}(x_0)$.*

Then, if $X_0 \neq \emptyset$, each map N_λ, $\lambda \in [0,1]$, has at least one fixed point.

For maps not necessarily continuous it is true the following result.

Theorem 7 *[22] Let M be a complete metric space, $X \subset M$ a closed set, $\psi : M \times [0,1] \to \mathbf{R}_+$ a lower semicontinuous function, and $N : X \times [0,1] \to M$ a map. Suppose that the following conditions hold:*

(i) *X_λ is closed for every $\lambda \in [0,1]$;*

(ii) *$d(x, N_\lambda(x)) \leq \psi_\lambda(x) - \psi_\lambda(N_\lambda(x))$ for all $x \in X$ and $\lambda \in [0,1]$;*

(iii) *$\psi_\lambda(x) \leq d(x, \partial X)$ for all $\lambda \in [0,1]$ and whenever $N_\eta(x) = x$ for some $\eta \in [0,1]$.*

Then, if $X_0 \neq \emptyset$, each map N_λ, $\lambda \in [0,1]$, has at least one fixed point.

3 Continuation results for nonexpansive mappings

For the beginning we recall a continuation result from Dugundji-Granas [5] in Hilbert spaces:

Proposition 3 *[5] Let H be a Hilbert space, B be the closed ball $\{x \in H : |x| \leq R\}$ and $T : B \to H$ be nonexpansive, i.e.*

$$|T(x) - T(y)| \leq |x - y|$$

for all $x, y \in B$. If

$$x \neq \lambda T(x)$$

for $|x| = R$, $\lambda \in (0,1)$, then T has at least one fixed point in B.

This result was generalized by Guennoun (see [6]), O'Regan [16] and Precup [21], independently, as follows:

Theorem 8 *Let E be a uniformly convex Banach space, U a bounded open convex set with $0 \in U$ and $T : \overline{U} \to E$ a nonexpansive map. If*

$$x \neq \lambda T(x)$$

for all $x \in \partial U$, $\lambda \in (0,1)$, then T has at least one fixed point in $\overline{U}$.

We showed in [21] that in case of Hilbert spaces, one may renounce at the assumption that U is convex and also that a much simpler proof is possible. Thus, the following result holds:

Theorem 9 *[21] Let $(H, \langle .,. \rangle)$ be a Hilbert space, U a bounded open set of H (not necessarily convex) with $0 \in U$ and $T : \overline{U} \to H$ a nonexpansive map. If*

$$x \neq \lambda T(x)$$

for all $x \in \partial U$, $\lambda \in (0,1)$, then T has at least one fixed point in $\overline{U}$.

Proof. Assume $x \neq \lambda T(x)$ for all $x \in \partial U$ and $\lambda \in [0,1]$. For each $\lambda \in (0,1)$, the map λT is a contraction. If we define $h : \overline{U} \times [0,1] \to H$, by $h(x,\mu) = \mu\lambda T$, then we easily see that all the assumptions of Theorem 1 are fulfilled. Hence there exists a unique $x_\lambda \in U$ with $x_\lambda - \lambda T(x_\lambda) = 0$. Let us denote by x_n the element x_λ for $\lambda = 1 - 1/n$, $n \in \mathbf{N} \setminus \{0\}$. We have

$$\begin{aligned}&\left\langle (n-1)^{-1} x_n - (m-1)^{-1} x_m, x_n - x_m \right\rangle \\ &= \langle T(x_n) - T(x_m), x_n - x_m \rangle - |x_n - x_m|^2 \leq 0\end{aligned}$$

for all integers $n, m > 1$. Let $r_n = (n-1)^{-1}$. Using the identity

$$2\langle r_n x_n - r_m x_m, x_n - x_m \rangle = (r_n + r_m)|x_n - x_m|^2 + (r_n - r_m)\left(|x_n|^2 - |x_m|^2\right)$$

we deduce

$$0 \leq (r_n + r_m) |x_n - x_m|^2 \leq (r_n - r_m) \left(|x_m|^2 - |x_n|^2 \right).$$

