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# On approximation of functions by positive linear operators

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## Abstract

The aim of the present paper is to point out basic results concerning the approximation of functions by using linear positive operators. We indicate the main research directions of this field and some of the most remarkable results obtained in the last half-century. Our presentation will bring to light classical and recent results in Korovkin-type approximation theory, obviously just as much as it can be done in a few pages.

**Key words and phrases:** positive linear operator**,** Korovkin-type approximation theory, statistical convergence.

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## 1. Introduction

With a great potential for applications to a wide variety of problems, Approximation Theory represents on old field of mathematical research. In the fifties, a new breath over it has been brought by a systematic study of the linear methods of approximation which are given by sequences of linear operators, the essential ingredient being that of positivity. These methods became a firmly entrenched part of Approximation Theory.

In this respect, his paper represents a journey in the world of positive approximation processes, meeting mathematicians, historical notes and outstanding classical results.

We will see that the theorem of K. Weierstrass can be elegant proved by using positive linear operators. At another stop, we recall one of the most powerful and spectacular criterion to decide if a sequence of positive linear operators towards the identity operator with respect to the uniform norm of the space $C([a,b])$. This celebrated result is known in literature as the Bohman-Korovkin theorem and our goal is to point that the result was independently earlier established by Tiberiu Popoviciu whose contribution remained unknown for a long time.

In order to bring together the most important results concerning this fruitful research direction, the exposed material reveals strong connection with probability theory, extensions of the above criterion springing from different sources.

## 2. Roots

At 70 years old, Karl Wilhelm Theodor Weierstrass (1815-1897) proved the density of algebraic polynomials in the space $C([a,b])$, and of trigonometric polynomials in $\widetilde{C}([a,b])$, the class of all functions *f* in $C([a,b])$ satisfying $f(a)=f(b)$. This class may be considered as the restriction to $[a,b]$ of functions belonging to $C(\mathbf{R})$ which are $(b-a)$-periodic. A selection of different proofs of this fundamental result can be found in the excellent survey paper [16] written by Allan Pinkus. Based on probabilistic considerations, the proof of Weierstrass theorem given by S.N. Bernstein [5] contains explicitely $(B_n f)_{n\geq 1}$ the sequence of polynomials associated to *f*, $f\in C([0,1])$, such that $\lim\limits_{n\to\infty}(B_n f)(x)=f(x)$, uniformly on $[0,1]$. We recall the Bernstein operators

$$(B_n f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \in [0,1],$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $k = \overline{0,n}$. A detailed study of these polynomials is due to G.G. Lorentz [15].

Our further interest is to study the approximation properties of general sequences of positive linear operators. If $(L_n)_{n\geq 1}$ is a such sequence, what are the sufficient conditions to guarantee that $(L_n f)_{n\geq 1}$ converges uniformly to *f*, for each continuous function *f* ? T. Popoviciu, H. Bohman and P.P. Korovkin found out the answer.

Setting $e_0(x) = 1$, $e_1(x) = x$, $e_2(x) = x^2$, $x \in [a,b]$, we present

**Theorem 1.** (*Popoviciu-Bohman-Korovkin*) *Let* $(L_n)_{n\geq 1}$ *be a sequence of positive linear operators mapping* $C([a,b])$ *into itself. If*

$$\lim_{n\to\infty} (L_n e_j)(x) = e_j(x) \text{ uniformly on } [a,b], \; j \in \{0,1,2\},$$

*then*

$$\lim_{n\to\infty} (L_n f)(x) = f(x) \text{ uniformly on } [a,b], \text{ for every } f \in C([a,b]).$$

Here are information about the contributions of the authors.

1) T. Popoviciu (a Romanian mathematician, 1906-1975): his result [17] was published in 1951 in Romanian language and thus his contribution regarding this criterion remained unknown for many researchers. In his proof, Popoviciu considered that the operators $L_n$, $n \in \mathbf{N}$, reproduce the constant functions.

