Chapter 1

Discrete Approximation Processes of King's Type

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Dedicated to Professor George Isac

Abstract. This survey paper is focused on linear positive operators having the degree of exactness null and fixing the monomial of the second degree. The starting point is represented by J.P. King's paper appeared in 2003. Our first aim is to sum up results obtained in the last five years. The second aim is to present a general class of discretizations following the features of the operators introduced by King.

1.1 Introduction

Let $C([a,b])$ be the Banach space of all real-valued and continuous functions defined on $[a,b]$, equipped with the norm $\|\cdot\|$ of the uniform convergence. Let e_n be the monomial of n degree, $n \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$. Bohman-Korovkin's theorem states: if $(L_n)_{n\geq 1}$ is a sequence of positive linear operators mapping $C([a,b])$ into itself such that $\lim\limits_n \|L_n e_i - e_i\| = 0$ for $i = 0, 1, 2$, then one has $\lim\limits_n \|L_n f - f\| = 0$ for every $f \in C([a,b])$.

Many classical linear positive operators have the degree of exactness one, this meaning they preserve the monomials e_0 and e_1. On the other hand, it is well known that if a linear positive operator reproduces all three test functions of the Bohman–Korovkin criterion, then it is the identity operator of the space. A question arises: What is known about operators which fix the monomials e_0 and e_2?

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P.M. Pardalos et al. (eds.), *Nonlinear Analysis and Variational Problems*,
Springer Optimization and Its Applications 35, DOI 10.1007/978-1-4419-0158-3_1,

In 2003, J.P. King [10] was the first to present an example of linear positive operators enjoying this property. These operators are of Bernstein type and are given as follows. $V_n : C([0,1]) \to C([0,1])$,

$$(V_n f)(x) = \sum_{x=0}^{n} \binom{n}{k} (r_n^*(x))^k (1 - r_n^*(x))^{n-k} f\left(\frac{k}{n}\right), \quad x \in [0,1], \tag{1.1}$$

where $r_n^* : [0,1] \to [0,1]$,

$$r_n^*(x) = \begin{cases} x^2, \; n = 1, \\ -\dfrac{1}{2(n-1)} + \sqrt{\dfrac{n}{n-1}x^2 + \dfrac{1}{4(n-1)^2}}, \; n = 2, 3, \ldots \end{cases} \tag{1.2}$$

From Approximation Theory point of view, this sequence is useful. In spite of the fact that the operators have the degree of exactness null, the order of approximation is at least as good as the order of Bernstein operators.

King's construction quickly gained the popularity and, during the last five years, several authors studied classes of linear positive operators which preserve the test functions e_0 and e_2. There are new papers on this topic constantly coming out and wide generalizations being studied.

The present paper aims to summarize the main results obtained in this area. Also, this survey should stimulate further researches.

1.2 Further results on V_n type operators

King also established quantitative estimates for V_n in terms of the classical first-order modulus $\omega_1(f;\cdot)$. New quantitative estimates involving the second modulus of continuity $\omega_2(f;\cdot)$ have been obtained by H. Gonska and P. Piţul. They proved [9; Theorem 2.1] the following result.

Theorem 1.1. *Let V_n, $n \in \mathbb{N}$, be defined by (1.1). For each $f \in C[0,1]$ and $x \in [0,1]$ one has*

$$|(V_n f)(x) - f(x)| \le \sqrt{x - r_n^*(x)}\,\omega_1(f; \sqrt{x - r_n^*(x)}) + (1+x)\omega_2(f; \sqrt{x - r_n^*(x)}).$$

Consequently, if $f \in C^1([0,1])$, one obtains the approximation order $\mathcal{O}(\sqrt{x - r_n^*(x)})$, $n \to \infty$. For $f \in C^2([0,1])$ the approximation order is $\mathcal{O}(x - r_n^*(x))$, $n \to \infty$.

Setting the powers of an operator L by $L^0 = I_X$, $L^1 = L$, $L^{n+1} = L \circ L^m$, $m \in \mathbb{N}$, where I_X indicates the identity operator on the space X, the iterates of V_n have been obtained [9; Theorem 3.2].

