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On the nonlocal initial value problem for first order differential systems

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Abstract. The aim of the is to study the existence of solutions of initial value problems for nonlinear first order differential systems with nonlocal conditions. The proof will rely on the Perov, Schauder and Leray-Schauder fixed point principles which are applied to a nonlinear integral operator splitted into two parts, one of Fredholm type for the subinterval containing the points involved by the nonlocal condition, and an another one of Volterra type for the rest of the interval. The novelty in this paper is that this approach is combined with the technique that uses convergent to zero matrices and vector norms.

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1. Introduction

In this paper we deal with the nonlocal initial value problem for the first order differential system

$$\begin{cases} x'(t) = f(t, x(t), y(t)) \\ y'(t) = g(t, x(t), y(t)) \qquad \text{(a.e. on } [0,1]) \\ x(0) + \sum_{k=1}^{m} a_k x(t_k) = 0 \\ y(0) + \sum_{k=1}^{m} \widetilde{a}_k y(t_k) = 0. \end{cases} \tag{1.1}$$

Here $f, g : [0,1] \times \mathbf{R}^2 \to \mathbf{R}$ are Carathéodory functions, t_k are given points with $0 \leq t_1 \leq t_2 \leq \ ... \ \leq t_m < 1$ and a_k, $\widetilde{a}_k$ are real numbers with

$$1 + \sum_{k=1}^{m} a_k \neq 0 \text{ and } 1 + \sum_{k=1}^{m} \widetilde{a}_k \neq 0.$$

Notice that the nonhomogeneous nonlocal initial conditions

$$\begin{cases} x(0) + \sum_{k=1}^{m} a_k x(t_k) = x_0 \\ y(0) + \sum_{k=1}^{m} \widetilde{a}_k y(t_k) = y_0 \end{cases}$$

can always be reduced to the homogeneous ones (with $x_0 = y_0 = 0$) by the change of variables $x_1(t) := x(t) - a\ x_0$ and $y_2(t) := y(t) - \widetilde{a}\ y_0$, where

$$a = \left(1 + \sum_{k=1}^{m} a_k\right)^{-1} \text{ and } \widetilde{a} = \left(1 + \sum_{k=1}^{m} \widetilde{a}_k\right)^{-1}.$$

According to [2], Problem (1.1) is equivalent to the following integral system in $C[0,1]^2$:

$$\begin{cases} x(t) = -a \sum_{k=1}^{m} a_k \int_0^{t_k} f(s, x(s), y(s))\, ds + \int_0^t f(s, x(s), y(s))\, ds \\ y(t) = -\widetilde{a} \sum_{k=1}^{m} \widetilde{a}_k \int_0^{t_k} g(s, x(s), y(s))\, ds + \int_0^t g(s, x(s), y(s))\, ds. \end{cases}$$

This can be viewed as a fixed point problem in $C[0,1]^2$ for the completely continuous operator $T = (T_1, T_2)$, $T : C[0,1]^2 \to C[0,1]^2$, where T_1 and T_2 are given by

$$T_1(x,y)(t) = -a \sum_{k=1}^{m} a_k \int_0^{t_k} f(s, x(s), y(s))\, ds + \int_0^t f(s, x(s), y(s))\, ds,$$

$$T_2(x,y)(t) = -\widetilde{a} \sum_{k=1}^{m} \widetilde{a}_k \int_0^{t_k} g(s, x(s), y(s))\, ds + \int_0^t g(s, x(s), y(s))\, ds.$$

Operators T_1 and T_2 appear as sums of two integral operators, one of Fredholm type, whose values depend only on the restrictions of functions to $[0, t_m]$, and the other one, a Volterra type operator depending on the restrictions to $[t_m, 1]$, as this was pointed out in [3]. Thus, T_1 can be rewritten as $T_1 = T_{F_1} + T_{V_1}$, where

$$T_{F_1}(x,y)(t) = \begin{cases} -a \sum_{k=1}^{m} a_k \int_0^{t_k} f(s, x(s), y(s))\, ds + \int_0^t f(s, x(s), y(s))\, ds, \\ \qquad \text{if } t < t_m \\ -a \sum_{k=1}^{m} a_k \int_0^{t_k} f(s, x(s), y(s))\, ds + \int_0^{t_m} f(s, x(s), y(s))\, ds, \\ \qquad \text{if } t \geq t_m \end{cases}$$

and

$$T_{V_1}(x,y)(t) = \begin{cases} 0, & \text{if } \ t < t_m \\ \displaystyle\int_{t_m}^{t} f\left(s, x\left(s\right), y(s)\right) ds, & \text{if } \ t \geq t_m. \end{cases}$$