Since (r_n) is a decreasing sequence, we get that $(|x_n|)$ is increasing. In addition, U being bounded, $(|x_n|)$ is also bounded and so convergent. Next, from

$$|x_n - x_m|^2 \leq \left(|x_m|^2 - |x_n|^2 \right) (r_n - r_m) / (r_n + r_m)$$

it follows that (x_n) is convergent. Obviously, its limit is a fixed point of T. ■

Theorems 8-9 were generalized for weakly inward nonexpansive maps in [23]. Also in [23] we gave some existence and approximation results for nonzero fixed points in cones of weakly inward nonexpansive maps. The set-valued analog of Theorem 8 is due to Frigon [6]. Applications can be found in [21] and [17].

4 Fixed point theorems in syntopogenous spaces

A *syntopogenous space* [3] is a pair $(X, \mathcal{S})$, where X is a non-empty set and $\mathcal{S}$ is a collection of relations $<$ defined on the set of all subsets of X, such that for every two relations $<, <' \in \mathcal{S}$, the following conditions are satisfied:

(S1) $\emptyset < \emptyset$ and $X < X$;

(S2) $A < B$ implies $A \subset B$;

(S3) $A' \subset A < B \subset B'$ implies $A' < B'$;

(S4) $A_i < B_i,\ i = 1, 2$ implies $A_1 \cup A_2 < B_1 \cup B_2$ and $A_1 \cap A_2 < B_1 \cap B_2$;

(S5) there exists $<'' \in \mathcal{S}$ with $< \cup <' \subset <''$;

(S6) there exists $<''' \in \mathcal{S}$ with $< \subset <'''^2$.

A syntopogenous space $(X, \mathcal{S})$ is *Hausdorff* if for every two distinct elements $x, y \in X$ there exists $< \in \mathcal{S}$ and $A \subset X$ such that $x < A$ and $y < X \setminus A$.

A system $\mathcal{R}$ of non-empty subsets of X is called a *filter base* if any intersection of two sets belonging to $\mathcal{R}$ contains a subset from $\mathcal{R}$. $\mathcal{R}$ is said to be a *Cauchy filter base* if for each $< \in \mathcal{S}$ there exists $R \in \mathcal{R}$ such that if $A < B$ then $A \cap R = \emptyset$ or $(X \setminus B) \cap R = \emptyset$. The sequence $(x_n) \subset X$ is called *Cauchy sequence* if the corresponding sequential filter base $\mathcal{R} = \{R_k : k \in \mathbf{N}\}$, $R_k = \{x_n : n \geq k\}$, is a Cauchy filter base. We say that the filter base $\mathcal{R}$ *converges* to $x \in X$ if for each *neighborhood* V of x, i.e. $V \subset X$ with $x < V$ for some $< \in \mathcal{S}$, there exists $R \in \mathcal{S}$ such that $R < V$. The syntopogenous space $(X, \mathcal{S})$ is said to be *sequentially complete* if every Cauchy sequence is convergent.

Let $(X, \mathcal{S})$, $(X', \mathcal{S}')$ be syntopogenous spaces. The map $T : X \to X'$ is said to be $(\mathcal{S}, \mathcal{S}')$-*continuous* if for each $<' \in \mathcal{S}'$ there exists $< \in \mathcal{S}$ such that $T^{-1}(A) < T^{-1}(B)$ whenever $A <' B$.

Let $\mathcal{S}$ and $\mathcal{S}'$ be two syntopogenous structures on X. We say that $\mathcal{S}'$ is *finer* than $\mathcal{S}$ and we denote this by $\mathcal{S} \subset \mathcal{S}'$, provided that for each $< \in \mathcal{S}$ there is $<' \in \mathcal{S}'$ with $< \subset <'$.

