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Construction of Upper and Lower Solutions with Applications to Singular Boundary Value Problems

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Abstract

An upper and lower solution theory is presented for the Dirichlet boundary value problem $y'' + f(t, y, y') = 0,\ 0 < t < 1$ with $y(0) = y(1) = 0$. Our nonlinearity may be singular in its dependent variable and is allowed to change sign.

Keywords: Boundary value problem, upper and lower solutions, singular, existence.

Subject Classes: 34B16.

1. Introduction

An approach based on upper and lower solutions and a truncation technique is presented for the singular boundary value problem

$$(1.1)\qquad \begin{cases} y'' + q(t)\, f(t, y, y') = 0, & 0 < t < 1 \\ y(0) = 0 = y(1), \end{cases}$$

where our nonlinearity f is allowed to change sign. In addition f may not be a Carathéodory function because of the singular behavior of the y variable i.e. f may be singular at $y = 0$. In the literature the case when f is independent of its third variable (i.e. when $f(t,y,z) \equiv f(t,y)$) has received almost all the attention, see [2-4, 6, 7] and the references therein. Only a few papers [1, 8] have appeared when f depends on the y' variable. This paper presents a new and very general result for (1.1) when $f:(0,1)\times(0,\infty)\times\mathbf{R}\to\mathbf{R}$. In addition our results are new even when f is independent of the third variable. It is also worth remarking here that we could consider Sturm-Liouville boundary data in (1.1); however since the arguments are essentially the same we will restrict our discussion to Dirichlet boundary data.

2. Existence Theory

In this section we present an upper and lower solution theory for the Dirichlet singular boundary value problem

$$\begin{cases} y'' + q(t)\,f(t,y,y') = 0, & 0<t<1 \\ y(0)=y(1)=0, \end{cases} \tag{2.1}$$

where our nonlinearity f may change sign.

Theorem 2.1. *Let $n_0 \in \{1,2,...\}$ be fixed and suppose the following conditions are satisfied:*

$$f:(0,1)\times(0,\infty)\times\mathbf{R}\to\mathbf{R} \;\; \textit{is continuous} \tag{2.2}$$

$$q\in C(0,1)\cap L^1[0,1] \;\; \textit{with} \;\; q>0 \;\; \textit{on} \;\; (0,1) \tag{2.3}$$

$$\begin{cases} \textit{let } n\in\{n_0,n_0+1,...\} \textit{ and associated with each } n \textit{ we} \\ \textit{have a constant } \rho_n \textit{ such that } \{\rho_n\} \textit{ is a nonincreasing} \\ \textit{sequence with } \lim_{n\to\infty}\rho_n = 0 \textit{ and such that for} \\ \frac{1}{2^{n+1}} \le t \le 1 \textit{ and } z\in\mathbf{R} \textit{ we have } f(t,\rho_n,z)\ge 0 \end{cases} \tag{2.4}$$

$$\begin{cases} \exists\,\alpha\in C[0,1]\cap C^2(0,1) \textit{ with } \alpha(0)=\alpha(1)=0, \\ \alpha>0 \textit{ on } (0,1) \textit{ such that} \\ q(t)\,f(t,\alpha(t),z)+\alpha''(t)\ge 0 \textit{ for } (t,z)\in(0,1)\times\mathbf{R} \end{cases} \tag{2.5}$$

$$\begin{cases} \exists\,\beta\in C^1[0,1]\cap C^2(0,1) \textit{ with } \beta(t)\ge\alpha(t),\; \beta(t)\ge\rho_{n_0} \\ \textit{for } t\in[0,1] \textit{ with } q(t)\,f(t,\beta(t),\beta'(t))+\beta''(t)\le 0 \\ \textit{for } t\in(0,1) \textit{ and } q(t)\,f\left(\frac{1}{2^{n_0+1}},\beta(t),\beta'(t)\right)+\beta''(t)\le 0 \\ \textit{for } t\in\left(0,\frac{1}{2^{n_0+1}}\right) \end{cases} \tag{2.6}$$

$$(2.7)\qquad \begin{cases} \textit{for any } \ \epsilon > 0,\ \epsilon < a_0 = \sup_{t\in[0,1]} \beta(t),\ \exists \ \ \textit{a function} \\ \psi_\epsilon \ \ \textit{continuous on } \ [0,\infty) \ \ \textit{with } \ |f(t,y,z)| \le \psi_\epsilon(|z|) \\ \textit{for } \ (t,y,z) \in (0,1)\times[\epsilon,a_0]\times \mathbf{R} \end{cases}$$

and

$$(2.8)\qquad \textit{for any } \ \epsilon > 0,\ \epsilon < a_0, \ \ \textit{we have } \ \int_0^1 q(s)\,ds < \int_0^\infty \frac{du}{\psi_\epsilon(u)}.$$

Then (2.1) *has a solution* $y \in C[0,1]\cap C^2(0,1)$ *with* $\alpha(t) \le y(t) \le \beta(t)$ *for* $t \in [0,1]$.

PROOF: For $n = n_0,\, n_0+1, ...$ let

$$e_n = \left[\frac{1}{2^{n+1}}, 1\right] \quad \text{and} \quad \theta_n(t) = \max\left\{\frac{1}{2^{n+1}}\,,\, t\right\},\ 0 \le t \le 1$$

and

$$f_n(t,x,z) = \max\left\{f(\theta_n(t),x,z)\,,\, f(t,x,z)\right\}.$$

Next we define inductively

$$g_{n_0}(t,x,z) = f_{n_0}(t,x,z)$$

and

$$g_n(t,x,z) = \min\left\{f_{n_0}(t,x,z), ..., f_n(t,x,z)\right\},\ n = n_0+1,\, n_0+2, ... \ .$$

Notice

$$f(t,x,z) \le ... \le g_{n+1}(t,x,z) \le g_n(t,x,z) \le ... \le g_{n_0}(t,x,z)$$

for $(t,x,z) \in (0,1)\times(0,\infty)\times\mathbf{R}$ and

$$g_n(t,x,z) = f(t,x,z) \quad \text{for} \quad (t,x,z) \in e_n \times (0,\infty)\times \mathbf{R}.$$

Without loss of generality assume $\rho_{n_0} \le \min_{t\in\left[\frac{1}{3},\frac{2}{3}\right]} \alpha(t)$. Fix $n \in \{n_0, n_0 + 1, ...\}$. Let $t_n \in \left[0,\frac{1}{3}\right]$ and $s_n \in \left[\frac{2}{3},1\right]$ be such that

$$\alpha(t_n) = \alpha(s_n) = \rho_n \quad \text{and} \quad \alpha(t) \le \rho_n \ \text{ for } \ t \in [0,t_n]\cup[s_n,1].$$