Similarly, $T_2 = T_{F_2} + T_{V_2}$, where

$$T_{F_2}(x,y)(t) = \begin{cases} -\widetilde{a}\displaystyle\sum_{k=1}^{m}\widetilde{a}_k \int_0^{t_k} g\left(s, x\left(s\right), y(s)\right) ds + \int_0^{t} g\left(s, x\left(s\right), y(s)\right) ds, \\ \qquad \text{if } t < t_m \\ -\widetilde{a}\displaystyle\sum_{k=1}^{m}\widetilde{a}_k \int_0^{t_k} g\left(s, x\left(s\right), y(s)\right) ds + \int_0^{t_m} g\left(s, x\left(s\right), y(s)\right) ds, \\ \qquad \text{if } t \geq t_m \end{cases}$$

and

$$T_{V_2}(x,y)(t) = \begin{cases} 0, & \text{if } \ t < t_m \\ \displaystyle\int_{t_m}^{t} g\left(s, x\left(s\right), y(s)\right) ds, & \text{if } \ t \geq t_m. \end{cases}$$

This allows us to split the growth condition on the nonlinear terms $f(t,x,y)$ and $g(t,x,y)$ into two parts, one for $t \in [0,t_m]$ and another one for $t \in [t_m, 1]$, in a such way that one reobtains the classical growth when $t_m = 0$, that is for the local initial condition $x(0) = 0$. In what follows, the notation $|x|_{C[a,b]}$ stands for the max-norm on $C\left[a,b\right]$

$$|x|_{C[a,b]} = \max\nolimits_{t\in[a,b]} \left|x(t)\right|,$$

while $\|x\|_{C[a,b]}$ denotes the Bielecki norm

$$\|x\|_{C[a,b]} = \left|x\left(t\right)e^{-\theta(t-a)}\right|_{C[a,b]}$$

for some suitable $\theta > 0$.

Nonlocal initial value problems were extensively discussed in the literature by different methods (see for example [2], [3], [5], [6], [8], [10]). The results in the present paper extend to systems those established for equations in [3]. The method could be adapted to treat systems of evolution equations as this was done for equations in [4].

In the next section three different fixed point principles will be used in order to prove the existence of solutions for the semilinear problem, namely the fixed point theorems of Perov, Schauder and Leray-Schauder (see [10]). In all three cases a key role will be played by the so called convergent to zero matrices. A square matrix M with nonnegative elements is said to be *convergent to zero* if

$$M^k \to 0 \quad \text{as} \quad k \to \infty.$$

It is known that the property of being convergent to zero is equivalent to each of the following three conditions (for details see [10], [11]):

(a) $I - M$ is nonsingular and $(I - M)^{-1} = I + M + M^2 + ...$ (where I stands for the unit matrix of the same order as M);

(b) the eigenvalues of M are located inside the unit disc of the complex plane;

(c) $I - M$ is nonsingular and $(I - M)^{-1}$ has nonnegative elements.

The following lemma, whose proof is immediate from characterization (b) of convergent to zero matrices, will be used in the sequel:

Lemma 1.1. *If A is a square matrix that converges to zero and the elements of an other square matrix B are small enough, then $A + B$ also converges to zero.*

We finish this introductory section by recalling (see [1], [10]) three fundamental results which will be used in the next sections. Let X be a nonempty set. By a *vector-valued metric* on X we mean a mapping $d : X \times X \to \mathbf{R}^n_+$ such that

(i) $d(u, v) \geq 0$ for all $u, v \in X$ and if $d(u, v) = 0$ then $u = v$;
(ii) $d(u, v) = d(v, u)$ for all $u, v \in X$;
(iii) $d(u, v) \leq d(u, w) + d(w, v)$ for all $u, v, w \in X$.

Here, if $x, y \in \mathbf{R}^n$, $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$, by $x \leq y$ we mean $x_i \leq y_i$ for $i = 1, 2, ..., n$. We call the pair (X, d) a *generalized metric space.* For such a space convergence and completeness are similar to those in usual metric spaces.

