Compression-expansion fixed point theorems in two norms

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ABSTRACT. In this paper we present compression-expansion fixed point theorems in cones where the compression and the expansion conditions are expressed in two norms.
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1 Introduction

Let $(E, |.|)$ be a normed linear space and $\|.\|$ will be another norm on E. Also $C \subset E$ will be a cone, i.e., a nonempty convex (not necessarily closed) set with $0 \notin C$ and $\lambda C \subset C$ for all $\lambda > 0$.We shall assume that there exist constants $c_1, c_2 > 0$ such that

$$c_1 |x| \leq \|x\| \leq c_2 |x| \quad \text{for all } x \in C. \tag{1}$$

Hence the norms $|.|$ and $\|.\|$ are topologically equivalent on C (but not necessarily on E).

In [5] the following two theorems are proved:

Theorem 1 *Assume* $0 < c_2 \rho < R$, $\|.\|$ *is increasing with respect to* C, *that is* $\|x + y\| > \|x\|$ *for all* $x, y \in C$, *and the map* $N : \{x \in C : \|x\| \leq R\} \to C$ *is compact. In addition assume that the following conditions are satisfied:*

(h1) $|N(x)| < |x|$ *for all* $x \in C$ *with* $|x| = \rho$,

(h2) $\|N(x)\| \geq \|x\|$ *for all* $x \in C$ *with* $\|x\| = R$.

Then N *has at least two fixed points* $x_1, x_2 \in C$ *with* $|x_1| < \rho \leq |x_2|$ *and* $\|x_2\| \leq R$.

Theorem 2 *Assume* $0 < \frac{1}{c_1}\rho < R$, $|.|$ *is increasing with respect to* C, *and the map* $N : \{x \in C : |x| \leq R\} \to C$ *is compact. In addition assume that the following conditions are satisfied:*

(h1) $\|N(x)\| < \|x\|$ *for all* $x \in C$ *with* $\|x\| = \rho$,

(h2) $|N(x)| \geq |x|$ *for all* $x \in C$ *with* $|x| = R$.

Then N *has at least two fixed points* $x_1, x_2 \in C$ *with* $\|x_1\| < \rho \leq \|x_2\|$ *and* $|x_2| \leq R$.

The aim of this paper is to show that similar results are true if the inequalities in (h1), (h2) are reversed.

2 Main results

Theorem 3 *Assume* $0 < c_2\,\rho < R$, $\|.\|$ *is increasing with respect to* C, *and the map* $N : D = \{x \in C : \|x\| \leq R\} \to C$ *is compact. In addition assume that the following conditions are satisfied:*

(H1) $\|N(x)\| \geq \|x\|$ *for all* $x \in C$ *with* $|x| = \rho$,

(H2) $|N(x)| < |x|$ *for all* $x \in C$ *with* $\|x\| = R$.

Then N *has at least one fixed point* $x \in C$ *with* $\rho \leq |x|$ *and* $\|x\| < R$.

Proof. Let $0 < \varepsilon < c_1\rho$ and let $N' : \{x \in C : \|x\| \leq R\} \to C$ be defined by

$$N'(x) = \begin{cases} \left(\frac{R}{\|x\|} + \frac{\rho}{|x|} - 1\right)^{-1} N\left(\left(\frac{R}{\|x\|} + \frac{\rho}{|x|} - 1\right)x\right) & \text{if } |x| \geq \rho \\ \frac{\|x\|}{R} N\left(\frac{R}{\|x\|}x\right) & \text{if } |x| \leq \rho,\ \|x\| \geq \varepsilon \\ \frac{\varepsilon}{R} N\left(\frac{R}{\varepsilon}x\right) & \text{if } |x| \leq \rho,\ \|x\| \leq \varepsilon. \end{cases}$$

For $|x| \geq \rho$, $\|x\| \leq R$, we have

$$\left|\left(\frac{R}{\|x\|} + \frac{\rho}{|x|} - 1\right)x\right| \geq \rho, \quad \left\|\left(\frac{R}{\|x\|} + \frac{\rho}{|x|} - 1\right)x\right\| \leq R.$$