Define

$$\alpha_n(t) = \begin{cases} \rho_n & \text{if } \ t \in [0,t_n]\cup[s_n,1] \\ \alpha(t) & \text{if } \ t \in (t_n,s_n). \end{cases}$$

We begin with the boundary value problem

$$(2.9)\qquad \begin{cases} y'' + q(t)\, g^{\star}_{n_0}(t,y,y') = 0,\ 0<t<1 \\ y(0) = y(1) = \rho_{n_0}; \end{cases}$$

here

$$g^{\star}_{n_0}(t,y,z) = \begin{cases} g_{n_0}(t,\alpha_{n_0}(t),z^{\star}) + r(\alpha_{n_0}(t)-y),\ \ y \le \alpha_{n_0}(t) \\ g_{n_0}(t,y,z^{\star}),\quad \alpha_{n_0}(t) \le y \le \beta(t) \\ g_{n_0}(t,\beta(t),z^{\star}) + r(\beta(t)-y),\ y \ge \beta(t), \end{cases}$$

with

$$z^{\star} = \begin{cases} M_{n_0},\ \ z > M_{n_0} \\ z,\ \ -M_{n_0} \le z \le M_{n_0} \\ -M_{n_0},\ \ z < -M_{n_0} \end{cases}$$

and $r : \mathbf{R} \to [-1,1]$ the radial retraction defined by

$$r(u) = \begin{cases} u, & |u| \le 1 \\ \frac{u}{|u|}, & |u| > 1, \end{cases}$$

and $M_{n_0} \ge \sup_{[0,1]} |\beta'(t)|$ is a predetermined constant (see (2.15)). Now Schauder's fixed point theorem [7] guarantees that there exists a solution $y_{n_0} \in C^1[0,1]$ to (2.9). We first show

$$(2.10)\qquad y_{n_0}(t) \ge \alpha_{n_0}(t),\ \ t \in [0,1].$$

Suppose (2.10) is not true. Then $y_{n_0} - \alpha_{n_0}$ has a negative absolute minimum at $\tau \in (0,1)$. Now since $y_{n_0}(0) - \alpha_{n_0}(0) = 0 = y_{n_0}(1) - \alpha_{n_0}(1)$ there exists $\tau_0,\, \tau_1 \in [0,1]$ with $\tau \in (\tau_1, \tau_2)$ and

$$y_{n_0}(\tau_0) - \alpha_{n_0}(\tau_0) = y_{n_0}(\tau_1) - \alpha_{n_0}(\tau_1) = 0$$

and

$$y_{n_0}(t) - \alpha_{n_0}(t) < 0,\ \ t \in (\tau_0, \tau_1).$$

We now claim

$$(2.11)\qquad (y_{n_0} - \alpha_{n_0})''(t) < 0 \ \text{ for a.e. } \ t \in (\tau_0, \tau_1).$$

If (2.11) is true then

$$y_{n_0}(t) - \alpha_{n_0}(t) = -\int_{\tau_0}^{\tau_1} G(t,s)\, [y''_{n_0}(s) - \alpha''_{n_0}(s)]\, ds \quad \text{for } \ t \in (\tau_0, \tau_1)$$

with

$$G(t,s)=\begin{cases}\frac{(s-\tau_0)\,(\tau_1-t)}{\tau_1-\tau_0}, & \tau_0\le s\le t\\ \frac{(t-\tau_0)\,(\tau_1-s)}{\tau_1-\tau_0}, & t\le s\le \tau_1\end{cases}$$

so we have

$$y_{n_0}(t)-\alpha_{n_0}(t)\;>\;0\quad\text{for}\quad t\in(\tau_0,\tau_1),$$

a contradiction. As a result if we show that (2.11) is true then (2.10) will follow. To see (2.11) we will show

$$(y_{n_0}-\alpha_{n_0})''(t)\;<\;0\quad\text{for}\quad t\in(\tau_0,\tau_1)\quad\text{provided}\quad t\neq t_{n_0}\quad\text{or}\quad t\neq s_{n_0}.$$

Fix $t\in(\tau_0,\tau_1)$ and assume $t\neq t_{n_0}$ or $t\neq s_{n_0}$. Then

$$\begin{aligned}(y_{n_0}-\alpha_{n_0})''(t)\;&=\;-\left[q(t)\left\{\,g_{n_0}(t,\alpha_{n_0}(t),(y'_{n_0}(t))^\star)\right.\right.\\ &\qquad\left.\left.+r(\alpha_{n_0}(t)-y_{n_0}(t))\,\right\}+\alpha''_{n_0}(t)\right]\\ &=\;\begin{cases}-\left[q(t)\left\{g_{n_0}(t,\alpha(t),(y'_{n_0}(t))^\star)+r(\alpha(t)-y_{n_0}(t))\right\}\right.\\ \qquad\left.+\alpha''(t)\right] & \text{if}\quad t\in(t_{n_0},s_{n_0})\\ -\left[q(t)\left\{g_{n_0}(t,\rho_{n_0},(y'_{n_0}(t))^\star)+r(\rho_{n_0}-y_{n_0}(t))\right\}\right] & \\ & \text{if}\quad t\in(0,t_{n_0})\cup(s_{n_0},1).\end{cases}\end{aligned}$$

Case (A). $t\in\left[\frac{1}{2^{n_0+1}},1\right)$.

Then since $g_{n_0}(t,x,z)=f(t,x,z)$ for $(x,z)\in(0,\infty)\times\mathbf{R}$ (note $t\in e_{n_0}$) we have

$$\begin{aligned}(y_{n_0}-\alpha_{n_0})''(t)\;&=\;\begin{cases}-\left[q(t)\left\{f(t,\alpha(t),(y'_{n_0}(t))^\star)+r(\alpha(t)-y_{n_0}(t))\right\}\right.\\ \qquad\left.+\alpha''(t)\right] & \text{if}\quad t\in(t_{n_0},s_{n_0})\\ -\left[q(t)\left\{f(t,\rho_{n_0},(y'_{n_0}(t))^\star)+r(\rho_{n_0}-y_{n_0}(t))\right\}\right] & \\ & \text{if}\quad t\in(0,t_{n_0})\cup(s_{n_0},1)\end{cases}\\ &<\;0,\end{aligned}$$

from (2.4) and (2.5).

Case (B). $t\in\left(0\,,\,\frac{1}{2^{n_0+1}}\right)$.