An operator $T : X \to X$ is said to be *contractive* (with respect to the vector-valued metric d on X) if there exists a convergent to zero matrix M such that

$$d(T(u), T(v)) \leq M d(u, v) \quad \text{for all } u, v \in X.$$

Theorem 1.2 (Perov). *Let (X, d) be a complete generalized metric space and $T : X \to X$ a contractive operator with Lipschitz matrix M. Then T has a unique fixed point u^* and for each $u_0 \in X$ we have*

$$d(T^k(u_0), u^*) \leq M^k (I - M)^{-1} d(u_0, T(u_0)) \;\; \textit{for all } k \in \mathbf{N}.$$

Theorem 1.3 (Schauder). *Let X be a Banach space, $D \subset X$ a nonempty closed bounded convex set and $T : D \to D$ a completely continuous operator (i.e., T is continuous and $T(D)$ is relatively compact). Then T has at least one fixed point.*

Theorem 1.4 (Leray–Schauder). *Let $(X, ||\,.\,||)$ be a Banach space, $R > 0$, $\overline{B}_R(0; X) = \{u \in X : \|u\| \leq R\}$ and $T : \overline{B}_R(0; X) \to X$ a completely continuous operator. If $||u|| < R$ for every solution u of the equation $u = \lambda T(u)$ and any $\lambda \in (0, 1)$, then T has at least one fixed point.*

Throughout the paper we shall assume that the following conditions are satisfied:

(H1) $1 + \sum_{k=1}^{m} a_k \neq 0$ and $1 + \sum_{k=1}^{m} \widetilde{a}_k \neq 0$.

(H2) $f, g : [0,1] \times \mathbf{R}^2 \to \mathbf{R}$ is such that $f(., x, y), g(., x, y)$ are measurable for each $(x, y) \in \mathbf{R}^2$ and $f(t, ., .), g(t, ., .)$ are continuous for almost all $t \in [0,1]$.

2. Nonlinearities with the Lipschitz property. Application of Perov's fixed point theorem

Here we show that the existence of solutions of problem (1.1) follows from Perov's fixed point theorem in case that f, g satisfy Lipschitz conditions in x and y :

$$|f(t,x,y) - f(t,\overline{x},\overline{y})| \leq \begin{cases} b_1 |x - \overline{x}| + \widetilde{b}_1 |y - \overline{y}| & \text{if } t \in [0, t_m] \\ c_1 |x - \overline{x}| + \widetilde{c}_1 |y - \overline{y}| & \text{if } t \in [t_m, 1], \end{cases} \tag{2.1}$$

$$|g(t,x,y) - g(t,\overline{x},\overline{y})| \leq \begin{cases} B_1 |x - \overline{x}| + \widetilde{B}_1 |y - \overline{y}| & \text{if } t \in [0, t_m] \\ C_1 |x - \overline{x}| + \widetilde{C}_1 |y - \overline{y}| & \text{if } t \in [t_m, 1] \end{cases} \tag{2.2}$$

for all $x, y, \overline{x}, \overline{y} \in \mathbf{R}$.

Theorem 2.1. *If f, g satisfy the Lipschitz conditions* (2.1), (2.2) *and the matrix*

$$M_0 := \begin{bmatrix} b_1 t_m A_1 & \widetilde{b}_1 t_m A_1 \\ B_1 t_m A_2 & \widetilde{B}_1 t_m A_2 \end{bmatrix} \tag{2.3}$$

converges to zero, then problem (1.1) *has a unique solution.*

Proof. We shall apply Perov's fixed point theorem in $C[0,1]^2$ endowed with the vector norm $\|.\|$ defined by

$$\|u\| = (\|x\|, \|y\|)$$

for $u = (x, y)$, where for $z \in C[0,1]$, we let

$$\|z\| = \max\left\{ |z|_{C[0,t_m]}, \|z\|_{C[t_m,1]} \right\}.$$

We have to prove that T is contractive, more exactly that

$$\|T(u) - T(\overline{u})\| \leq M_\theta \|u - \overline{u}\|$$

for all $u = (x, y), \overline{u} = (\overline{x}, \overline{y}) \in C[0,1]^2$ and some matrix M_θ converging to zero. To this end, let $u = (x, y), \overline{u} = (\overline{x}, \overline{y})$ be any elements of $C[0,1]^2$. For $t \in [0, t_m]$, if we denote

$$A_1 := 1 + |a| \sum_{k=1}^{m} |a_k|,$$

we have

$$\begin{aligned}
&\left|T_1(x,y)(t) - T_1(\overline{x},\overline{y})(t)\right| \\
= &\left| -a\sum_{k=1}^{m} a_k \int_0^{t_k} f\left(s, x\left(s\right), y(s)\right) ds + \int_0^t f\left(s, x\left(s\right), y(s)\right) ds \right. \\
&\left. +a\sum_{k=1}^{m} a_k \int_0^{t_k} f\left(s, \overline{x}\left(s\right), \overline{y}(s)\right) ds - \int_0^t f\left(s, \overline{x}\left(s\right), \overline{y}(s)\right) ds \right| \\
\leq\ & A_1 \int_0^{t_m} \left| f\left(s, x\left(s\right), y(s)\right) - f\left(s, \overline{x}\left(s\right), \overline{y}(s)\right)\right| ds \\
\leq\ & b_1 t_m A_1 \left|x - \overline{x}\right|_{C[0,t_m]} + \widetilde{b}_1 t_m A_1 \left|y - \overline{y}\right|_{C[0,t_m]}.
\end{aligned}$$