Theorem 1.2. *Let V_n, $n \in \mathbb{N}$, be defined by (1.1). If $n \in \mathbb{N}$ is fixed, then for all $f \in C([0,1])$ and $x \in [0,1]$, one has*

$$\lim_{m\to\infty}(V_n^m f)(x) = f(0) + (f(1) - f(0))x^2 = (V_1 f)(x).$$

As regards the approximation process $(V_n)_n$, the transition from uniform convergence to A-statistical convergence was done by O. Duman and C. Orhan [6]. At first, we briefly recall elements of this type of convergence. Let $A = (a_{j,n})_{j\geq 1, n\geq}$ be an infinite summability matrix. For a given sequence $x := (x_n)_{n\geq 1}$ the A-transform of x, denoted by $Ax := ((Ax)_j)$, is defined by $(Ax)_j = \sum_{n=1}^{\infty} a_{j,n}x_n$ provided the series converges for each j. A is regular if $\lim_n x_n = L$ implies $\lim_j (Ax)_j = L$. Further on, we assume that A is a non-negative regular summability matrix and K is a subset of $\mathbb{N}$. The A-density of K, denoted by $\delta_A(K)$, is defined by $\delta_A(K) := \lim_j \sum_{n\in K} a_{j,n}$ provided the limit exists. A sequence $x := (x_n)_{n\geq 1}$ is said to be A-statistical convergent to the real number L if, for every $\varepsilon > 0$,

$$\delta_A(\{n \in \mathbb{N} : \, |x_n - L| \geq \varepsilon\}) = 0$$

takes place. This limit is denoted by $st_A - \lim x = L$.

From this moment, assume that A is a non-negative regular summability matrix such that $\lim_j \max_n \{a_{j,n}\} = 0$ holds. On the basis of [11], we can choose an infinite subset K of $\mathbb{N} \setminus \{1\}$ such that $\delta_A(K) = 0$. Define the functions p_n, $n \in \mathbb{N}$, by

$$p_n(x) = \begin{cases} x^2, & \text{if } n = 1, \\ r_n^*(x), & \text{if } n \notin K \cup \{1\}, \\ 0, & \text{otherwise}, \end{cases} \tag{1.3}$$

where r_n^* is defined by (1.2). Each function p_n is continuous on $[0,1]$ and $p_n([0,1]) \subset [0,1]$. One has $st_A - \lim_n p_n(x) = x$, $x \in [0,1]$. An analog of King's result proved in [6; Theorem 2.3] will be read as follows.

Theorem 1.3. *Let V_n be defined by (1.1) such that r_n^* is replaced by p_n described at (1.3). Then, for all $f \in C([0,1])$ and all $x \in [0,1]$,*

$$st_A - \lim_n |(V_n f)(x) - f(x)| = 0$$

holds.

Another recent paper [4] centers around a family of sequences of linear Bernstein-type operators depending on a real parameter $\alpha \geq 0$ and preserving e_0 and the polynomial $e_2 + \alpha e_1$. Let $\alpha \geq 0$ be fixed. For each $n \geq 2$, the authors consider the function $r_{n,\alpha} : [0,1] \to \mathbb{R}$ defined by

$$r_{n,\alpha}(x) := -\frac{n\alpha + 1}{2(n-1)} + \sqrt{\frac{(n\alpha+1)^2}{4(n-1)^2} + \frac{n(\alpha x + x^2)}{n-1}} \tag{1.4}$$

and the operator $B_{n,\alpha} : C([0,1]) \to C([0,1])$ given by

$$(B_{n,\alpha}f)(x) := \sum_{k=0}^{n} p_{n,k,\alpha}(x) f\left(\frac{k}{n}\right), \tag{1.5}$$

$$p_{n,k,\alpha}(x) := \binom{n}{k} r_{n,\alpha}^k(x)(1 - r_{n,\alpha}(x))^{n-k}$$

If we choose $\alpha = 0$, $B_{n,0}$ becomes V_n operator introduced by King. Also, the relations (1.4), (1.5) guarantee $B_{n,\alpha}e_0 = e_0$, $B_{n,\alpha}e_1 = r_{n,\alpha}$ and $B_{n,\alpha}(e_2 + \alpha e_1) = e_2 + \alpha e_1$.

Besides qualitative and quantitative results regarding the sequence $(B_{n,\alpha})_{n\geq 2}$, the authors obtain the following asymptotic formula of Voronovskaja type.