2) H. Bohman (a Swedish statistician, 1920-1996) proved the mentioned result [6] only for positive linear operators of the form

$$(L_n f)(x) = \sum_{i=0}^{n} u_{n,i}(x) f(x_{n,i}), \quad x \in [a,b],$$

where $u_{n,i}$, $i = \overline{0,n}$, are non-negative functions defined on $[a,b]$ and $(x_{n,i})_{i=\overline{0,n}}$ is a net of the same interval.

3) P.P. Korovkin (a Russian mathematician, 1913-1987) proved the same result [13] for integral type operators. In time, he extended his theory, see [14].

### 3. Research directions

*A. Classical approaches*. The following three aspects are most important in this field.

1° The construction of these processes by using various methods as algebraic and trigonometric identities, convolution products, probability schemes.

2° The study of the degree of approximation by using moduli of smoothness, *K*-functionals and establishing asymptotic formulas of Voronovskaja-type.

3° The capacity of these processes to mimic qualitative properties of the approximated function such as monotonicity, convexity, shape preservation.

During the last five decades, a fruitful research direction has consisted in extensions of Theorem 1 to abstract spaces. Practically, a new theory was born that we may call KAT, this meaning *Korovkin-type approximation theory*.

*B. More about KAT*. In what follows we mention some spaces in which Korovkin-type approximation theory has been developed: in $C(X)$, the space of all continuous real-valued functions defined on a compact Hausdorff space *X*; in $C_0(X)$, all continuous real-valued functions defined on a locally compact Hausdorff space $X$ which vanish at infinity; in $L^p(X,F,\mu)$ where $(X,F,\mu)$ is a measure space or in $L^p(X,\mu)$ where $\mu$ belongs to the cone of all positive Radon measures and $1\le p\le\infty$; in the spaces of continuous affine functions defined on a compact convex set; in Banach lattices, also in ordered topological vector spaces. In this direction, the monograph of F. Altomare and M. Campiti [2] contains a complete picture of what has been achieved in this field.

**4. Outstanding results**

We focus upon the quantitative forms of Popoviciu-Bohman-Korovkin theorem by using the modulus of continuity $\omega_h$ of the function $h\in C([a,b])$ defined as follows

$$\omega_h(\delta):=\sup\{|h(x')-h(x'')|: x',x''\in[a,b],|x'-x''|\le\delta\},\ \text{for every } \delta>0.$$

**Theorem 2.** (*O. Shisha, B. Mond* [18]) *Let* $L_n : C([a,b])\to C([c,d])$, $[c,d]\subset[a,b]$, $n\in\mathbf{N}$, *be linear positive operators.*

*(i) For each* $f\in C([a,b])$ *and* $x\in[c,d]$ *one has*

$$\begin{aligned}|(L_nf)(x)-f(x)|&\le|f(x)||(L_ne_0)(x)-1|\\&\quad+\left((L_ne_0)(x)+\sqrt{(L_ne_0)(x)}\right)\omega_f(\mu_{2,n}^{1/2}(x)).\end{aligned}$$

*(ii) If f is differentiable on* $[a,b]$ *and* $f'\in C([a,b])$, *then*

$$\begin{aligned}|(L_nf)(x)-f(x)|&\le|f(x)||(L_ne_0)(x)-1|+|f'(x)||L_n(\cdot-x;x)|\\&\quad+\left(1+\sqrt{(L_ne_0)(x)}\right)\sqrt{\mu_{2,n}(x)}\,\omega_{f'}(\mu_{2,n}^{1/2}(x)).\end{aligned}$$

*In the above,* $\mu_{2,n}(x):=L_n((\cdot-x)^2;x)$ *representing the second central moment of the operator* $L_n$.