Taking the supremum, we obtain that

$$\left|T_1(x,y) - T_1(\overline{x},\overline{y})\right|_{C[0,t_m]} \leq b_1 t_m A_1 \left|x - \overline{x}\right|_{C[0,t_m]} + \widetilde{b}_1 t_m A_1 \left|y - \overline{y}\right|_{C[0,t_m]}. \tag{2.4}$$

For $t \in [t_m, 1]$ and any $\theta > 0$, we have

$$\begin{aligned}
&\left|T_1(x,y)(t) - T_1(\overline{x},\overline{y})(t)\right| \\
= &\left| -a\sum_{k=1}^{m} a_k \int_0^{t_k} f\left(s, x\left(s\right), y(s)\right) ds + \int_0^t f\left(s, x\left(s\right), y(s)\right) ds \right. \\
&\left. +a\sum_{k=1}^{m} a_k \int_0^{t_k} f\left(s, \overline{x}\left(s\right), \overline{y}(s)\right) ds - \int_0^t f\left(s, \overline{x}\left(s\right), \overline{y}(s)\right) ds \right| \\
\leq\ & b_1 t_m A_1 \left|x - \overline{x}\right|_{C[0,t_m]} + \widetilde{b}_1 t_m A_1 \left|y - \overline{y}\right|_{C[0,t_m]} \\
&+ \int_{t_m}^t \left(c_1 \left|x(s)x - \overline{x}(s)\right| + \widetilde{c}_1 \left|y(s) - \overline{y}(s)\right|\ \right) ds.
\end{aligned}$$

The last integral can be further estimated as follows:

$$\begin{aligned}
&\int_{t_m}^t \left(c_1 \left|x(s)x - \overline{x}(s)\right| + \widetilde{c}_1 \left|y(s) - \overline{y}(s)\right|\ \right) ds \\
= \ & c_1 \int_{t_m}^t \left|x(s) - \overline{x}(s)\right| \cdot e^{-\theta(s-t_m)} \cdot e^{\theta(s-t_m)} ds \\
&+\widetilde{c}_1 \int_{t_m}^t \left|y(s) - \overline{y}(s)\right| \cdot e^{-\theta(s-t_m)} \cdot e^{\theta(s-t_m)} ds \\
\leq\ & \frac{c_1}{\theta} e^{\theta(t-t_m)} \left\|x - \overline{x}\right\|_{C[t_m,1]} + \frac{\widetilde{c}_1}{\theta} e^{\theta(t-t_m)} \left\|y - \overline{y}\right\|_{C[t_m,1]}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\left|T_1(x,y)(t) - T_1(\overline{x},\overline{y})(t)\right| \\
\leq\ & b_1 t_m A_1 \left|x - \overline{x}\right|_{C[0,t_m]} + \widetilde{b}_1 t_m A_1 \left|y - \overline{y}\right|_{C[0,t_m]} \\
&+ \frac{c_1}{\theta} e^{\theta(t-t_m)} \left\|x - \overline{x}\right\|_{C[t_m,1]} + \frac{\widetilde{c}_1}{\theta} e^{\theta(t-t_m)} \left\|y - \overline{y}\right\|_{C[t_m,1]}.
\end{aligned}$$

Dividing by $e^{\theta(t-t_m)}$ and taking the supremum when $t \in [t_m, 1]$, we obtain

$$\begin{aligned} & \|T_1(x,y) - T_1(\overline{x},\overline{y})\|_{C[t_m,1]} \qquad (2.5) \\ \leq \; & b_1 t_m A_1 \left| x - \overline{x} \right|_{C[0,t_m]} + \widetilde{b}_1 t_m A_1 \left| y - \overline{y} \right|_{C[0,t_m]} \\ \leq \; & \frac{c_1}{\theta} \left\| x - \overline{x} \right\|_{C[t_m,1]} + \frac{\widetilde{c}_1}{\theta} \left\| y - \overline{y} \right\|_{C[t_m,1]} . \end{aligned}$$

Now (2.4) and (2.5) imply that

$$\|T_1(x,y) - T_1(\overline{x},\overline{y})\| \leq \left(b_1 t_m A_1 + \frac{c_1}{\theta} \right) \|x - \overline{x}\| + \left(\widetilde{b}_1 t_m A_1 + \frac{\widetilde{c}_1}{\theta} \right) \|y - \overline{y}\| . \qquad (2.6)$$