Theorem 1.4. *Let $B_{n,\alpha}$ be defined by (1.5). For $x \in (0,1)$ one has*

$$\lim_{n} 2n((B_{n,\alpha}f)(x) - f(x)) = x(1-x)\left(f''(x) - \frac{2}{2x+\alpha} f'(x)\right)$$

provided f has the required regularity conditions at the point x.

We conclude this Section with q-Bernstein operators. To formulate the results we need the following definitions.

Let $q > 0$. For any $n \in \mathbb{N}_0$, the q-integer $[n]_q$ is defined by

$$[n]_q := 1 + q + \cdots + q^{n-1} \quad (n \in \mathbb{N}), \quad [0]_q := 0$$

and the q-factorial $[n]_q!$ by

$$[n]_q! := [1]_q [2]_q \ldots [n]_q \quad (n \in \mathbb{N}), \quad [0]_q! := 1.$$

For each integer $k \in \{0, 1, \ldots, n\}$, the q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

Clearly, for $q = 1$, one gets $[n]_1 = n$, $[n]_1! = n!$, $\begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}$.

The q-Bernstein polynomials of $f : [0,1] \to \mathbb{C}$ introduced by G.M. Phillips [13] are defined as follows

$$(B_n f)(q; x) := \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{\nu=0}^{n-1-k} (1 - q^\nu x), \quad x \in [0,1],\ n \in \mathbb{N}. \tag{1.6}$$

We mention that an empty product is taken to be equal to 1.

For $q = 1$, the polynomials $(B_n f)(q; \cdot)$ are classical Bernstein polynomials. In what follows we consider $0 < q < 1$; in this case q-Bernstein polynomials are positive linear operators on $C([0,1])$. These operators satisfy the following properties

$$(B_n e_0)(q;x) = 1, \quad (B_n e_1)(q;x) = x, \quad (B_n e_2)(q;x) = x^2 + \frac{x - x^2}{[n]_q}. \tag{1.7}$$

The Phillips' results are the basis of many research papers, and the comprehensive survey due to S. Ostrovska [12] gives a good perspective of these achievements.

Our aim is to modify the q-Bernstein operators $(B_n f)(q;\cdot)$, $n \geq 2$, into King variant. Setting

$$r_{n,q}(x) := -\frac{1}{2([n]_q - 1)} + \sqrt{\frac{[n]_q}{[n]_q - 1}x^2 + \frac{1}{4([n]_q - 1)^2}}, \quad x \in [0,1], \tag{1.8}$$

for each $n \geq 2$, we consider the operator

$$(B_n^* f)(q;x) := (B_n f)(q; r_{n,q}(x)), \quad f \in \mathbb{R}^{[0,1]}, \tag{1.9}$$

where $(B_n f)(q;\cdot)$ is given by (1.5).

For the particular case $q = 1$, $(B_n^* f)(1;\cdot)$ turns into $V_n f$, King's example.

Since $(B_n^* e_1)(q;\cdot) = r_{n,q}$ and $\lim_n r_{n,q}(x) = (\sqrt{4q^2x^2 + (1-q)^2} - 1 + q)/2q$, $q \in (0,1)$, based on Bohman-Korovkin theorem, it is obvious that our sequence does not form an approximation process on the space $C([0,1])$. In order to satisfy this property, for each $n \geq 2$, the constant $q \in (0,1)$ will be replaced by a number $q_n \in (0,1)$.

Theorem 1.5. *Let $(q_n)_{n\geq 2}$, $0 < q_n < 1$, be a sequence such that $\lim_n q_n = 1$ and $\lim_n q_n^n$ exists. Let $(B_n^* f)(q_n;\cdot)$ be defined as in (1.9). For any $f \in C([0,1])$ one has*

$$\lim_n (B_n^* f)(q_n;x) = f(x), \text{ uniformly in } x \in [0,1].$$

Proof. The assumptions made upon the sequence $(q_n)_{n\geq 2}$ guarantee that $\lim_n [n]_{q_n} = \infty$. Examining (1.8), this implies

$$\lim_n r_{n,q_n}(x) = x, \text{ uniformly in } x \in [0,1],$$

and, consequently, $\lim_n (B_n^* e_1)(q_n;\cdot) = e_1$. Also, relations (1.9) and (1.7) ensure $\lim_n (B_n^* e_j)(q_n;\cdot) = e_j$, $j \in \{0,2\}$. Since the requirements of Bohman-Korovkin theorem are satisfied, the conclusion follows. □