Then since

$$g_{n_0}(t,x,z)=\max\left\{f\left(\frac{1}{2^{n_0+1}},x,z\right)\,,\,f(t,x,z)\right\}$$

we have

$$g_{n_0}(t,x,z)\ge f(t,x)\quad\text{and}\quad g_{n_0}(t,x,z)\ge f\left(\frac{1}{2^{n_0+1}},x,z\right)$$

for $(x,z) \in (0,\infty) \times \mathbf{R}$. Thus we have

$$
\begin{aligned}
(y_{n_0}-\alpha_{n_0})''(t) &\leq \begin{cases} -\left[q(t)\left\{f(t,\alpha(t),(y'_{n_0}(t))^\star)+r(\alpha(t)-y_{n_0}(t))\right\}\right. \\ \quad \left.+\alpha''(t)\right] \qquad\qquad \text{if} \quad t \in (t_{n_0}, s_{n_0}) \\ -\left[q(t)\left\{f\left(\frac{1}{2^{n_0+1}},\rho_{n_0},(y'_{n_0}(t))^\star\right)\right.\right. \\ \quad \left.\left.+r(\rho_{n_0}-y_{n_0}(t))\right\}\right] \qquad \text{if} \quad t\in(0,t_{n_0})\cup(s_{n_0},1) \end{cases} \\
&< 0,
\end{aligned}
$$

from (2.4) and (2.5).

Consequently (2.11) (and so (2.10)) holds and now since $\alpha(t) \leq \alpha_{n_0}(t)$ for $t \in [0,1]$ we have

$$\alpha(t) \leq \alpha_{n_0}(t) \leq y_{n_0}(t) \quad \text{for} \quad t\in[0,1]. \tag{2.12}$$

Next we show

$$y_{n_0}(t) \leq \beta(t) \quad \text{for} \quad t \in [0,1]. \tag{2.13}$$

If (2.13) is not true then $y_{n_0} - \beta$ would have a positive absolute maximum at say $\tau_0 \in (0,1)$, in which case $(y_{n_0}-\beta)'(\tau_0) = 0$ and $(y_{n_0}-\beta)''(\tau_0) \leq 0$. There are two cases to consider, namely $\tau_0 \in \left[\frac{1}{2^{n_0+1}},1\right)$ and $\tau_0 \in \left(0,\frac{1}{2^{n_0+1}}\right)$.

Case (A). $\tau_0 \in \left[\frac{1}{2^{n_0+1}},1\right)$.

Then $y_{n_0}(\tau_0) > \beta(\tau_0)$, $y'_{n_0}(\tau_0) = \beta'(\tau_0)$ together with $g_{n_0}(\tau_0,x,z) = f(\tau_0,x,z)$ for $(x,z)\in(0,\infty)\times\mathbf{R}$ and $M_{n_0} \geq \sup_{[0,1]} |\beta'(t)|$ gives

$$
\begin{aligned}
(y_{n_0}-\beta)''(\tau_0) &= -q(\tau_0)\left[g_{n_0}(\tau_0,\beta(\tau_0),(y'_{n_0}(\tau_0))^\star)+r(\beta(\tau_0)-y_{n_0}(\tau_0))\right] \\
&- \beta''(\tau_0) \\
&= -q(\tau_0)\left[f(\tau_0,\beta(\tau_0),\beta'(\tau_0))+r(\beta(\tau_0)-y_{n_0}(\tau_0))\right] \\
&- \beta''(\tau_0) \\
&> 0
\end{aligned}
$$

from (2.6), a contradiction.

Case (B). $\tau_0 \in \left(0,\frac{1}{2^{n_0+1}}\right)$.

Now

$$g_{n_0}(\tau_0,x,z) = \max\left\{f\left(\frac{1}{2^{n_0+1}},x,z\right)\ ,\ f(\tau_0,x,z)\right\}$$

for $(x,z)\in(0,\infty)\times\mathbf{R}$ gives

$$
\begin{aligned}
(y_{n_0}-\beta)''(\tau_0) &= -q(\tau_0)\left[\max\{f(\frac{1}{2^{n_0+1}},\beta(\tau_0),\beta'(\tau_0)),f(\tau_0,\beta(\tau_0),\beta'(\tau_0))\}\right. \\
&+ \left. r(\beta(\tau_0)-y_{n_0}(\tau_0))\right]-\beta''(\tau_0) \\
&> 0
\end{aligned}
$$

from (2.6), a contradiction.

Thus (2.13) holds. Next we show

$$|y'_{n_0}|_\infty = \sup_{[0,1]} |y'_{n_0}(t)| \le M_{n_0}. \tag{2.14}$$

With $\epsilon = \min_{[0,1]} \alpha_{n_0}(t)$, then (2.7) guarantees the existence of ψ_ϵ (as described in (2.7)) with

$$|f(t,y,z)| \le \psi_\epsilon(|z|) \quad \text{for} \quad (t,y,z) \in (0,1) \times [\epsilon, a_0] \times \mathbf{R}$$

where $a_0 = \sup_{[0,1]} \beta(t)$. Let $M_{n_0} \ge \sup_{[0,1]} |\beta'(t)|$ be chosen so that

$$\int_0^1 q(s)\,ds < \int_0^{M_{n_0}} \frac{du}{\psi_\epsilon(u)} \tag{2.15}$$

holds. Suppose (2.14) is false. Without loss of generality assume $y'_{n_0}(t) \not\le M_{n_0}$ for some $t \in [0,1]$. Then since $y_{n_0}(0) = y_{n_0}(1) = \rho_{n_0}$ there exists $\tau_1 \in (0,1)$ with $y'_{n_0}(\tau_1) = 0$, and so there exists τ_2, $\tau_3 \in (0,1)$ with $y'_{n_0}(\tau_3) = 0$, $y'_{n_0}(\tau_2) = M_{n_0}$ and $0 \le y'_{n_0}(s) \le M_{n_0}$ for s between τ_3 and τ_2. Without loss of generality assume $\tau_3 < \tau_2$. Now since $\alpha_{n_0}(t) \le y_{n_0}(t) \le \beta(t)$ for $t \in [0,1]$ and

$$g_{n_0}(t,x,z) = \max\left\{ f\left(\frac{1}{2^{n_0+1}}, x, z\right),\ f(t,x,z)\right\}$$

for $(t,x,z) \in (0,1) \times (0,\infty) \times \mathbf{R}$, we have for $s \in (\tau_3, \tau_2)$ that

$$y''_{n_0}(s) \le q(s)\,\psi_\epsilon(y'_{n_0}(s)),$$

and so

$$\int_0^{M_{n_0}} \frac{du}{\psi_\epsilon(u)} = \int_{\tau_3}^{\tau_2} \frac{y''_{n_0}(s)}{\psi_\epsilon(y'_{n_0}(s))}\,ds \le \int_0^1 q(s)\,ds.$$

This contradicts (2.15). The other cases are treated similarly. As a result $\alpha(t) \le y_{n_0}(t) \le \beta(t)$ for $t \in [0,1]$ and $|y'_{n_0}|_\infty \le M_{n_0}$. Thus y_{n_0} satisfies $y''_{n_0} + q\,g_{n_0}(t, y_{n_0}, y'_{n_0}) = 0$ on $(0,1)$.