Similarly

$$\|T_2(x,y) - T_2(\overline{x},\overline{y})\| \leq \left(B_1 t_m A_2 + \frac{C_1}{\theta} \right) \|x - \overline{x}\| + \left(\widetilde{B}_1 t_m A_2 + \frac{\widetilde{C}_1}{\theta} \right) \|y - \overline{y}\| , \qquad (2.7)$$

where

$$A_2 = 1 + |\widetilde{a}| \sum_{k=1}^{m} |\widetilde{a}_k| .$$

Using the vector norm we can put both inequalities (2.6), (2.7) under the vector inequality

$$\|T(u) - T(\overline{u})\| \leq M_\theta \|u - \overline{u}\| ,$$

where

$$M_\theta = \begin{bmatrix} b_1 t_m A_1 + \frac{c_1}{\theta} & \widetilde{b}_1 t_m A_1 + \frac{\widetilde{c}_1}{\theta} \\ B_1 t_m A_2 + \frac{C_1}{\theta} & \widetilde{B}_1 t_m A_2 + \frac{\widetilde{C}_1}{\theta} \end{bmatrix} . \qquad (2.8)$$

Clearly matrix M_θ can be represented as $M_\theta = M_0 + M_1$, where

$$M_1 = \begin{bmatrix} \frac{c_1}{\theta} & \frac{\widetilde{c}_1}{\theta} \\ \frac{C_1}{\theta} & \frac{\widetilde{C}_1}{\theta} \end{bmatrix} .$$

Since M_0 is assumed to be convergent to zero, from Lemma 1.1 we have that M_θ also converges to zero for large enough $\theta > 0$. The result follows now from Perov's fixed point theorem. □

3. Nonlinearities with growth at most linear. Application of Schauder's fixed point theorem

Here we show that the existence of solutions of problem (1.1) follows from Schauder's fixed point theorem in case that f, g satisfy instead of the Lipschitz condition the more relaxed condition of at most linear growth:

$$|f(t,x,y)| \leq \begin{cases} b_1 |x| + \widetilde{b}_1 |y| + d_1 & \text{if } t \in [0, t_m] \\ c_1 |x| + \widetilde{c}_1 |y| + d_2 & \text{if } t \in [t_m, 1] , \end{cases} \qquad (3.1)$$

$$|g(t,x,y)| \leq \begin{cases} B_1 |x| + \widetilde{B}_1 |y| + D_1 & \text{if } t \in [0,t_m] \\ C_1 |x| + \widetilde{C}_1 |y| + D_2 & \text{if } t \in [t_m,1]. \end{cases} \tag{3.2}$$

Theorem 3.1. *If* f, g *satisfy* (3.1), (3.2) *and the matrix* (2.3) *converges to zero, then problem* (1.1) *has at least one solution.*

Proof. In order to apply Schauder's fixed point theorem, we look for a nonempty, bounded, closed and convex subset B of $C[0,1]^2$ so that $T(B) \subset B$. Let x, y be any elements of $C[0,1]$.
For $t \in [0,t_m]$, we have

$$\begin{aligned} |T_1(x,y)(t)| &= \left| -a\sum_{k=1}^{m} a_k \int_0^{t_k} f(s,x(s),y(s))\,ds + \int_0^t f(s,x(s),y(s))\,ds \right| \\ &\leq A_1 \int_0^{t_m} |f(s,x(s),y(s))|\,ds \\ &\leq b_1 t_m A_1 |x|_{C[0,t_m]} + \widetilde{b}_1 t_m A_1 |y|_{C[0,t_m]} + d_1 t_m A_1. \end{aligned}$$

Taking the supremum, we obtain that

$$|T_1(x,y)|_{C[0,t_m]} \leq b_1 t_m A_1 |x|_{C[0,t_m]} + \widetilde{b}_1 t_m A_1 |y|_{C[0,t_m]}. \tag{3.3}$$