1.3 A general class in study

The object of this Section is to present a class of discrete operators reproducing the third test function of Bohman-Korovkin theorem. This class is defined on certain subspaces of $C(J)$, $J \subset \mathbb{R}$. We take into account two kinds of intervals: $J = [0,1]$

and $J=\mathbb{R}_+:=[0,\infty)$, respectively. Let $I_n\subset\mathbb{N}$ be a set of indices. Following [1], we consider a sequence $(L_n)_{n\ge1}$ of linear positive operators acting on $C(J)$ and defined by

$$(L_nf)(x)=\sum_{k\in I_n}u_{n,k}(x)f(x_{n,k}),\quad x\in J,\ f\in\mathcal{F}(J),\tag{1.10}$$

where $u_{n,k}\in C(J)$ is a positive real valued function for each $(n,k)\in\mathbb{N}\times I_n$, $(x_{n,k})_{k\in I_n}$ is a mesh of nodes on J and

$$\mathcal{F}(J):=\{f\in C(J):\ \text{the series in (1.10) is convergent}\}.$$

We note that the right-hand side of (1.10) could be a finite sum. In this case, $\mathcal{F}(J)$ is just $C(J)$. For each $n\in\mathbb{N}$, we assume that the following identities

$$(L_ne_0)(x)=1,\quad (L_ne_1)(x)=x,\quad (L_ne_2)(x)=a_nx^2+b_nx,\quad x\in J,\tag{1.11}$$

are fulfilled, where $a_n>0$, $b_n\ge0$. Knowing that $u_{n,k}\ge0$, $k\in I_n$, the first identity from (1.11) implies that each $u_{n,k}$ belongs to $C_B(J)$, the space of all real-valued continuous and bounded functions on J. In regard with the sequences of real numbers $(a_n)_{n\ge1}$ and $(b_n)_{n\ge1}$ we assume

$$\lim_{n\to\infty}a_n=1,\quad \lim_{n\to\infty}b_n=0.\tag{1.12}$$

Relations (1.11) and (1.12) guarantee that $(L_n)_{n\ge1}$ is a strong approximation process on any compact $\mathcal{K}\subset J$, this meaning $\lim\limits_n(L_nf)(x)=f(x)$ uniformly for every $x\in\mathcal{K}$ and $f\in\mathcal{F}(J)$.

For each $n\in\mathbb{N}$, we define the continuous function $v_n:J\to\mathbb{R}_+$,

$$v_n(x)=\frac{1}{2a_n}\left(\sqrt{b_n^2+4a_nx^2}-b_n\right),\quad x\in J.\tag{1.13}$$

Starting with (1.10), we introduce the operators

$$(L_n^*f)(x)=\sum_{k\in I_n}u_{n,k}(v_n(x))f(x_{n,k}),\quad x\in J,\ f\in\mathcal{F}(J).\tag{1.14}$$

On the basis of (1.11), the following identities [1]

$$L_n^*e_0=e_0,\quad L_n^*e_1=v_n,\quad L_n^*e_2=e_2\tag{1.15}$$

hold. Consequently, one has $\lim\limits_n L_n^*f=f$ uniformly on compact subintervals of J for every $f\in\mathcal{F}(J)$.

Considering $\varphi_x:J\to\mathbb{R}$, $\varphi_x(t)=t-x$, $x\in J$, the second central moment of L_n^* has the form

$$(L_n^*\varphi_x)^2(x)=2x(x-v_n(x)),\quad x\in J.$$

By using (1.13) and the fact that L_n^* is a positive operator, one gets

$$0 \le v_n(x) \le x, \quad x \in J.$$

Our aim is to explore the rate of convergence. For $J = [0,1]$ we use the modulus of continuity. For $J = \mathbb{R}_+$ we use a weighted modulus associated to the Banach space $(E_\alpha, \|\cdot\|_\alpha)$, $\alpha \ge 2$, where

$$E_\alpha := \{f \in C(\mathbb{R}_+) : \ w_\alpha(x) f(x) \text{ is convergent as } x \to \infty\},$$

and $\|f\|_\alpha := \sup\limits_{x \ge 0} w_\alpha(x)|f(x)|$. The weight w_α is given by $w_\alpha(x) = (1 + x^\alpha)^{-1}$. Since $\alpha \ge 2$, the test functions e_j, $j \in \{0,1,2\}$, belong to E_α.