Let X be a non-empty set. A system φ of real functions defined on X is said to be *ordering system* on X if φ contains all constant functions on X and $f + c$, $\max\{f, g\}$, $\min\{f, g\} \in \varphi$ whenever $f,\ g \in \varphi$ and $c \in \mathbf{R}$. A non-empty collection Φ of ordering systems on X is said to be *ordering structure* on X. To each ordering structure on a set X we may attach a syntopogenous structure on X, namely

$$\mathcal{S}_\Phi = \{<_{\varphi,\varepsilon} \colon \varphi \in \Phi,\, \varepsilon > 0\},$$

where $A <_{\varphi,\varepsilon} B$ if and only if $f(A) <_\varepsilon \mathbf{R} \setminus f(X \setminus B)$ for some $f \in \varphi$. Here $C <_\varepsilon D$ $(C, D \subset \mathbf{R})$ means $\sup C + \varepsilon \leq \inf(\mathbf{R} \setminus D)$. It is known [3] that for each syntopogenous structure $\mathcal{S}$ on X there exists an ordering structure Φ on X *compatible* with $\mathcal{S}$, i.e. $\mathcal{S} \sim \mathcal{S}_\Phi$ in the sense that $\mathcal{S}_\Phi \subset \mathcal{S} \subset \mathcal{S}_\Phi$. Thus each syntopogenous structure $\mathcal{S}$ can be identified with any ordering structure Φ compatible with $\mathcal{S}$.

Definition 1 *[18] Let Φ be an ordering structure on X. We say that a map $T : X \to X$ is a contractive on X with respect to Φ (Φ-contractive), if there exist two functions $\alpha : \Phi \to \mathbf{R}_+$, $\beta : \Phi \to \Phi$ such that for every $\varphi \in \Phi$ one has: (a)*

$$A <_{\varphi,\varepsilon} B \text{ implies } T^{-1}(A) <_{\beta(\varphi), \frac{\varepsilon}{\alpha(\varphi)}} T^{-1}(B)$$

and (b) for every $x, y \in X$, the family of series

$$\left\{ \sum_{n=0}^{\infty} \alpha(\varphi)\, \alpha(\beta(\varphi)) \ldots \alpha(\beta^n(\varphi)) \left| f_n(x) - f_n(y) \right| :\ f_n \in \beta^{n+1}(\varphi) \right\}$$

is uniformly convergent.

We now recall our generalizations to syntopogenous spaces of the Banach and Maia theorems on contractive maps in metric spaces.

Theorem 10 *[18] Let $(X, \mathcal{S})$ be a sequentially complete, Hausdorff syntopogenous space. Let also Φ be an ordering structure on X compatible with the syntopogenous structure $\mathcal{S}$. If the map $T : X \to X$ is contractive with respect to Φ, then T has a unique fixed point which can be found by the method of successive approximations starting from any element of X.*

Theorem 11 *[19] Let X be a non-empty set endowed with two syntopogenous structures $\mathcal{S}$ and $\mathcal{S}'$. Let also Φ be an ordering structure on X compatible $\mathcal{S}$ and $T : X \to X$ a map. Assume that the following conditions are satisfied:*
(i) $(X, \mathcal{S}')$ is a sequentially complete, Hausdorff syntopogenous space;
(ii) there exists $k \in \mathbf{N}$ such that T^k is $(\mathcal{S}, \mathcal{S}')$-continuous;
(iii) T is $(\mathcal{S}', \mathcal{S}')$-continuous;
(iv) T is Φ-contractive.
Then T has a unique fixed point which can be obtained in $(X, \mathcal{S}')$ by successive approximations starting from any element of X.

Remark 1 *If $\mathcal{S}' \subset \mathcal{S}$, then (ii), (iii) are guaranteed if*
(v) T is $(\mathcal{S}', \mathcal{S})$-continuous.
Assumption (v) also guarantees that the successive approximation sequences are not only $\mathcal{S}'$-convergent, but also $\mathcal{S}$-convergent.

The set-valued analog of Theorem 10 was also given in [18].