For $t \in [t_m,1]$ and any $\theta > 0$, we have

$$\begin{aligned} |T_1(x,y)(t)| &= \left| -a\sum_{k=1}^{m} a_k \int_0^{t_k} f(s,x(s),y(s))\,ds + \int_0^t f(s,x(s),y(s))\,ds \right| \\ &\leq b_1 t_m A_1 |x|_{C[0,t_m]} + \widetilde{b}_1 t_m A_1 |y|_{C[0,t_m]} + d_1 t_m A_1 \\ &+ \int_{t_m}^t (c_1 |x(s)| + \widetilde{c}_1 |y(s)| + d_2)\,ds \\ &\leq b_1 t_m A_1 |x|_{C[0,t_m]} + \widetilde{b}_1 t_m A_1 |y|_{C[0,t_m]} + d_1 t_m A_1 + (1-t_m) d_2 \\ &+ c_1 \int_{t_m}^t |x(s)| \cdot e^{-\theta(s-t_m)} \cdot e^{\theta(s-t_m)} ds \\ &+ \widetilde{c}_1 \int_{t_m}^t |y(s)| \cdot e^{-\theta(s-t_m)} \cdot e^{\theta(s-t_m)} ds \\ &\leq b_1 t_m A_1 |x|_{C[0,t_m]} + \widetilde{b}_1 t_m A_1 |y|_{C[0,t_m]} + c_0 \\ &+ \frac{c_1}{\theta} e^{\theta(t-t_m)} \|x\|_{C[t_m,1]} + \frac{\widetilde{c}_1}{\theta} e^{\theta(t-t_m)} \|y\|_{C[t_m,1]}, \end{aligned}$$

where $c_0 = d_1 t_m A_1 + (1-t_m) d_2$. Dividing by $e^{\theta(t-t_m)}$ and taking the supremum, it follows that

$$\begin{aligned} \|T_1(x,y)\|_{C[t_m,1]} &\leq b_1 t_m A_1 |x|_{C[0,t_m]} + \widetilde{b}_1 t_m A_1 |y|_{C[0,t_m]} \\ &\quad + \frac{c_1}{\theta} \|x\|_{C[t_m,1]} + \frac{\widetilde{c}_1}{\theta} \|y\|_{C[t_m,1]} + c_0. \end{aligned} \tag{3.4}$$

Clearly (3.3), (3.4) give

$$\|T_1(x,y)\| \leq \left(b_1 t_m A_1 + \frac{c_1}{\theta}\right)\|x\| + \left(\widetilde{b}_1 t_m A_1 + \frac{\widetilde{c}_1}{\theta}\right)\|y\| + \widetilde{c}_0, \tag{3.5}$$

where $\widetilde{c}_0 = \max\{d_1 t_m A_1, c_0\}$. Similarly

$$\|T_2(x,y)\| \leq \left(B_1 t_m A_2 + \frac{C_1}{\theta}\right)\|x\| + \left(\widetilde{B}_1 t_m A_2 + \frac{\widetilde{C}_1}{\theta}\right)\|y\| + \widetilde{C}_0, \tag{3.6}$$

with $\widetilde{C}_0 = \max\{D_1 t_m A_2, C_0\}$. Now (3.5), (3.6) can be put together as

$$\begin{bmatrix} \|T_1(x,y)\| \\ \|T_2(x,y)\| \end{bmatrix} \leq M_\theta \begin{bmatrix} \|x\| \\ \|y\| \end{bmatrix} + \begin{bmatrix} \widetilde{c}_0 \\ \widetilde{C}_0 \end{bmatrix},$$

where matrix M_θ is given by (2.8) and converges to zero for large enough $\theta > 0$. Next we look for two positive numbers R_1, R_2 such that if $\|x\| \leq R_1, \|y\| \leq R_2$, then $\|T_1(x,y)\| \leq R_1$, $\|T_2(x,y)\| \leq R_2$. To this end it is sufficient that

$$\begin{cases} \left(b_1 t_m A_1 + \frac{c_1}{\theta}\right) R_1 + \left(\widetilde{b}_1 t_m A_1 + \frac{\widetilde{c}_1}{\theta}\right) R_2 + \widetilde{c}_0 \leq R_1 \\ \left(B_1 t_m A_2 + \frac{C_1}{\theta}\right) R_1 + \left(\widetilde{B}_1 t_m A_2 + \frac{\widetilde{C}_1}{\theta}\right) R_2 + \widetilde{C}_0 \leq R_2, \end{cases} \tag{3.7}$$

or equivalently

$$M_\theta \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} + \begin{bmatrix} \widetilde{c}_0 \\ \widetilde{C}_0 \end{bmatrix} \leq \begin{bmatrix} R_1 \\ R_2 \end{bmatrix},$$

whence

$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \geq (I - M_\theta)^{-1} \begin{bmatrix} \widetilde{c}_0 \\ \widetilde{C}_0 \end{bmatrix}.$$

