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On a generalization of Bleimann, Butzer and Hahn operators based on q-integers

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ABSTRACT

We propose a class of linear positive operators based on q-integers. These operators depend on a non-negative parameter and represent a generalization of the classical Bleimann, Butzer and Hahn operators. Approximation properties are presented and bounds of the error of approximation are established.

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1. Introduction

To make the exposure self-contained, we recall basic formulas in q-Calculus. Let $q > 0$. For each $k \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$, the q-integer $[k]_q$ and the q-factorial $[k]_q!$ are respectively defined by

$$[k]_q = 1 + q + \cdots + q^{k-1}, \qquad [k]_q! = [1]_q[2]_q \cdots [k]_q, \quad k \in \mathbb{N},$$

and $[0]_q = 0$, $[0]_q! = 1$. For integers $k \in \{0, 1, \ldots, n\}$, the q-binomial coefficients are denoted by $\begin{bmatrix} n \\ k \end{bmatrix}_q$ and are defined as follows

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

As particular cases, $[n]_1 = n$, $[n]_1! = n!$ and $\begin{bmatrix} n \\ k \end{bmatrix}_1$ represent $\binom{n}{k}$ the ordinary binomial coefficients. We also use the standard notation

$$(x-a)_q^0 = 1, \qquad (x-a)_q^n = \prod_{s=0}^{n-1}(x - aq^s), \quad n \in \mathbb{N}. \tag{1.1}$$

In order to approximate a continuous function on $\mathbb{R}_+ = [0, \infty)$, Bleimann et al. [1] introduced the following positive linear operators defined on the space $\mathbb{R}^{[0,\infty)}$

$$(L_n f)(x) = \frac{1}{(1+x)^n} \sum_{k=0}^{n} \binom{n}{k} x^k f\left(\frac{k}{n-k+1}\right), \quad x \geq 0. \tag{1.2}$$

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Over time, this class of operators has been deeply investigated and different generalizations have been obtained in numerous papers, see, for instance, [2–6] and the literature cited therein.

In 2007, Aral and Doğru [7] introduced and investigated the following q-BBH operators

$$(L_{n,q}f)(x) = \frac{1}{(1+x)_q^n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2}x^k f\left(\frac{[k]_q}{[n-k+1]_q q^k}\right), \tag{1.3}$$

see also [8,9]. Another q-analogue of BBH operators was introduced by Mahmudov and Sabancigil [10]. This class, let us say $H_{n,q}$, is related to $L_{n,q}$ as follows $(H_{n,q}f)(x) = (L_{n,q}f)(qx)$, $x \in \mathbb{R}_+$, and its construction is based on Lupaş q-Bernstein operator [11]. The authors showed that many properties of the original BBH operator are inherited by $H_{n,q}$.

In the present note, we define a general family of BBH-type operators depending on a parameter and constructed via q-Calculus. For $q > 0$, $\alpha \geq 0$ and each $n \in \mathbb{N}$, we consider the operators

$$(L_{n,q}^{\langle\alpha\rangle}f)(x) = \frac{1}{l_{n,q}^{\langle\alpha\rangle}(x)}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q f_k \prod_{i=0}^{k-1}(q^i x + \alpha[i]_q)\prod_{j=0}^{n-k-1}(1+\alpha[j]_q), \tag{1.4}$$

where

$$l_{n,q}^{\langle\alpha\rangle}(x) = \prod_{s=0}^{n-1}(1+q^s x + \alpha[s]_q), \qquad f_k = f\left(\frac{[k]_q}{[n-k+1]_q q^k}\right) \tag{1.5}$$

and $x \in \mathbb{R}_+$. Note, an empty product is considered to be 1.

One observes that these operators are positive and linear. For $\alpha = 0$, the operators (1.4) reduce to the q-BBH operators defined by (1.3). Moreover, $L_{n,1}^{\langle 0\rangle}$ represents the classical BBH operator defined by (1.2). On the other hand, for $q = 1$, $L_{n,1}^{\langle\alpha\rangle}$ turns into the operator L_n^{α} associated with the Pólya–Eggenberger distribution and introduced by Adell et al. [3].

Such q-generalizations depending on a real parameter have been recently achieved, see, for instance, [12,13].

The aim of this note is to study some properties of the operators defined by (1.4). By using the q-difference operator, we indicate another form of this class. Also, we give sufficient conditions which ensure that it becomes an approximation process and the rate of convergence is established in different function spaces.

2. Preliminary results

We continue to recall elements of q-Calculus. The book [14] can be a good guide. Let $q > 0$. We denote the q-derivative of a function $f : \mathbb{R} \to \mathbb{R}$ by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad x \neq 0, \qquad D_q f(0) = \lim_{x\to 0} D_q f(x),$$

and the high q-derivatives $D_q^0 f = f$, $D_q^n f = D_q(D_q^{n-1}f)$, $n \in \mathbb{N}$.