Definition 2 *[18] Let Φ be an ordering structure on the set X and $T : X \to 2^X \setminus \{\emptyset\}$ a map. We say that T is contractive on X with respect to Φ (Φ-contractive), if there exist two functions $\alpha : \Phi \to \mathbf{R}_+$, $\beta : \Phi \to \Phi$ such that for every $\varphi \in \Phi$ condition (b) and (a*):*

$$A <_{\varphi,\varepsilon} B \text{ implies } T_-^{-1}(A) <_{\beta(\varphi),\frac{\varepsilon}{\alpha(\varphi)}} T_+^{-1}(B)$$

hold. Here

$$T_-^{-1}(A) = \{x \in X : \Gamma(x) \cap A \neq \emptyset\}, \; T_+^{-1}(A) = \{x \in X : \Gamma(x) \subset A\}.$$

Theorem 12 *[18] Let $(X, \mathcal{S})$ be a sequentially complete, Hausdorff syntopogenous space. Let also Φ be an ordering structure on X compatible with the syntopogenous structure $\mathcal{S}$. If the map $T : X \to 2^X \setminus \{\emptyset\}$ is contractive with respect to Φ, then T has a unique fixed point x^* which is the limit of the sequence of the iterations of any selection of T, and $T(x^*) = \{x^*\}$.*

Obviously, a set-valued variant of Theorem 11 can also be given.

We conclude this section by a result given in [20] which represents a natural generalization to syntopogenous spaces of Granas' topological transversality theorem [13].

Let $(X, \mathcal{S})$ be a syntopogenous space with a countable syntopogenous structure $\mathcal{S} = \{<_n : n \in \mathbf{N}\}$ satisfying the additional condition

$$<_n \subset <_{n+1}^2 \tag{9}$$

for every $n \in \mathbf{N}$.

Definition 3 *[4] A function $f : X \to I := [0,1]$ is said to be associated with the sequence $\{<_n : n \in \mathbf{N}\}$ if*

$$P, Q \subset I, \; d(P,Q) > 1/2^n \text{ implies } f^{-1}(P) <_{n+2} f^{-1}(I \setminus Q)$$

for every $n \in \mathbf{N}$.

Lemma 1 *[4] Let $(X, \mathcal{S})$ be a syntopogenous space with a countable syntopogenous structure $\mathcal{S} = \{<_n : n \in \mathbf{N}\}$ satisfying (9). If $M <_0 N$, then there exists a function f associated with $\mathcal{S}$ such that $f(x) = 0$ for all $x \in M$ and $f(x) = 1$ for all $x \in X \setminus N$.*

Let $Y \subset X$ and $\emptyset \neq A \subset Y$. Consider a class of maps

$$\mathcal{A}_A(Y; X) \subset \{T : Y \to X : \, Fix\,(T) \cap A = \emptyset\}.$$

Here $Fix\,(T)$ stands for the set of all fixed points of T.

For every relation $< \in \mathcal{S}$ we shall denote by $<|_Y$ the *restriction of* $<$ to Y (see [3], (6.19)), i.e.

$$M <|_Y N \text{ if } M, N \subset Y \text{ and } M < N \cup (X \setminus Y).$$

It is easily seen that the sequence $\mathcal{S}|_Y := \{<_n|_Y : n \in \mathbf{N}\}$ is a syntopogenous structure on Y which also satisfies (9).

Definition 4 *A map $T \in \mathcal{A}_A(Y;X)$ is said to be essential if for each $T' \in \mathcal{A}_A(Y;X)$ having the same restriction to A as T, i.e. $T'|_A = T|_A$, one has Fix $(T') \neq \emptyset$. Otherwise, T is said to be inessential.*

Let us consider an equivalence relation $\simeq$ on $\mathcal{A}_A(Y;X)$ such that the following two conditions are satisfied:

(i) if $T'|_A = T|_A$ then $T' \simeq T$;

(ii) if $T' \simeq T$ then there exists a map $H : Y \times I \to X$ such that $H(.,0) = T'$, $H(.,1) = T$,

$$Z := \cup \{Fix\ (H(.,\lambda)) : \lambda \in I\} <_0 X \setminus A$$

and $H(.,\theta(.)) \in \mathcal{A}_A(Y;X)$ for every function $\theta : Y \to I$ associated to $\mathcal{S}|_Y$ with $\theta(x) = 1$ for all $x \in A$.