Theorem 1.6. *Let L_n^*, $n \in \mathbb{N}$, be defined by (1.14), where $J = [0,1]$. For every $f \in C(J)$, one has*

$$|(L_n^* f)(x) - f(x)| \le \left(1 + \frac{1}{\delta}\widetilde{v}_n(x)\right)\omega(f;\delta), \quad x \in J, \ \delta > 0, \tag{1.16}$$

where

$$\widetilde{v}_n(x) = \sqrt{2x(x - v_n(x))}, \quad x \in J. \tag{1.17}$$

The proof can be found in [1, Theorem 3.1.(ii)].

Since $x \in J = [0,1]$, the following upper bound for $e_1 - v_n$ can be easily established

$$x - v_n(x) = x + \frac{b_n}{2a_n} - \sqrt{\frac{x^2}{a_n} + \frac{b_n^2}{4a_n^2}} \le \left(1 - \frac{1}{\sqrt{a_n}}\right)x + \frac{b_n}{2a_n} \le \frac{|a_n - 1|}{\sqrt{a_n}} + \frac{b_n}{2a_n}.$$

Consequently, $\widetilde{v}_n(x) \le \left(\frac{2|a_n - 1|}{\sqrt{a_n}} + \frac{b_n}{a_n}\right)^{1/2}$ which tends to zero for n tending to infinity. Choosing in (1.16) δ to be equal with this quantity, we obtain

$$|(L_n^* f)(x) - f(x)| \le 2\omega\left(f; \left(\frac{2|a_n - 1|}{\sqrt{a_n}} + \frac{b_n}{a_n}\right)^{1/2}\right), \quad x \in [0,1].$$

For $f \in E_\alpha$ we consider the following weighed modulus

$$\Omega_{w_\alpha}(f;\delta) := \sup_{\substack{x \ge 0 \\ 0 < h \le \delta}} w_\alpha(x+h)|f(x+h) - f(x)|, \quad \delta > 0.$$

Theorem 1.7. *Let L_n^*, $n \in \mathbb{N}$, be defined by (1.14), where $J = \mathbb{R}_+$. For every $f \in \mathcal{F}(J) \cap E_\alpha$, one has*

$$|L(L_n^* f)(x) - f(x)| \le \sqrt{(L_n^* \mu_x^2)(x)}\left(1 + \frac{\widetilde{v}_n(x)}{\delta}\right)\Omega_{w_\alpha}(f;\delta), \quad x \ge 0, \ \delta > 0,$$

where $\widetilde{v}_n$ is given at (1.17) and $\mu_x(t) := 1 + (x + |t - x|)^\alpha$, $t \ge 0$.

The proof can be found in [1; Theorem 3.2].

Let $n \geq 2$. Choosing $I_n = \{0, 1, \ldots, n\}$, $J = [0, 1]$, $x_{n,k} = k/n$,

$$u_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \ 0 \leq k \leq n,$$

L_n becomes Bernstein operator and we get

$$a_n = 1 - \frac{1}{n}, \quad b_n = \frac{1}{n}, \quad v_n(x) = r_n^*(x), \quad x \in [0, 1],$$

where r_n^* is defined by (1.2). Consequently, L_n^* turns into King operator V_n.

In what follows, starting from classical Baskakov operators we present the modified variant L_n^*.

Example 1.8. In (1.10) we choose $J = [0, \infty)$, $I_n = \mathbb{N}$, $x_{n,k} = k/n$ and

$$u_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k}, \ x \geq 0.$$

The requirements (1.11) are fulfilled. It is known that $a_n = 1 + 1/n$ and $b_n = 1/n$, $n \in \mathbb{N}$, consequently (1.12) holds. We obtain

$$v_n(x) = \frac{\sqrt{1 + 4n(n+1)x^2} - 1}{2(n+1)}, \quad x \geq 0, \ n \in \mathbb{N},$$

and, following (1.14), the modified Baskakov operators are defined by

$$(L_n^* f)(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{v_n^k(x)}{(1+v_n(x))^{n+k}} f\left(\frac{k}{n}\right), \quad x \geq 0, \ n \in \mathbb{N},$$

where $f \in E_2$.