Next we consider the boundary value problem

$$\begin{cases} y'' + q(t)\,g^\star_{n_0+1}(t,y,y') = 0,\ 0 < t < 1 \\ y(0) = y(1) = \rho_{n_0+1} \end{cases} \tag{2.16}$$

where

$$g^\star_{n_0+1}(t,y,z) = \begin{cases} g_{n_0+1}(t, \alpha_{n_0+1}(t), z^\star) + r(\alpha_{n_0+1}(t) - y), & y \le \alpha_{n_0+1}(t) \\ g_{n_0+1}(t,y,z^\star), \quad \alpha_{n_0+1}(t) \le y \le y_{n_0}(t) \\ g_{n_0+1}(t, y_{n_0}(t), z^\star) + r(y_{n_0}(t) - y),\ y \ge y_{n_0}(t) \end{cases}$$

with

$$z^\star = \begin{cases} M_{n_0+1}, & z > M_{n_0+1} \\ z, & -M_{n_0+1} \le z \le M_{n_0+1} \\ -M_{n_0+1}, & z < -M_{n_0+1}; \end{cases}$$

here $M_{n_0+1} \ge M_{n_0}$ is a predetermined constant (see (2.22)). Now Schauder's fixed point theorem guarantees that there exists a solution $y_{n_0+1} \in C^1[0,1]$ to (2.16). We first show

$$y_{n_0+1}(t) \ge \alpha_{n_0+1}(t), \ \ t \in [0,1]. \tag{2.17}$$

Suppose (2.17) is not true. Then there exists $\tau_0,\, \tau_1 \in [0,1]$ with

$$y_{n_0+1}(\tau_0) - \alpha_{n_0+1}(\tau_0) = y_{n_0+1}(\tau_1) - \alpha_{n_0+1}(\tau_1) = 0$$

and

$$y_{n_0+1}(t) - \alpha_{n_0+1}(t) < 0, \ \ t \in (\tau_0, \tau_1).$$

If we show

$$(y_{n_0+1} - \alpha_{n_0+1})''(t) \, < \, 0 \ \text{ for a.e. } \ t \in (\tau_0, \tau_1), \tag{2.18}$$

then as before (2.17) is true. Fix $t \in (\tau_0, \tau_1)$ and assume $t \ne t_{n_0+1}$ or $t \ne s_{n_0+1}$. Then

$$(y_{n_0+1} - \alpha_{n_0+1})''(t) \;=\; \begin{cases} -\left[q(t)\left\{g_{n_0+1}(t, \alpha(t), (y'_{n_0+1}(t))^\star)\right.\right. \\ \quad \left.\left. + r(\alpha(t) - y_{n_0+1}(t))\right\} + \alpha''(t)\right] \\ \qquad \text{if} \ \ t \in (t_{n_0+1}, s_{n_0+1}) \\ -\left[\, q(t)\left\{g_{n_0+1}(t, \rho_{n_0+1}, (y'_{n_0+1}(t))^\star)\right.\right. \\ \quad \left.\left. + r(\rho_{n_0+1} - y_{n_0+1}(t))\right\}\,\right] \\ \qquad \text{if} \ \ t \in (0, t_{n_0+1}) \cup (s_{n_0+1}, 1). \end{cases}$$

Case (A). $t \in \left[\frac{1}{2^{n_0+2}}, 1\right)$.

Then since $g_{n_0+1}(t,x,z) = f(t,x,z)$ for $(x,z) \in (0,\infty) \times \mathbf{R}$ (note $t \in e_{n_0+1}$) we have

$$\begin{aligned} (y_{n_0+1} - \alpha_{n_0+1})''(t) \;&=\; \begin{cases} -\left[q(t)\left\{f(t, \alpha(t), (y'_{n_0+1}(t))^\star)\right.\right. \\ \quad \left.\left. + r(\alpha(t) - y_{n_0+1}(t))\right\} + \alpha''(t)\right] \\ \qquad \text{if} \ \ t \in (t_{n_0+1}, s_{n_0+1}) \\ -\left[\, q(t)\left\{f(t, \rho_{n_0+1}, (y'_{n_0+1}(t))^\star)\right.\right. \\ \quad \left.\left. + r(\rho_{n_0+1} - y_{n_0+1}(t))\right\}\,\right] \\ \qquad \text{if} \ \ t \in (0, t_{n_0+1}) \cup (s_{n_0+1}, 1) \end{cases} \\ &<\; 0, \end{aligned}$$

from (2.4) and (2.5).

Case (B). $t \in \left(0\,,\,\frac{1}{2^{n_0+2}}\right)$.

Then since $g_{n_0+1}(t,x,z)$ equals

$$\min\{\max\{f\left(\frac{1}{2^{n_0+1}},x,z\right)\,,\,f(t,x,z)\}\,,\,\max\{f\left(\frac{1}{2^{n_0+2}},x,z\right)\,,\,f(t,x,z)\}\}$$

we have

$$g_{n_0+1}(t,x,z) \geq f(t,x,z)$$

and

$$g_{n_0+1}(t,x,z) \geq \min\left\{f\left(\frac{1}{2^{n_0+1}},x,z\right)\,,\,f\left(\frac{1}{2^{n_0+2}},x,z\right)\right\}$$

for $(x,z) \in (0,\infty)\times\mathbf{R}$. Thus we have

$$\begin{aligned}(y_{n_0+1}-\alpha_{n_0+1})''(t) \quad &\leq \quad \begin{cases} -\left[q(t)\left\{f(t,\alpha(t),(y'_{n_0+1}(t))^{\star})\right.\right. \\ \left.\left.+r(\alpha(t)-y_{n_0+1}(t))\right\}+\alpha''(t)\right] \\ \qquad \text{if} \quad t\in(t_{n_0+1},s_{n_0+1}) \\ -\left[q(t)\left\{\min\{f\left(\frac{1}{2^{n_0+1}},\rho_{n_0+1},(y'_{n_0+1}(t))^{\star}\right)\,,\right.\right. \\ f\left(\frac{1}{2^{n_0+2}},\rho_{n_0+1},(y'_{n_0+1}(t))^{\star}\right)\} \\ \left.\left.+r(\rho_{n_0+1}-y_{n_0+1}(t))\right\}\right] \\ \qquad \text{if} \quad t\in(0,t_{n_0+1})\cup(s_{n_0+1},1)\end{cases} \\ &< \quad 0,\end{aligned}$$

from (2.4) and (2.5) since $f\left(\frac{1}{2^{n_0+1}},\rho_{n_0+1},(y'_{n_0+1}(t))^{\star}\right)\geq 0$ because

$$f\left(t,\rho_{n_0+1},(y'_{n_0+1}(t))^{\star}\right)\geq 0 \quad \text{for} \quad t\in\left[\frac{1}{2^{n_0+2}}\,,\,1\right]$$

and

$$\frac{1}{2^{n_0+1}} \in \left[\frac{1}{2^{n_0+2}}\,,\,1\right].$$

Consequently (2.17) is true so

$$\alpha(t)\leq\alpha_{n_0+1}(t)\leq y_{n_0+1}(t) \quad \text{for} \quad t\in[0,1]. \tag{2.19}$$