Lemma 2 *[20] The map $T \in \mathcal{A}_A(Y;X)$ is inessential if and only if there exists $T' \in \mathcal{A}_A(Y;X)$ with $T' \simeq T$ and Fix $(T') = \emptyset$.*

Proof. The necessity follows from the definition of an inessential map and condition (i). Now assume that $T' \simeq T$ and $Fix\ (T') = \emptyset$. Let H be a map like in (ii). If $Z = \emptyset$, then $Fix\ (H(.,1)) = \emptyset$, hence $T = H(.,1)$ is inessential. Suppose $Z \neq \emptyset$. Since $Z <_0 X \setminus A$ one has $Z <_0|_Y Y \setminus A$. Then by Lemma 1 there exists a function $\theta : Y \to I$ associated to $\mathcal{S}|_Y$ such that $\theta(x) = 0$ for all $x \in Z$ and $\theta(x) = 1$ for all $x \in A$. Let $H^* : Y \to X$ be given by

$$H^*(x) = H(x, \theta(x)).$$

According to (ii), $H^* \in \mathcal{A}_A(Y;X)$. In addition $H^*|_A = H(.,1)|_A = T|_A$ and $Fix\ (H^*) = Fix\ (T') = \emptyset$. Hence T is inessential. ■

Now we can state and prove the topological transversality theorem in syntopogenous spaces.

Theorem 13 *[20] Assume T, T' belong to $\mathcal{A}_A(Y;X)$ and $T \simeq T'$. Then T and T' are both essential or both inessential.*

Proof. Assume T is inessential. Then, by Lemma 2, there exists $T'' \in \mathcal{A}_A(Y;X)$ with $T'' \simeq T$ and $Fix\ (T'') = \emptyset$. From $T'' \simeq T$, $T \simeq T'$ and the transitivity of the relation $\simeq$ it follows that $T'' \simeq T'$. According to Lemma 2, this together with $Fix\ (T'') = \emptyset$, guarantees that T' is inessential. For the converse implication: T' inessential implies T inessential, use the symmetry of $\simeq$. ■

Remark 2 *The assumption $Z <_0 X \setminus A$ in (ii) is satisfied if we require that*

$$Y \setminus A <_0 X \setminus A \text{ and } Fix\ (H(.,\lambda)) \cap A = \emptyset \text{ for all } \lambda \in I.$$

Indeed, this last condition guarantees $Z \subset Y \setminus A$. This together with $Y \setminus A <_0 X \setminus A$ implies $Z <_0 X \setminus A$.

As a consequence of Theorem 13, we have the topological transversality theorem in normal topological spaces.

Theorem 14 *[20] Let X be a normal topological space, $\emptyset \neq A \subset Y \subset X$ and A, Y closed in X. Let*

$$\mathcal{A}_A(Y; X) \subset \{T : Y \to X : \ Fix\ (T) \cap A = \emptyset\}$$

and let $\simeq$ be an equivalence relation on $\mathcal{A}_A(Y; X)$ satisfying (i) and

(ii') if $T' \simeq T$, then there exists $H : Y \times I \to X$ such that $H(.,0) = T'$, $H(.,1) = T$, $cl(\cup\{Fix\ (H(.,\lambda)) : \lambda \in I\}) \cap A = \emptyset$ and $H(.,\theta(.)) \in \mathcal{A}_A(Y; X)$ for every continuous function $\theta : Y \to I$ with $\theta(x) = 1$ for all $x \in A$.

If $T \simeq T'$, then T and T' are both essential or both inessential.

5 Contractive maps on quasi-uniform spaces

Since the category of quasi-uniform spaces is isomorphic to a subcategory of that of syntopogenous spaces, all the results presented in the above section yield, in particular, fixed point theorems in quasi-uniform spaces.

By a *quasi-uniform* space we mean a pair (X, Σ) of a non-empty set X and a non-empty family Σ of quasi-metrics on X. Here a *quasi-metric* on X is a map $d : X \times X \to \mathbf{R}_+$ such that for every $x, y, z \in X$ one has: $d(x,x) = 0$ and $d(x,z) \leq d(x,y) + d(y,z)$.