Remark 1.9 For $J = \mathbb{R}_+$, our operator L_n^* maps $C_B(\mathbb{R}_+)$ in $C_B(\mathbb{R}_+)$ because of the first identity of relation (1.15). Here $C_B(\mathbb{R}_+)$ denotes the space of all real-valued continuous and bounded functions defined on $\mathbb{R}_+$. On the basis of [8; Theorem 1], relations (1.15) lead us to the following result.

If $st - \lim_n \|v_n - e_1\|_K = 0$, then $st - \lim_n \|L_n^* f - f\|_K = 0$, for any function f belonging to $C_B(\mathbb{R}_+)$, where $K \subset \mathbb{R}_+$ is a compact and $\|\cdot\|_K$ is the norm of the uniform convergence on K.

Remark 1.10. For general operators of King's type, a generalization to the m-dimensional case will be read as follows. Let K_m be a compact and convex subset of the Euclidean space $\mathbb{R}^m$. It was shown by Volkov [15] that the following $m+2$ functions: $\mathbf{1}, pr_1, \ldots, pr_n, \sum_{j=1}^{m} pr_j^2$, are test functions for $C(K_m)$. Here $\mathbf{1}$ stands for the constant function on K_m of value 1 and pr_j, $1 \leq j \leq m$, represent the canonical projections on K_m, i.e., $pr_j(x) = x_j$ for every $x = (x_i)_{1\le i\le m} \in K_m$. Considering $\widetilde{L}_n^* : C(K_m) \to C(K_m)$ such that $\widetilde{L}_n^*\mathbf{1} = \mathbf{1}$ and $\widetilde{L}_n^*\left(\sum_{j=1}^m pr_j^2\right) = \sum_{j=1}^m pr_j^2$, we get

$$\widetilde{L}_n^*(\|\cdot - x\|^2; x) = \widetilde{L}_n^*\left(\sum_{j=1}^m (\cdot - x_j)^2; x\right) = 2\|x\|^2 - 2\sum_{j=1}^m x_j E_{n,j}(x),$$

where $\|\cdot\|$ stands for the Euclidean norm in $\mathbb{R}^m$. Here $E_{n,j}(x) := \widetilde{L}_n^*(pr_j; x)$, $x \in K_m$. Let $\mu_n := 2\|x\|^2 - 2\sum_{j=1}^m x_j E_{n,j}(x)$.

Taking into account a result established by Censor [5; Eq. (5)], we obtain

$$\|\widetilde{L}_n^* f - f\|_{K_n} \le 2\omega(f; \sqrt{\mu_n}),$$

where $\omega(f;\cdot)$ is the modulus of continuity defined for $f \in C(K_m)$ as follows

$$\omega(f;\delta) = \max_{\substack{x,y\in K_m \\ d(x,y)\le\delta}} |f(x) - f(y)|,$$

$d(\cdot,\cdot)$ being the Euclidean distance.

At the end of this section we emphasize other achievements as regards the sequences of operators of King's type. In [3], for a compact interval J and $I_n = \{0, 1, \ldots, n\}$, the limit of iterates of L_n^* operators defined as in (1.14) has been established. The tensor product extension of L_n^* to the bidimensional case was investigated in [2]. As a particular case, a modified variant of the bivariate Bernstein-Chlodovsky operators was presented. Recently, O. Duman and M.A. Özarslan [7] studied the modified Szász-Mirakjan operators S_n^* which preserve e_0 and e_2. They also proved that the order of approximation of a function f by $S_n^* f$ is at least as good as the order of approximation of f by $S_n f$.

Important results have been obtained by L. Rempulska and K. Tomczak [14] who entered upon certain sequences of linear positive operators acting on $C_p(I)$, $p \in \mathbb{N}_0$, the space of all functions $f : I \to \mathbb{R}$ with the property that $f w_p$ is bounded and uniformly continuous on I, where $w_0(x) = 1$ and $w_p(x) = (1 + x^p)^{-1}$, for $p \in \mathbb{N}$.

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