Notice that $I - M_\theta$ is invertible and its inverse $(I - M_\theta)^{-1}$ has nonnegative elements since M_θ converges to zero. Thus, if

$$B = \left\{(x,y) \in C[0,1]^2 : \|x\| \leq R_1, \|y\| \leq R_2\right\},$$

then $T(B) \subset B$ and Schauder's fixed point theorem can be applied. □

4. More general nonlinearities. Application of the Leray-Schauder principle

We now consider that nonlinearities f, g satisfy more general growth conditions, namely:

$$|f(t,u)| \leq \begin{cases} \omega_1(t, |u|_e) & \text{if } t \in [0, t_m] \\ \alpha(t)\beta_1(|u|_e), & \text{if } t \in [t_m, 1], \end{cases} \tag{4.1}$$

$$|g(t,u)| \leq \begin{cases} \omega_2(t, |u|_e) & \text{if } t \in [0, t_m] \\ \alpha(t)\beta_2(|u|_e) & \text{if } t \in [t_m, 1], \end{cases} \tag{4.2}$$

for all $u=(x,y)\in\mathbf{R}^2$, where by $|u|_e$ we mean the Euclidean norm in $\mathbf{R}^2$. Here ω_1,ω_2 are Carathéodory functions on $[0,t_m]\times\mathbf{R}_+$, nondecreasing in their second argument, $\alpha\in L^1[t_m,1]$, while $\beta_1,\beta_2:\mathbf{R}_+\to\mathbf{R}_+$ are nondecreasing and $1/\beta_1, 1/\beta_2\in L^1_{loc}(\mathbf{R}_+)$.

Theorem 4.1. *Assume that conditions* (4.1), (4.2) *hold. In addition assume that there exists a positive number R_0 such that for $\rho=(\rho_1,\rho_2)\in(0,\infty)^2$*

$$\begin{cases} \frac{1}{\rho_1}\int_0^{t_m}\omega_1(t,|\rho|_e)dt\geq\dfrac{1}{A_1} \\ \frac{1}{\rho_2}\int_0^{t_m}\omega_2(t,|\rho|_e)dt\geq\dfrac{1}{A_2}\end{cases} \quad \textit{implies} \quad |\rho|_e\leq R_0 \tag{4.3}$$

and

$$\int_{R^*}^{\infty}\frac{d\tau}{\beta_1(\tau)+\beta_2(\tau)}>\int_{t_m}^{1}\alpha(s)ds, \tag{4.4}$$

where $R^*=\left[\left(A_1\int_0^{t_m}\omega_1(t,R_0)dt\right)^2+\left(A_2\int_0^{t_m}\omega_2(t,R_0)dt\right)^2\right]^{1/2}.$
Then problem (1.1) *has at least one solution.*

Proof. The result will follow from the Leray-Schauder fixed point theorem once we have proved the boundedness of the set of all solutions to equations $u=\lambda T(u)$, for $\lambda\in[0,1]$. Let $u=(x,y)$ be such a solution. Then, for $t\in[0,t_m]$, we have

$$\begin{aligned} |x(t)| &= |\lambda T_1(x,y)(t)| \\ &= \lambda\left|-a\sum_{k=1}^m a_k\int_0^{t_k}f(s,x(s),y(s))\,ds+\int_0^t f(s,x(s),y(s))\,ds\right| \\ &\leq \left(1+|a|\sum_{k=1}^m|a_k|\right)\int_0^{t_m}|f(s,x(s),y(s))|\,ds \\ &= A_1\int_0^{t_m}|f(s,u(s))|\,ds \\ &\leq A_1\int_0^{t_m}\omega_1(s,|u(s)|_e)ds. \end{aligned} \tag{4.5}$$

Similarly

$$|y(t)|\leq A_2\int_0^{t_m}\omega_2(s,|u(s)|_e)ds. \tag{4.6}$$

Let $\rho_1=|x|_{C[0,t_m]}$, $\rho_2=|y|_{C[0,t_m]}$. Then from (4.5), (4.6), we deduce

$$\begin{cases}\rho_1\leq A_1\int_0^{t_m}\omega_1(t,|\rho|_e)dt \\ \rho_1\leq A_1\int_0^{t_m}\omega_1(t,|\rho|_e)dt.\end{cases}$$

This by (4.3) guarantees

$$|\rho|_e \le R_0. \tag{4.7}$$

Next we let $t \in [t_m, 1]$. Then

$$\begin{aligned} |x(t)| &= |\lambda T_1(x,y)(t)| \\ &= \lambda \left| -a \sum_{k=1}^{m} a_k \int_0^{t_k} f(s, x(s), y(s))\, ds + \int_0^t f(s, x(s), y(s))\, ds \right| \\ &\le A_1 \int_0^{t_m} \omega_1(t, R_0) dt + \int_{t_m}^t |f(s, x(s), y(s))|\, ds \\ &\le A_1 \int_0^{t_m} \omega_1(t, R_0) dt + \int_{t_m}^t \alpha(s)\beta_1(|u(s)|_e) ds \\ &= : \phi_1(t) \end{aligned}$$

and similarly

$$\begin{aligned} |y(t)| &\le A_1 \int_0^{t_m} \omega_2(t, R_0) dt + \int_{t_m}^t \alpha(s)\beta_2(|u(s)|_e) ds \\ &= : \phi_2(t). \end{aligned}$$