A quasi-metric d on X is called *pseudo-metric (or a gauge)* if $d(x,y) = d(y,x)$ for all $x, y \in X$. A pair (X, Σ) of a non-empty set and a non-empty family of pseudo-metrics on X is said to be a *uniform space*.

Definition 5 *[11] Let (X, Σ) be a quasi-uniform space with $\Sigma = \{d_j : j \in J\}$. A map $T : X \to X$ is said to be contractive with respect to Σ (Σ-contractive) if there exist two functions $\alpha : J \to \mathbf{R}_+$, $\beta : J \to J$ such that for all $j \in J$ and $x, y \in X$ one has: (a)*

$$d_j(Tx, Ty) \leq \alpha(j)\, d_{\beta(j)}(x,y)$$

and (b) the series

$$\sum_{n=0}^{\infty} \alpha(j)\, \alpha(\beta(j)) \ldots \alpha(\beta^n(j))\, d_{\beta^{n+1}(j)}(x,y) \tag{10}$$

is convergent.

The Banach contraction principle was extended to locally convex spaces by Marinescu [15] and to uniform spaces by Colojoară [2] and Gheorghiu [11]. The analog for set-valued maps on uniform spaces is due to Avramescu [1]. A generalization of Maia's theorem to uniform spaces endowed with two uniform structures was first given by Gheorghiu [12].

There is a broad literature in metric fixed point theory (see Rus [27]) on the so called *generalized contractions.* In [19] we showed that in many cases, such a generalized contraction T is in fact a usual contraction with respect to a suitable quasi-uniform (or uniform) structure associated to T. Consequently, the fixed point theorems in quasi-uniform spaces yield fixed point results for generalized contractions. Now we describe some types of generalized contractions.

Let (X, d) be a metric space and $T : X \to X$ be a map.

Theorem 15 *[19] Assume T satisfies*

$$d(Tx,Ty) \leq a\,d(x,Tx) + b\,d(y,Ty) + c\,d(x,y)$$

for all $x, y \in X$, where a, b, c are non-negative numbers with $a+b+c < 1$ and $a+bc > 0$. Then T is contractive with respect to $\Sigma = \{d_n : n \in \mathbf{N}\}$, where

$$\begin{aligned} d_n(x,y) &= \frac{r^n - c^n}{r - c}\left[a\,d(x,Tx) + b\,d(y,Ty)\right] + c^n d(x,y) \ \textit{for } x \neq y \\ &= 0 \ \textit{for } x = y. \end{aligned}$$

Here $r = (a+c)/(1-b)$ and $c < r < 1$.

Proof. Take $\alpha : \mathbf{N} \to \mathbf{R}_+,\ \alpha(n) = 1,\ \beta : \mathbf{N} \to \mathbf{N},\ \beta(n) = n+1$ and observe that for an arbitrary $d_m \in \Sigma$, the series (10) is $\sum_{n=0}^{\infty} d_{m+n+1}(x,y)$, which since $0 \leq c < 1$ and $c < r < 1$, is obviously convergent. ■

Theorem 16 *[19] Assume T satisfies*

$$d(Tx,Ty) \leq a\ \max\{d(x,y),\ d(x,Tx),\ d(y,Ty),\ d(x,Ty),\ d(y,Tx)\} \tag{11}$$

for all $x, y \in X$ and some $a \in [0,1)$. Then T is contractive with respect to $\Sigma = \{d_n : n \in \mathbf{N}\}$, where d_n is the pseudo-metric on X given by

$$\begin{aligned} d_n(x,y) &= \max\{d\left(T^i x, T^j x\right),\ d\left(T^i y, T^j y\right),\ d\left(T^i x, T^j y\right) :\ i,j = 0,1,...,n\} \\ &\qquad \textit{for } x \neq y \\ &= 0 \ \textit{for } x = y. \end{aligned}$$