Also, $\left\|\frac{R}{\|x\|}x\right\| = R$ and for $\|x\| \leq \varepsilon$, $\left\|\frac{R}{\varepsilon}x\right\| \leq R$. Hence N' is well defined. It is easy to see that N' is continuous. In addition, since N is compact and the coefficients $\left(\frac{R}{\|x\|} + \frac{\rho}{|x|} - 1\right)^{-1}$, $\frac{\|x\|}{R}$ are located between two positive constants, more exactly

$$1 \leq \left(\frac{R}{\|x\|} + \frac{\rho}{|x|} - 1\right)^{-1} \leq \frac{R}{c_1\rho} \quad \text{for } |x| \geq \rho$$

$$\frac{\varepsilon}{R} \leq \frac{\|x\|}{R} \leq 1 \text{ for } \|x\| \geq \varepsilon,$$

we deduce that N' is also compact.

Now if $|x| = \rho$, then $N'(x) = \frac{\|x\|}{R} N\left(\frac{R}{\|x\|}x\right)$. From (H2) we have

$$\left|N\left(\frac{R}{\|x\|}x\right)\right| < \frac{R}{\|x\|}|x|.$$

Hence $|N'(x)| < |x|$ and so condition (h1) in Theorem 1 holds for N'. Furthermore, if $\|x\| = R$, then $N'(x) = \frac{|x|}{\rho} N\left(\frac{\rho}{|x|}x\right)$ and from (H1),

$$\left\|N\left(\frac{\rho}{|x|}x\right)\right\| \geq \frac{\rho}{|x|}\|x\|.$$

Thus $\|N'(x)\| \geq \|x\|$ which shows that condition (h2) in Theorem 1 also holds for N'. Thus Theorem 1 applies to N'. Let x_0 be the fixed point of N' with $|x_0| \geq \rho$ and $\|x_0\| \leq R$. If $|x_0| = \rho$, then from (1) we deduce that $\|x_0\| \geq c_1\rho > \varepsilon$. Then $N'(x_0) = \frac{\|x_0\|}{R} N\left(\frac{R}{\|x_0\|}x_0\right)$, whence $x' = \frac{R}{\|x_0\|}x_0$ is a fixed point of N. This is however impossible since $\|x'\| = R$ and N has no fixed points with $\|x\| = R$ as shows (H2). Therefore $|x_0| > \rho$. Consequently, $x := \left(\frac{R}{\|x_0\|} + \frac{\rho}{|x_0|} - 1\right)x_0$ is the fixed point of N we look for. ■

Theorem 4 *Assume* $0 < \frac{1}{c_1}\rho < R$, $|.|$ *is increasing with respect to* C, *and the map* $N : \{x \in C : |x| \leq R\} \to C$ *is compact. In addition assume that the following conditions are satisfied:*

(h1) $|N(x)| \geq |x|$ *for all* $x \in C$ *with* $\|x\| = \rho$

(h2) $\|N(x)\| < \|x\|$ *for all* $x \in C$ *with* $|x| = R$.

Then N *has at least one fixed point* $x \in C$ *with* $\rho \leq \|x\|$ *and* $|x| < R$.

3 Example

We shall illustrate the use of Theorem 3 on the two point boundary value problem

$$\begin{cases} -u''(t) = f(u(t)), & t \in [0,1] \\ u(0) = u(1) = 0. \end{cases} \tag{2}$$

Assume $f : \mathbf{R} \to \mathbf{R}$ is continuous. Then (2) is equivalent to the fixed point problem

$$u = N(u), \quad u \in C[0,1]$$

where $N : C[0,1] \to C[0,1]$, $N(u)(t) = \int_0^1 G(t,s) f(u(s))\, ds$ and G is the Green function $G(t,s) = t(1-s)$ if $0 \le t \le s \le 1$, $G(t,s) = s(1-t)$ if $0 \le s \le t \le 1$. One has

$$\begin{aligned} G(t,s) &\le G(s,s) \text{ for all } t,s \in [0,1] \\ \frac{1}{4} G(s,s) &\le G(t,s) \text{ for all } t \in \left[\frac{1}{4}, \frac{3}{4}\right],\ s \in [0,1]. \end{aligned}$$