Denote $\psi(t) := \left(\phi_1^2(t) + \phi_2^2(t)\right)^{1/2}$. Then

$$\begin{cases} \phi_1^{'}(t) = \alpha(t)\beta_1(|u(t)|_e) \le \alpha(t)\beta_1(\psi(t)) \\ \phi_2^{'}(t) = \alpha(t)\beta_2(|u(t)|_e) \le \alpha(t)\beta_2(\psi(t))). \end{cases} \tag{4.8}$$

Consequently

$$\begin{aligned} \psi^{'}(t) &= \frac{\phi_1(t)\phi_1^{'}(t) + \phi_2(t)\phi_2^{'}(t)}{\psi(t)} \\ &\le \alpha(t) \cdot \frac{\phi_1(t)}{\psi(t)} \cdot \beta_1(\psi(t)) + \alpha(t) \cdot \frac{\phi_2(t)}{\psi(t)} \cdot \beta_2(\psi(t)) \\ &\le \alpha(t)\left[\beta_1(\psi(t)) + \beta_2(\psi(t))\right]. \end{aligned}$$

It follows that

$$\int_{t_m}^t \frac{\psi'(s)}{\beta_1(\psi(s)) + \beta_2(\psi(s))} ds \le \int_{t_m}^t \alpha(s) ds.$$

Furthermore, also using (4.4) we obtain

$$\int_{\psi(t_m)}^{\psi(t)} \frac{d\tau}{\beta_1(\tau) + \beta_2(\tau))} \le \int_{t_m}^t \alpha(s) ds \le \int_{t_m}^1 \alpha(s) ds < \int_{R^*}^\infty \frac{d\tau}{\beta_1(\tau) + \beta_2(\tau)}. \tag{4.9}$$

Note that $\psi(t_m) = R^*$. Then from (4.9) it follows that there exists R_1 such that

$$\psi(t) \le R_1$$

for all $t \in [t_m, 1]$. Then $|x(t)| \le R_1$ and $|y(t)| \le R_1$, for all $t \in [t_m, 1]$, whence

$$|x|_{C[t_m,1]} \le R_1, \quad |y|_{C[t_m,1]} \le R_1. \tag{4.10}$$

Let $R = \max\{R_0, R_1\}$. From (4.7), (4.10) we have $|x|_{C[0,1]} \leq R$ and $|y|_{C[0,1]} \leq R$. □

Remark 4.2. If $\omega_1(t,\tau) = \alpha_0(t)\beta_0(\tau)$, then the first inequality in (4.3) implies that $\beta_0(\tau) \leq c\tau + c'$ for all $\tau \in R_+$ and some constants c and c', i.e. the growth of β_0 is at most linear. However, β_1 may have a superlinear growth. Thus we may say that under the assumptions of Theorem 4.1, the growth of $f(t,u)$ in u is at most linear for $t \in [0, t_m]$ and can be superlinear for $t \in [t_m, 1]$. The same can be said about $g(t,u)$.

In particular, when $t_m = 0$, that is when problem (1.1) becomes the classical local initial value problem

$$\begin{cases} x' = f(t,x,y) \\ y' = g(t,x,y) \quad \text{(a.e. } t \in [0,1]) \\ x(0) = y(0) = 0, \end{cases} \tag{4.11}$$

our assumptions reduce to the classical conditions (see [7], [9]) and Theorem 4.1 gives the following result:

Corollary 4.3. *Assume that*

$$\begin{aligned} |f(t,u)| &\leq \alpha(t)\beta_1(|u|_e), \\ |g(t,u)| &\leq \alpha(t)\beta_2(|u|_e) \end{aligned}$$

for $t \in [0,1]$ *and* $u \in \mathbf{R}^2$, *where* $\alpha \in L^1[0,1]$, *while* $\beta_1, \beta_2 : \mathbf{R}_+ \to \mathbf{R}_+$ *are nondecreasing and* $1/\beta_1, 1/\beta_2 \in L^1_{loc}(\mathbf{R}_+)$. *In addition assume that*

$$\int_0^\infty \frac{d\tau}{\beta_1(\tau) + \beta_2(\tau)} > \int_0^1 \alpha(s)ds.$$

Then, problem (4.11) *has at least one solution.*

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