Next we show

$$y_{n_0+1}(t)\leq y_{n_0}(t) \quad \text{for} \quad t\in[0,1]. \tag{2.20}$$

If (2.20) is not true then $y_{n_0+1}-y_{n_0}$ would have a positive absolute maximum at say $\tau_0\in(0,1)$, in which case

$$(y_{n_0+1}-y_{n_0})'(\tau_0)=0 \quad \text{and} \quad (y_{n_0+1}-y_{n_0})''(\tau_0)\leq 0.$$

Then $y_{n_0+1}(\tau_0) > y_{n_0}(\tau_0)$ together with $g_{n_0}(\tau_0,x,z) \geq g_{n_0+1}(\tau_0,x,z)$ for $(x,z) \in (0,\infty)\times \mathbf{R}$ gives (note $(y'_{n_0+1}(\tau_0))^\star = (y'_{n_0}(\tau_0))^\star = y'_{n_0}(\tau_0)$ since $M_{n_0+1} \geq M_{n_0}$ and $|y'_{n_0}|_\infty \leq M_{n_0}$),

$$\begin{aligned}(y_{n_0+1}-y_{n_0})''(\tau_0) &= -\,q(\tau_0)\,[g_{n_0+1}(\tau_0,y_{n_0}(\tau_0),(y'_{n_0+1}(\tau_0))^\star)\\ &+\ r(y_{n_0}(\tau_0)-y_{n_0+1}(\tau_0))] - y''_{n_0}(\tau_0)\\ &\geq -\,q(\tau_0)\,[g_{n_0}(\tau_0,y_{n_0}(\tau_0),y'_{n_0}(\tau_0))\\ &+\ r(y_{n_0}(\tau_0)-y_{n_0+1}(\tau_0))] - y''_{n_0}(\tau_0)\\ &= -q\,(\tau_0)\ [r(y_{n_0}(\tau_0)-y_{n_0+1}(\tau_0))]\\ &> 0,\end{aligned}$$

a contradiction. Thus (2.20) holds. Next we show

$$|y'_{n_0+1}|_\infty \leq M_{n_0+1}. \tag{2.21}$$

With $\epsilon = \min_{[0,1]} \alpha_{n_0+1}(t)$, then (2.7) guarantees the existence of ψ_ϵ (as described in (2.7)) with

$$|f(t,y,z)| \leq \psi_\epsilon(|z|) \quad \text{for} \quad (t,y,z)\in(0,1)\times[\epsilon,a_0]\times\mathbf{R}$$

where $a_0 = \sup_{[0,1]}\beta(t)$. Let $M_{n_0+1} \geq M_{n_0}$ be chosen so that

$$\int_0^1 q(s)\,ds < \int_0^{M_{n_0+1}} \frac{du}{\psi_\epsilon(u)}. \tag{2.22}$$

Essentially the same argument as before guarantees that (2.21) holds. Thus $y''_{n_0+1} + q\,g_{n_0+1}(t,y_{n_0+1},y'_{n_0+1}) = 0$ on $(0,1)$.

Now proceed inductively to construct $y_{n_0+2}, y_{n_0+3}, ...$ as follows. Suppose we have y_k for some $k\in\{n_0+1,n_0+2,...\}$ with $\alpha(t)\leq\alpha_k(t)\leq y_k(t)\leq y_{k-1}(t)\,(\leq\beta(t))$ for $t\in[0,1]$. Then consider the boundary value problem

$$\begin{cases} y'' + q(t)\,g^\star_{k+1}(t,y,y') = 0,\ 0<t<1\\ y(0)=y(1)=\rho_{k+1}\end{cases} \tag{2.23}$$

where

$$g^\star_{k+1}(t,y,z) = \begin{cases} g_{k+1}(t,\alpha_{k+1}(t),z^\star) + r(\alpha_{k+1}(t)-y), & y\leq\alpha_{k+1}(t)\\ g_{k+1}(t,y,z^\star), \quad \alpha_{k+1}(t)\leq y\leq y_k(t)\\ g_{k+1}(t,y_k(t),z^\star)+r(y_k(t)-y),\ y\geq y_k(t)\end{cases}$$

with

$$z^\star = \begin{cases} M_{k+1}, & z > M_{k+1}\\ z, & -M_{k+1}\leq z\leq M_{k+1}\\ -M_{k+1}, & z<-M_{k+1};\end{cases}$$

here $M_{k+1} \geq M_k$ is a predetermined constant. Now Schauder's fixed point theorem guarantees that (2.23) has a solution $y_{k+1} \in C^1[0,1]$, and essentially the same reasoning as above yields

$$\alpha(t) \leq \alpha_{k+1}(t) \leq y_{k+1}(t) \leq y_k(t) \quad \text{for} \quad t \in [0,1], \ \ |y'_{k+1}|_\infty \leq M_{k+1},$$

so $y''_{k+1} + q\, g_{k+1}(t, y_{k+1}, y'_{k+1}) = 0$ on $(0,1)$.

Now lets look at the interval $[\frac{1}{2^{n_0+1}}, 1 - \frac{1}{2^{n_0+1}}]$. We claim

$$\left\{ \begin{array}{l} \{y_n^{(j)}\}_{n=n_0+1}^\infty, j = 0,1, \ \text{ is a bounded, equicontinuous} \\ \text{family on } \left[\frac{1}{2^{n_0+1}}, 1 - \frac{1}{2^{n_0+1}}\right]. \end{array} \right. \tag{2.24}$$

Firstly note

$$|y_n|_\infty \leq |y_{n_0}|_\infty \leq \sup_{[0,1]} \beta(t) = a_0 \quad \text{for} \quad t \in [0,1] \quad \text{and} \quad n \geq n_0 + 1. \tag{2.25}$$

Let

$$\epsilon = \min_{t \in \left[\frac{1}{2^{n_0+1}}, 1 - \frac{1}{2^{n_0+1}}\right]} \alpha(t).$$