These inequalities guarantee that

$$u(t) \ge \frac{1}{4} |u|_r \quad \text{for all } t \in \left[\frac{1}{4}, \frac{3}{4}\right],\ r \in [1,\infty] \tag{3}$$

and any solution u of the problem

$$\begin{cases} -u''(t) = v(t), & t \in [0,1] \\ u(0) = u(1) = 0 \end{cases}$$

with $v \in C([0,1]; \mathbf{R}_+)$. Here $|u|_r$ stands for the usual norm on $L^r[0,1]$. From (3) we see that for $1 \le p \le q \le \infty$, we have

$$\frac{1}{4} |u|_q \le |u|_p \le |u|_q$$

whenever u belongs to the cone

$$C := \left\{ u \in C([0,1]; \mathbf{R}_+) : u \ne 0,\ u(t) \ge \frac{1}{4} |u|_r \text{ for } t \in \left[\frac{1}{4}, \frac{3}{4}\right],\ r \in [1,\infty] \right\}.$$

It is easily seen that $N : C \to C$ and N is completely continuous provided that $f(\tau) \ge \varepsilon$ for all $\tau \in \mathbf{R}_+$ and some $\varepsilon > 0$.

Theorem 5 *Assume $f : \mathbf{R}_+ \to \mathbf{R}_+$ is continuous and nondecreasing on $\mathbf{R}_+$, and there exist ρ, R with $0 < \rho < R$ and $1 \le p \le q \le \infty$ such that:*

$$\frac{f(4R)}{4R} < \frac{1}{4} \left(\int_0^1 \left(\int_0^1 G(t,s)\, ds \right)^q dt \right)^{-\frac{1}{q}},$$

$$\frac{f\left(\frac{1}{4}\rho\right)}{\frac{1}{4}\rho} > 4 \left(\int_0^1 \left(\int_{\frac{1}{4}}^{\frac{3}{4}} G(t,s)\, ds \right)^p dt \right)^{-\frac{1}{p}}.$$

Then (2) has a solution u with $\rho \le |u|_q$ and $|u|_p \le R$.

Proof. We shall assume that $f(\tau) \geq \varepsilon$ for all $\tau \geq 0$ and some small $\varepsilon > 0$ (otherwise we take $f^{\varepsilon} = f + \varepsilon$ instead of f and we use a compactness argument as $\varepsilon \to 0$). Let $u \in C$ with $|u|_q = \rho$. Then, for every $t \in [0,1]$, one has

$$\begin{aligned} N(u)(t) &= \int_0^1 G(t,s) f(u(s))\, ds \geq \int_{\frac{1}{4}}^{\frac{3}{4}} G(t,s) f(u(s))\, ds \\ &\geq f\left(\frac{1}{4}\rho\right) \int_{\frac{1}{4}}^{\frac{3}{4}} G(t,s)\, ds. \end{aligned}$$

Consequently

$$|N(u)|_p \geq f\left(\frac{1}{4}\rho\right) \left(\left(\int_{\frac{1}{4}}^{\frac{3}{4}} G(t,s)\, ds \right)^p dt \right)^{\frac{1}{p}} \geq \rho \geq |u|_p$$

Hence (H1) holds. Furthermore, if $u \in C$ and $|u|_p = R$, then

$$N(u)(t) \leq \int_0^1 G(t,s) f(|u|_\infty)\, ds \leq f(4R) \int_0^1 G(t,s)\, ds.$$

It follows that

$$|N(u)|_q \leq f(4R) \left(\left(\int_0^1 G(t,s)\, ds \right)^q dt \right)^{\frac{1}{q}} < R \leq |u|_q$$

which proves (H2). ■

A similar result can be obtained from Theorem 4.

For related topics and applications of the compression-expansion theorems to integral and differential equations, see [1-6]. For other results based on the idea of using two norms, see [7].

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