Proof. We have $d_0 = d$ and from (11), we deduce

$$\begin{aligned} &d\left(T^i x, T^j x\right) \leq a\,d_n(x,y),\ d\left(T^i y, T^j y\right) \leq a\,d_n(x,y) \\ &d\left(T^i x, T^j y\right) \leq a\,d_n(x,y) \end{aligned}$$

for all $x, y \in X$ and $i, j \in \{1, 2, ..., n\}$. It follows

$$d_n(Tx,Ty) \leq a\,d_{n+1}(x,y)$$

and also

$$d_n(x,y) = \max\{d\left(x, T^i x\right), d\left(y, T^i y\right), d\left(x, T^i y\right), d\left(y, T^i x\right) :\ i = 0,1,...,n\}.$$

If, for example, $d_n(x,y) = d\left(x, T^i x\right)$ for some $i \in \{1, 2, ..., n\}$, then

$$d_n(x,y) \leq d(x,Tx) + d\left(Tx, T^i x\right) \leq d(x,Tx) + a\,d_n(x,y).$$

Hence

$$d_n(x,y) \leq \frac{1}{1-a} d(x,Tx) \leq \frac{1}{1-a} d_1(x,y).$$

Generally, we can prove similarly

$$d_n(x,y) \leq \frac{1}{1-a} d_1(x,y) \tag{12}$$

for all $x, y \in X$ and $n \in \mathbf{N}$.

In this case $\alpha : \mathbf{N} \to \mathbf{R}_+$, $\alpha(n) = a$, $\beta : \mathbf{N} \to \mathbf{N}$, $\beta(n) = n+1$. Also, for an arbitrary $d_m \in \Sigma$, the series (10) is $\sum_{n=0}^{\infty} a^{n+1} d_{m+n+1}(x,y)$, which is convergent as follows from (12). ■

Theorem 17 *[19] Let (X,d) be a generalized metric space endowed with a vector-valued metric $d : X \times X \to \mathbf{R}^r$. Assume $T : X \to X$ is contractive in the Perov sense, i.e.*

$$d(Tx, Ty) \leq A\, d(x,y)$$

for all $x, y \in X$ and some matrix $A \in M_{r\times r}(\mathbf{R}_+)$ with $A^n \to 0$ as $n \to \infty$.

Let d_i, $i = 1, 2, ..., r$, be the pseudo-metrics for which $d = (d_1, d_2, ..., d_r)$. Also, if $A^n = \left(a_{ij}^n\right)$, we define the pseudo-metrics

$$d_{in} = \sum_{j=1}^{r} a_{ij}^n d_j$$

for $i = 1, 2, ..., r$ and $n \in \mathbf{N}$.

Then T is contractive with respect to $\Sigma = \{d_{in} : (i,n) \in \{1, 2, ..., r\} \times \mathbf{N}\}$.

Proof. Take $\alpha(i,n) = 1$ and $\beta(i,n) = (i, n+1)$. Also observe that from $A^n \to 0$ as $n \to \infty$, we have

$$\sum_{n=0}^{\infty} A^{m+n+1} = A^{m+1}(I-A)^{-1}$$

whence

$$\sum_{n=0}^{\infty} A^{m+n+1} d(x,y) = A^{m+1}(I-A)^{-1} d(x,y).$$

Hence, for all $x, y \in X$ and $(i,n) \in \{1, 2, ..., r\} \times \mathbf{N}$, the series

$$\sum_{n=0}^{\infty} \sum_{j=1}^{r} a_{ij}^{m+n+1} d_j(x,y) = \sum_{n=0}^{\infty} d_{i,m+n+1}(x,y)$$

which coincides with the series (10), is convergent. ■

Finally, we mention that a continuation type result in separated complete uniform spaces was recently obtained by Frigon-Granas [9] (see also Frigon [7]) for particular contractive maps, with $\beta(j) = j$ for all $j \in J$. Thus, an open problem is to give extensions of the Frigon-Granas theorem to general contractive maps on quasi-uniform spaces, and to derive specific continuation results for several classes of generalized contractions.

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