Now (2.7) guarantees the existence of ψ_ϵ (as described in (2.7)) with

$$|f(t,y,z)| \leq \psi_\epsilon(|z|) \quad \text{for} \quad (t,y,z) \in (0,1) \times [\epsilon, a_0] \times \mathbf{R}.$$

This implies

$$|g_n(t, y_n(t), y'_n(t))| \leq \psi_\epsilon(|y'_n(t)|) \quad \text{for} \quad t \in [a,b] \equiv \left[\frac{1}{2^{n_0+1}}, 1 - \frac{1}{2^{n_0+1}}\right] \subseteq e_{n_0}$$

and $n \geq n_0 + 1$. As a result

$$|y''_n(t)| \leq q(t)\, \psi_\epsilon(|y'_n(t)|) \quad \text{for} \quad t \in [a,b] \quad \text{and} \quad n \geq n_0 + 1. \tag{2.26}$$

The mean value theorem implies that there exists $\tau_{1,n} \in (a,b)$ with

$$|y'(\tau_{1,n})| = \frac{|y_(b) - y_n(a)|}{b-a} \leq \frac{2\, a_0}{b-a} = d_{n_0} \quad \text{for} \quad n \geq n_0.$$

Fix $n \geq n_0 + 1$ and let $t \in [a,b]$. Without loss of generality assume $y'_n(t) > d_{n_0}$. Then there exists $\tau_1 \in (a,b)$ with $y'_n(\tau_1) = d_{n_0}$ and $y'_n(s) > d_{n_0}$ for s between τ_1 and t. Without loss of generality assume $\tau_1 < s$. From (2.26) we have

$$\frac{y''_n(s)}{\psi_\epsilon(y'_n(s))} \leq q(s) \quad \text{for} \quad s \in (\tau_1, t),$$

so integration from τ_1 to t yields

$$\int_{d_{n_0}}^{y_n'(t)} \frac{du}{\psi_\epsilon(u)} \le \int_0^1 q(s)\,ds.$$

Let $I_{n_0}(z) = \int_{d_{n_0}}^{z} \frac{du}{\psi_\epsilon(u)}$, so

$$|y_n'(t)| \le I_{n_0}^{-1}\left(\int_0^1 q(s)\,ds\right) \equiv R_{n_0}.$$

A similar bound is obtained for the other cases, so

$$|y_n'(s)| \le R_{n_0} \text{ for } s \in [a,b] = \left[\frac{1}{2^{n_0+1}}, 1 - \frac{1}{2^{n_0+1}}\right] \tag{2.27}$$

and $n \ge n_0 + 1$. Now (2.25), (2.26) and (2.27) guarantee that (2.24) holds. The Arzela–Ascoli theorem guarantees the existence of a subsequence N_{n_0} of integers and a function $z_{n_0} \in C^1\left[\frac{1}{2^{n_0+1}}, 1 - \frac{1}{2^{n_0+1}}\right]$ with $y_n^{(j)}$, $j = 0, 1$, converging uniformly to $z_{n_0}^{(j)}$ on $\left[\frac{1}{2^{n_0+1}}, 1 - \frac{1}{2^{n_0+1}}\right]$ as $n \to \infty$ through N_{n_0}. Similarly

$$\left\{ \begin{array}{l} \{y_n^{(j)}\}_{n=n_0+2}^{\infty}, j = 0, 1, \quad \text{is a bounded, equicontinuous} \\ \text{family on } \left[\frac{1}{2^{n_0+2}}, 1 - \frac{1}{2^{n_0+2}}\right], \end{array} \right. \tag{2.28}$$

so there is a subsequence N_{n_0+1} of N_{n_0} and a function

$$z_{n_0+1} \in C^1\left[\frac{1}{2^{n_0+2}}, 1 - \frac{1}{2^{n_0+2}}\right]$$

with $y_n^{(j)}$, $j = 0, 1$, converging uniformly to $z_{n_0+1}^{(j)}$ on $\left[\frac{1}{2^{n_0+2}}, 1 - \frac{1}{2^{n_0+2}}\right]$ as $n \to \infty$ through N_{n_0+1}. Note $z_{n_0+1} = z_{n_0}$ on $\left[\frac{1}{2^{n_0+1}}, 1 - \frac{1}{2^{n_0+1}}\right]$ since $N_{n_0+1} \subseteq N_{n_0}$. Proceed inductively to obtain subsequences of integers

$$N_{n_0} \supseteq N_{n_0+1} \supseteq \dots \supseteq N_k \supseteq \dots$$

and functions

$$z_k \in C^1\left[\frac{1}{2^{k+1}}, 1 - \frac{1}{2^{k+1}}\right]$$

with

$$y_n^{(j)},\ j = 0,\ 1, \text{ converging uniformly to } z_k^{(j)} \text{ on } \left[\frac{1}{2^{k+1}}, 1 - \frac{1}{2^{k+1}}\right]$$

as $n \to \infty$ through N_k, and

$$z_k = z_{k-1} \quad \text{on} \quad \left[\frac{1}{2^k}, 1 - \frac{1}{2^k}\right].$$

Define a function $y : [0,1] \to [0,\infty)$ by $y(x) = z_k(x)$ on $\left[\frac{1}{2^{k+1}}, 1 - \frac{1}{2^{k+1}}\right]$ and $y(0) = y(1) = 0$. Notice y is well defined and $\alpha(t) \le y(t) \le y_{n_0}(t) \le \beta(t)$ for $t \in (0,1)$. Next fix $t \in (0,1)$ (without loss of generality assume $t \neq \frac{1}{2}$) and let $m \in \{n_0, n_0+1, ...\}$ be such that $\frac{1}{2^{m+1}} < t < 1 - \frac{1}{2^{m+1}}$. Let $N_m^\star = \{n \in N_m : \ n \ge m\}$. Now y_n, $n \in N_m^\star$, satisfies the integral equation

$$\begin{aligned} y_n(x) &= y_n\left(\frac{1}{2}\right) + y_n'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) + \int_{\frac{1}{2}}^{x} (s-x) q(s) g_n(s, y_n(s), y_n'(s))\, ds \\ &= y_n\left(\frac{1}{2}\right) + y_n'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) + \int_{\frac{1}{2}}^{x} (s-x) q(s) f(s, y_n(s), y_n'(s))\, ds \end{aligned}$$

for $x \in \left[\frac{1}{2^{m+1}}, 1 - \frac{1}{2^{m+1}}\right]$. Let $n \to \infty$ through $N_m^\star$ to obtain

$$z_m(x) = z_m\left(\frac{1}{2}\right) + z_m'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) + \int_{\frac{1}{2}}^{x} (s-x) q(s) f(s, z_m(s), z_m'(s))\, ds,$$

so in particular

$$y(t) = y\left(\frac{1}{2}\right) + y'\left(\frac{1}{2}\right)\left(t - \frac{1}{2}\right) + \int_{\frac{1}{2}}^{t} (s-t)\, q(s)\, f(s, y(s), y'(s))\, ds.$$

We can do this argument for each $t \in (0,1)$, so $y''(t) + q(t)\, f(t, y(t), y'(t)) = 0$ for $t \in (0,1)$. It remains to show y is continuous at 0 and 1.