In [11] H.H. Gonska obtained quantitative Korovkin type theorems for approximation by positive linear operators acting on the Banach lattice of real-valued continuous functions defined on the compact metric space $(X,d)$. This was achieved by using a smoothing approach and the least concave majorant $\widetilde{\omega}_h$ of the modulus $\omega_h$, $h \in C(X)$. We recall

$$\widetilde{\omega}_h(\delta) := \begin{cases} \sup\limits_{\substack{0 \le x \le \delta \le y \le \rho(X) \\ x \ne y}} \dfrac{(\delta - x)\omega_h(y) + (y - \delta)\omega_h(x)}{y - x} & \text{for} \quad 0 \le \delta \le \rho(X), \\ \omega_h(\rho(X)) \quad \text{for} \quad t > \rho(X), \end{cases}$$

where $\rho(X) < \infty$ is the diameter of the compact space *X*.

However, the classical moduli of smoothness $\omega^r(f,t) = \sup\limits_{0<h\le t} \| \Delta_h^r f \|$, $r \in \mathbf{N}$, have proved to be useful for measuring the error of approximation. In time, another type of modulus of smoothness became a better tool to measure the rate of the best approximation. It is defined by

$$\omega_\varphi^r(f;t)_p := \sup_{0<h\le t} \| \Delta_{h\varphi}^r f \|_{L_p(D)},$$

$x \in D = (a,b)$, $-\infty \le a < b \le \infty$, where the step-weight function $\varphi$ and the interval in question are related to the problem at hand. Here $\Delta_{h\varphi}^r f$ represents the *r*-th symmetric difference of the function *f*,

$$\Delta_{h\varphi}^r f(x) := \sum_{k=0}^{r} (-1)^k \binom{r}{k} f\left( x + \left( \frac{r}{2} - k \right) h\varphi(x) \right).$$

Z. Ditzian and V. Totik, see the monograph [7], proved the equivalence between this modulus and a certain Peetre *K*-functional,

$$K_{r,\varphi}(f;t^r)_p := \inf_g \{ \| f - g \|_p + t^r \| \varphi^r g^{(r)} \|_p : g^{(r-1)} \in A.C._{loc} \},$$

where $g^{(r-1)} \in A.C._{loc}$ means that $g$ is $r-1$ times differentiable and $g^{(r-1)}$ is absolutely continuous on every compact included in *D*. This equivalence was the starting point in exploring by many mathematicians the rate of convergence of different classes of linear positive operators in terms both of *K*-functionals and of certain weighted moduli.

Further on, we gather the results established in 1989 by D. Andrica and C. Mustăța [4] as regards Korovkin-type theorems in an abstract space.

**Theorem 3.** *Let $(X,d)$ be a compact metric space and $L_n$, $n \in \mathbf{N}$, linear positive operators from $C(X)$ into itself which preserve the constant functions. For each $f \in Lip(X)$ one has*

$$\| L_n f - f \| \le K_f \| \alpha_n \|,$$

*where $\alpha_n(x) := L_n(d(\cdot,x),x)$ and $K_f$ is a Lipschitz constant for f.*

*Let H be a real vector space with an inner-product $\langle \cdot,\cdot \rangle$. For each $f \in Lip(X)$, $X \subset H$ compact, one has*

$$\| L_n f - f \| \le K_f \sqrt{\| a_n - 2b_n \|},$$

*where $a_n(x) := (L_n e)(x) - e(x)$, $b_n(x) := (L_n e_x)(x) - e(x)$ with $e(x) = \langle x,x \rangle$ and $e_t(x) = \langle x,t \rangle$, $(x,t) \in H \times H$.*

At the end, we notice that a comprehensive study of the computational aspects of the moduli of smoothness and the global smoothness preservation property can be found in the book of G. Anastassiou and S. Gal [3].

## 5. A probabilistic approach

In the frame of a probability space $(\Omega, A, P)$, let $M_2(\Omega)$ be the space of all real square-integrable random variables on Ω. A *random scheme* on the interval $I$ is a mapping $Z:\mathbf{N}\times I\to M_2(\Omega)$. In the sequel, if $Z$ is a random scheme, then we set

$$\alpha_{n,x}:=E(Z(n,x)),\quad \sigma_{n,x}^2:=Var(Z(n,x)),\quad (n,x)\in\mathbf{N}\times I, \tag{2}$$

the expectation, respectively the variance of $Z$.

For a given random scheme we associate the sequence $(P_n)_{n\geq 1}$ defined as follows $P_n:C_B(\mathbf{R})\to B(I)$,

$$(P_nf)(x)=E(f\circ Z(n,x))=\int_{\mathbf{R}} fdP_{Z(n,x)},\quad x\in I. \tag{3}$$

$C_B(I)$ stands for the space of all real valued continuous and bounded functions defined on $I$, equipped with the usual sup-norm.

Clearly, these operators are linear and positive. The following theorem establishes under which conditions the sequence is an approximation process.

**Theorem 4.** *Let* $(P_n)_{n\geq 1}$ *be defined by (3). If* $\lim\limits_{n\to\infty}\alpha_{n,x}=x$ *and* $\lim\limits_{n\to\infty}\sigma_{n,x}^2=0$ *uniformly in* $x\in I$*, then, for every* $f\in C_B(\mathbf{R})$*,* $\lim\limits_{n\to\infty}P_nf=f$ *uniformly on compact subintervals of I.*

The proof can be found, e.g. [2; *Theorem 5.2.2*]. We mention that at the first step the above result must be proved for a uniform continuous function *f* on **R** and this fact was established by W. Feller in 1966.

Among the pioneers of approaching Korovkin type results by using probabilistic tools we quote W. Feller [9], D.D. Stancu [19] and J.P. King [12].

If we require $P(Z(n,x)\in I)=1$ for each $(n,x)\in\mathbf{N}\times I$, then we can restrict the operators (3) to a suitable function space on $I$. Now, we consider

$$P_n:C_B(I)\to B(I),\quad (P_nf)(x)=\int_I fdP_{Z(n,x)}\,.$$

In the particular case $I=[a,b]$, taking into account (2), we have

$$\alpha_{n,x}=\int_a^b tP_{Z(n,x)}(t)=(P_n e_1)(x),$$

$$\sigma_{n,x}^2=\int_a^b (t-E(Z(n,x))^2 dP_{Z(n,x)}(t)=(P_n e_2)(x)-\alpha_{n,x}^2 .$$

These identities help us to establish the strong connection between the requirements of Theorem 4 and Popoviciu-Bohman-Korovkin result.

$$\lim_{n\to\infty}\alpha_{n,x}=x \Leftrightarrow \lim_{n\to\infty}(P_n e_1)(x)=x \text{, uniformly in } x\in[a,b].$$

Under the above assumption, we also have

$$\lim_{n\to\infty}\sigma_{n,x}^2=0 \Leftrightarrow \lim_{n\to\infty}(P_n e_2)(x)=x^2 \text{, uniformly in } x\in[a,b].$$

## 6. On the statistical convergence

In 2002 A.D. Gadjiev and C. Orhan [10] proved Korovkin theorem via statistical convergence. We recall: if $S$ is a subset of $\mathbf{N}$ and $\aleph_S$ is its characteristic function, then the density of $S$ is defined by

$$\delta(S)=\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}\aleph_S(j),$$

provided the limit exists. The sequence $\widetilde{a}=(a_n)_{n\geq 1}$ of real numbers is *statistically convergent* to the number $L$ if the following relation

$$\delta(\{n\in\mathbf{N}:|a_n-L|\geq\varepsilon\})=0$$

holds for every $\varepsilon>0$. In this case we write $st-\lim_{n\to\infty}a_n=L$. In analogy with the classical Korovkin theorem, in [10] the authors obtained sufficient conditions which guarantee that a sequence $(A_n)_{n\geq 1}$ of positive linear operators verifies $st-\lim_n \| A_n f - f \|=0$, for any function $f$ which is continuous on the interval $[a,b]$ and bounded on the entire line. The statistical approximation on the space of $2\pi$-periodic and continuous functions on the whole real axis was also studied, see [8].

Recently, in [1] the following result is proved.