Let $\epsilon > 0$ be given. Now since $\lim_{n\to\infty} y_n(0) = 0$ there exists $n_1 \in \{n_0, n_0+1, ...\}$ with $y_{n_1}(0) < \frac{\epsilon}{2}$. Since $y_{n_1} \in C[0,1]$ there exists $\delta_{n_1} > 0$ with

$$y_{n_1}(t) < \frac{\epsilon}{2} \quad \text{for} \quad t \in [0, \delta_{n_1}].$$

Now for $n \ge n_1$ we have, since $\{y_n(t)\}$ is nonincreasing for each $t \in [0,1]$,

$$\alpha(t) \le y_n(t) \le y_{n_1}(t) < \frac{\epsilon}{2} \quad \text{for} \quad t \in [0, \delta_{n_1}].$$

Consequently

$$\alpha(t) \le y(t) \le \frac{\epsilon}{2} < \epsilon \quad \text{for} \quad t \in (0, \delta_{n_1}]$$

and so y is continuous at 0. Similarly y is continuous at 1. As a result $y \in C[0,1]$. $\square$

Suppose (2.2)–(2.5) hold and in addition assume the following conditions are satisfied:

$$(2.29)\qquad \begin{cases} q(t)\,f(t,y,\alpha'(t))+\alpha''(t)>0 \ \text{ for} \\ (t,y)\in(0,1)\times\{\,y\in(0,\infty):\,y<\alpha(t)\,\} \end{cases}$$

and

$$(2.30)\qquad \begin{cases} \text{there exists a function } \ \beta\in C[0,1]\cap C^2(0,1) \\ \text{with } \ \beta(t)\geq\rho_{n_0} \ \text{ for } \ t\in[0,1] \ \text{ and with} \\ q(t)\,f(t,\beta(t),\beta'(t))+\beta''(t)\leq 0 \ \text{ for } t\in(0,1) \ \text{ and} \\ q(t)\,f\left(\frac{1}{2^{n_0+1}},\beta(t),\beta'(t)\right)+\beta''(t)\leq 0 \ \text{ for } t\in\left(0,\frac{1}{2^{n_0+1}}\right). \end{cases}$$

Also if (2.7) and (2.8) hold, then the result in Theorem 2.1 is again true. This follows immediately from Theorem 2.1 once we show (2.6) holds i.e. once we show $\beta(t)\geq\alpha(t)$ for $t\in[0,1]$. Suppose it is false. Then $\alpha-\beta$ would have a positive absolute maximum at say $\tau_0\in(0,1)$, so $(\alpha-\beta)'(\tau_0)=0$ and $(\alpha-\beta)''(\tau_0)\leq 0$. Now $\alpha(\tau_0)>\beta(\tau_0)$ and (2.29) implies

$$q(\tau_0)\,f(\tau_0,\beta(\tau_0),\beta'(\tau_0))+\alpha''(\tau_0)=q(\tau_0)\,f(\tau_0,\beta(\tau_0),\alpha'(\tau_0))+\alpha''(\tau_0)\,>\,0,$$

and this together with (2.30) yields

$$(\alpha-\beta)''(\tau_0)=\alpha''(\tau_0)-\beta''(\tau_0)\geq\alpha''(\tau_0)+q(\tau_0)\,f(\tau_0,\beta(\tau_0),\beta'(\tau_0))\,>\,0,$$

a contradiction. Thus we have

Corollary 2.2. *Let $n_0\in\{1,2,...\}$ be fixed and suppose (2.2) − (2.5), (2.7), (2.8), (2.29) and (2.30) hold. Then (2.1) has a solution $y\in C[0,1]\cap C^2(0,1)$ with $\alpha(t)\leq y(t)\leq\beta(t)$ for $t\in[0,1]$.*

Remark 2.1. (i). If in (2.4) we replace $\frac{1}{2^{n+1}}\leq t\leq 1$ with $0\leq t\leq 1-\frac{1}{2^{n+1}}$ then one would replace (2.6) with

$$(2.31)\qquad \begin{cases} \exists\,\beta\in C^1[0,1]\cap C^2(0,1) \ \text{ with } \ \beta(t)\geq\alpha(t), \ \beta(t)\geq\rho_{n_0} \\ \text{for } \ t\in[0,1] \ \text{ with } \ q(t)\,f(t,\beta(t),\beta'(t))+\beta''(t)\leq 0 \\ \text{for } \ t\in(0,1) \ \text{ and } \ q(t)\,f\left(1-\frac{1}{2^{n_0+1}},\beta(t),\beta'(t)\right)+\beta''(t)\leq 0 \\ \text{for } \ t\in\left(1-\frac{1}{2^{n_0+1}},1\right). \end{cases}$$

(ii). If in (2.4) we replace $\frac{1}{2^{n+1}}\leq t\leq 1$ with $\frac{1}{2^{n+1}}\leq t\leq 1-\frac{1}{2^{n+1}}$ then one would replace (2.6) with

$$(2.32)\qquad \begin{cases} \exists\,\beta\in C^1[0,1]\cap C^2(0,1) \ \text{ with } \ \beta(t)\geq\alpha(t), \ \beta(t)\geq\rho_{n_0} \\ \text{for } \ t\in[0,1] \ \text{ with } \ q(t)\,f(t,\beta(t),\beta'(t))+\beta''(t)\leq 0 \\ \text{for } \ t\in(0,1) \ \text{ and } \ q(t)\,f\left(\frac{1}{2^{n_0+1}},\beta(t),\beta'(t)\right)+\beta''(t)\leq 0 \\ \text{for } \ t\in\left(0,\frac{1}{2^{n_0+1}}\right), \ \ q(t)\,f\left(1-\frac{1}{2^{n_0+1}},\beta(t),\beta'(t)\right)+\beta''(t)\leq 0 \\ \text{for } \ t\in\left(1-\frac{1}{2^{n_0+1}},1\right). \end{cases}$$

This is clear once one changes the definition of e_n and θ_n. For example in case (ii), take

$$e_n = \left[\frac{1}{2^{n+1}}, 1 - \frac{1}{2^{n+1}}\right] \quad \text{and} \quad \theta_n(t) = \max\left\{\frac{1}{2^{n+1}}\,,\ \min\left\{t, 1 - \frac{1}{2^{n+1}}\right\}\right\}.$$