**Theorem 5.** *Let* $L_n$, $n \in \mathbf{N}$, *be positive linear operators acting on* $C(X)$, *where* $(X,d)$ *is a compact metric space. Set* $\alpha_n(x) := L_n(d(\cdot,x),x)$. *If*

$$st-\lim_n \| L_n e_0 - e_0 \| = 0 \quad \text{and} \quad st-\lim_n \| \alpha_n \| = 0,$$

*then, for every* $f \in C(X)$, *the following identity holds true*

$$st-\lim_n \| L_n f - f \| = 0.$$

To conclude, we point out that our goal was not a dictionary or an encyclopedia but instead a brief biography of linear positive operators with a beginning and an end, and some substance in between. Our chief aim was to present, first of all, basic results concerning this research field.

## REFERENCES


[1] O. Agratini, *On statistical approximation in spaces of continuous functions*, Math. Inequalities and Applications, **9**(2006), in print.

[2] F. Altomare and M. Campiti, *Korovkin-type Approximation Theory and its Applications,* de Gruyter Studies in Mathematics, Vol.17, Walter de Gruyter & Co., Berlin-New York, 1994.

[3] G. Anastassiou and S. Gal, *Approximation Theory*, Birkhäuser, Boston, 2000.

[4] D. Andrica and C. Mustăța, *An abstract Korovkin type theorem and applications*, Studia Univ. Babeş-Bolyai, Mathematica, **34**(1989), 44-51.

[5] S.N. Bernstein, *Démonstration du théorème de Weierstrass fondée sur le calcul des probabilités*, Commun. Soc. Math. Kharkow (2), **13**(1912), 1-2.

[6] H. Bohman, *On approximation of continuous and of analytic functions*, Ark. Mat., **2**(1952-54), 43-56.

[7] Z. Ditzian and V. Totik, *Moduli of smoothness*, Springer-Verlag, New York Inc., 1987.

[8] O. Duman, *Statistical approximation for periodic functions*, Demonstratio Mathematica, **36**(2003), f.4, 873-878.

[9] W. Feller, *An Introduction to Probability Theory and its Applications*, Vol.II, Wiley, 1966.

[10] A.D. Gadjiev and C. Orhan, *Some approximation theorems via statistical convergence,* Rocky Mountain Journal of Mathematics, **32**(2002), f.1, 129-138.

[11] H.H. Gonska, *On approximation in spaces of continuous functions*, Bull. Austral. Math. Soc., **28**(1983), 411-432.

[12] J.P. King, *Probabilistic interpretation of some positive linear operators*, Rev. Roum. Math. Pures Appl., **25**(1980), f.1, 77-82.

[13] P.P. Korovkin, *On convergence of linear positive operators in the space of continuous functions* (Russian), Dokl. Akad. Nauk SSSR, **90**(1953), 961-964.

[14] P.P. Korovkin, *Linear Operators and Approximation Theory*, translated from the Russian ed.1959, Russian Monographs and Texts

on Advanced Mathematics and Physics, Vol.III, Gordon and Breach Publishers, Inc., New York, Hindustan Publishing Corp. (India), Delhi, 1960.

[15] G.G. Lorentz, *Bernstein Polynomials*, Toronto: Univ. of Toronto Press, 1953.

[16] A. Pinkus, *Weierstrass and Approximation Theory*, Journal of Approximation Theory, **107**(2000), 1-66.

[17] T. Popoviciu, *Asupra demonstrației teoremei lui Weierstrass cu ajutorul polinoamelor de interpolare*, Lucrările Sesiunii Gen. Şt. Acad. Române, 2-12 iunie 1950, Editura Academiei Republicii Populare Române, 1951, pp. 1664-1667. [Translated in English by Daniela Kacsó: *On the proof of Weierstrass' theorem using interpolation polynomials*, East Journal on Approximations, **4**(1998), f.1, 107-110.]

[18] O. Shisha and B. Mond, *The degree of convergence of linear positive operators*, Proc. Nat. Acad. Sci. USA, **60**(1968), 1196-1200.

[19] D.D. Stancu, *Use of probabilistic methods in the theory of uniform approximation of continuous functions*, Rev. Roum. Math. Pures et Appl., **14**(1969), f.5, 673-691.

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