Finally we discuss condition (2.5) and (2.29). Suppose the following condition is satisfied:

$$(2.33) \qquad \begin{cases} \text{let } n \in \{n_0, n_0+1, ...\} \text{ and associated with each } n \text{ we} \\ \text{have a constant } \rho_n \text{ such that } \{\rho_n\} \text{ is a decreasing} \\ \text{sequence with } \lim_{n\to\infty} \rho_n = 0 \text{ and there exists a constant} \\ k_0 > 0 \text{ such that for } \frac{1}{2^{n+1}} \le t \le 1,\ 0 < y \le \rho_n \text{ and } z \in \mathbf{R} \\ \text{we have } q(t)\, f(t,y,z) \ge k_0. \end{cases}$$

We will show if (2.33) holds then (2.5) (and of course (2.4)) and (2.29) are satisfied (we also note that $\frac{1}{2^{n+1}} \le t \le 1$ in (2.33) could be replaced by $0 \le t \le 1 - \frac{1}{2^{n+1}}$ (respectively $\frac{1}{2^{n+1}} \le t \le 1 - \frac{1}{2^{n+1}}$) and (2.5), (2.29) hold with $\frac{1}{2^{n+1}} \le t \le 1$ replaced by $0 \le t \le 1 - \frac{1}{2^{n+1}}$ (respectively $\frac{1}{2^{n+1}} \le t \le 1 - \frac{1}{2^{n+1}}$)).

To show (2.5) and (2.29) recall the following Lemma from [5].

Lemma 2.3. *Let e_n be as described in Theorem 2.1 (or Remark 2.1) and let $0 < \epsilon_n < 1$ with $\epsilon_n \downarrow 0$. Then there exists $\lambda \in C^2[0,1]$ with $\sup_{[0,1]} |\lambda''(t)| > 0$ and $\lambda(0) = \lambda(1) = 0$ with*

$$0 < \lambda(t) \le \epsilon_n \quad \textit{for} \quad t \in e_n \backslash e_{n-1},\ \ n \ge 1.$$

Let $\epsilon_n = \rho_n$ (and $n \ge n_0$) and let λ be as in Lemma 2.3. From (2.33) there exists $k_0 > 0$ with

$$(2.34) \qquad \begin{cases} q(t)\, f(t,y,z) \ge k_0 \quad \text{for} \\ (t,y,z) \in (0,1) \times \{y \in (0,\infty) :\ y \le \lambda(t)\} \times \mathbf{R}, \end{cases}$$

since if $t \in e_n \setminus e_{n-1}$ $(n \ge n_0)$ then $y \le \lambda(t)$ implies $y \le \rho_n$. Let

$$M = \sup_{[0,1]} |\lambda''(t)|,\ \ m = \min\left\{1\,,\ \frac{k_0}{M+1}\right\} \quad \text{and} \quad \alpha(t) = m\,\lambda(t),\ t \in [0,1].$$

In particular since $\alpha(t) \le \lambda(t)$ we have from (2.34) that

$$q(t)\, f(t, \alpha(t), z) + \alpha''(t) \ge k_0 + \alpha''(t) \ge k_0 - \frac{k_0\, |\lambda''(t)|}{M+1} > 0$$

for $(t,z) \in (0,1) \times \mathbf{R}$, and also

$$q(t)\,f(t,y,\alpha'(t)) + \alpha''(t) \geq k_0 + \alpha''(t) > 0$$

for $(t,y) \in (0,1) \times \{y \in (0,\infty) : y \leq \alpha(t)\}$. Thus (2.5) and (2.29) hold.

Theorem 2.4. *Let $n_0 \in \{1,2,...\}$ be fixed and suppose* (2.2), (2.3), (2.7), (2.8), (2.30) *and* (2.33) *hold. Then* (2.1) *has a solution $y \in C[0,1] \cap C^2(0,1)$ with $y(t) > 0$ for $t \in (0,1)$.*

Example. Consider the boundary value problem

$$\begin{cases} y'' + \frac{t}{y^2} + |y'|^\alpha - \mu^2 = 0, & 0 < t < 1 \\ y(0) = y(1) = 0 \end{cases} \tag{2.35}$$

with $\mu > 0$ and $0 \leq \alpha \leq 1$. Then (2.35) has a solution $y \in C[0,1] \cap C^2(0,1)$ with $y(t) > 0$ for $t \in (0,1)$.

To see that (2.35) has the desired solution we will apply Theorem 2.4 with $q = 1$, $f(t,y,z) = \frac{t}{y^2} + |z|^\alpha - \mu^2$ and

$$\rho_n = \left(\frac{1}{2^{n+1}\,(\mu^2+1)} \right)^{\frac{1}{2}}, \quad k_0 = 1 \ \text{ and } \ n_0 = 1.$$

Clearly (2.2) and (2.3) hold and notice also if $n \in \{1,2,...\}$, $\frac{1}{2^{n+1}} \leq t \leq 1$, $0 < y \leq \rho_n$ and $z \in \mathbf{R}$ we have

$$q(t)\,f(t,y,z) \geq \frac{t}{\rho_n^2} - \mu^2 \geq \frac{1}{2^{n+1}\,\rho_n^2} - \mu^2 = (\mu^2+1) - \mu^2 = 1,$$

so (2.33) is also true. Next let $\beta(t) = M + \rho_1$ where M is chosen large enough so that

$$\frac{1}{(M+\rho_1)^2} \leq \mu^2.$$

Notice (2.30) is immediate since

$$q(t)\,f(t,\beta(t),\beta'(t)) + \beta''(t) = \frac{t}{[\beta(t)]^2} - \mu^2 \leq \frac{1}{(M+\rho_1)^2} - \mu^2 \leq 0$$

for $t \in (0,1)$, and

$$q(t)\,f\left(\frac{1}{2^{n_0+1}}, \beta(t), \beta'(t)\right) + \beta''(t) = \frac{1}{4\,[\beta(t)]^2} - \mu^2 \leq \frac{1}{(M+\rho_1)^2} - \mu^2 \leq 0$$

for $t \in \left(0, \frac{1}{4}\right)$. Next let

$$\psi_\epsilon(z) = \frac{1}{\epsilon^2} + \mu^2 + z^\alpha$$

and notice (2.7) and (2.8) are satisfied since

$$|f(t,y,z)| \leq \frac{1}{\epsilon^2} + \mu^2 + |z|^\alpha = \psi_\epsilon(|z|) \quad \text{for} \quad t \in (0,1), \; y \geq \epsilon, \; z \in \mathbf{R}$$

and

$$\int_0^\infty \frac{du}{\psi_\epsilon(u)} = \infty \quad \text{since} \quad 0 \leq \alpha \leq 1.$$

Existence now follows from Theorem 2.4.

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