Let $q \in (0, 1)$. The most known q-analogue of integration for a function $f : \mathbb{R}_+ \to \mathbb{R}$ is given by

$$I_q(f; 0, a) = \int_0^a f(t)\mathrm{d}_q t = (1-q)a\sum_{j=0}^{\infty} q^j f(aq^j), \quad a > 0,$$

provided that the series of the right hand side is convergent. Over a general interval $[a, b]$, $0 < a < b$, one defines

$$I_q(f; a, b) = I_q(f; 0, b) - I_q(f; 0, a).$$

Also, one has [14, Eq. (20.2)]

$$\int_a^b D_q f(x)\mathrm{d}_q x = f(b) - f(a).$$

The q-version of Taylor's formula will be read as follows, [14, Eq. (4.1)]. For any polynomial $f(t)$ of degree N and any number c, we get

$$f(t) = \sum_{j=0}^{N}(D_q^j f)(c)\frac{(t-c)_q^j}{[j]_q!}. \tag{2.1}$$

Lemma 2.1. *For each $n \in \mathbb{N}$, one has*

$$\prod_{s=0}^{n-1}(1+\alpha[s]_q) = \sum_{l=0}^{n}(-1)^l q^{l(l-1)/2}\begin{bmatrix} n \\ l \end{bmatrix}_q \prod_{j=0}^{l-1}(q^j y + \alpha[j]_q)\prod_{i=l}^{n-1}(1+q^i y + \alpha[i]_q), \tag{2.2}$$

where $\alpha \geq 0$, $q > 0$ and $y \in \mathbb{R}$.

Proof. We expand $f(t)=(t+a)_q^n$ by using formula (2.1). Since

$$(D_q^l f)(t)=[n]_q[n-1]_q\cdots[n-l+1]_q(t+a)_q^{n-l}$$

for any $l\le n$, the q-Taylor formula gives

$$(t+a)_q^n=\sum_{l=0}^{n}\begin{bmatrix} n\\ l\end{bmatrix}_q (t-c)_q^l(c+a)_q^{n-l}. \tag{2.3}$$

In the above we replace q by q^{-1}. Taking into account both relation (1.1) and the identity $\begin{bmatrix} n\\ l\end{bmatrix}_{1/q}=\begin{bmatrix} n\\ l\end{bmatrix}_q q^{-l(n-l)}$, we can rewrite (2.3) as

$$\prod_{s=0}^{n-1}(t+q^{-n+1}aq^s)=\sum_{l=0}^{n}\begin{bmatrix} n\\ l\end{bmatrix}_q\prod_{i=0}^{l-1}(t-cq^{-i})\prod_{j=0}^{n-l-1}(q^{-l}c+aq^{-l-j}).$$

In this identity making the change $l:=n-l$, we can write successively

$$\begin{aligned}\prod_{s=0}^{n-1}(t+q^{-n+1}aq^s)&=\sum_{l=0}^{n}\begin{bmatrix} n\\ l\end{bmatrix}_q\prod_{j=0}^{l-1}(q^{-n+l}c+aq^{-n+l-j})\prod_{i=0}^{n-l-1}(t-cq^{-i})\\&=\sum_{l=0}^{n}\begin{bmatrix} n\\ l\end{bmatrix}_q\prod_{j=0}^{l-1}(q^{-n+l}c+aq^{-n+1+j})\prod_{i=0}^{n-l-1}(t-cq^{-i})\\&=\sum_{l=0}^{n}\begin{bmatrix} n\\ l\end{bmatrix}_q q^{l(l-1)/2}\prod_{j=0}^{l-1}(q^{-j-n+l}c+aq^{-n+1})\prod_{i=0}^{n-l-1}(t-cq^{-n+l+1+i})\\&=\sum_{l=0}^{n}\begin{bmatrix} n\\ l\end{bmatrix}_q q^{l(l-1)/2}\prod_{j=0}^{l-1}(q^jq^{-n+1}c+aq^{-n+1})\prod_{i=l}^{n-1}(t-cq^iq^{-n+1}).\end{aligned}$$

Putting

$$t=1+\frac{\alpha}{1-q},\qquad a=-\frac{\alpha}{1-q}q^{n-1},\qquad c=\left(\frac{\alpha}{1-q}-y\right)q^{n-1},$$

we arrive at (2.2). □

For any $k\in\mathbb{N}_0$, the kth order q-difference operator of a function f is recursively defined by

$$\Delta_q^0 f_j=f_j\quad\text{and}\quad\Delta_q^{k+1}f_j=\Delta_q^k f_{j+1}-q^k\Delta_q^k f_j, \tag{2.4}$$

where f_j denotes the value of the function f on the knot x_j belonging to a certain net of an interval included in the domain of f. The kth q-difference $\Delta_q^k f_j$ can be expressed as a sum of multiples of values of f as follows, see [15, p. 46],

$$\Delta_q^k f_j=\sum_{r=0}^{k}(-1)^r q^{r(r-1)/2}\begin{bmatrix} k\\ r\end{bmatrix}_q f_{j+k-r}. \tag{2.5}$$

For our purposes, we consider f_j given by (1.5). In this circumstance, formula (2.5) is valid for any $j\in\{0,1,\ldots,n\}$ and $k\in\{0,1,\ldots,n-j-1\}$.

In what follows we indicate another look at our operators.

Theorem 2.1. *The operators defined by (1.4) can be expressed in the form*

$$(L_{n,q}^{\langle\alpha\rangle}f)(x)=\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q\Delta_q^k f_0\prod_{i=0}^{k-1}\frac{q^ix+\alpha[i]_q}{1+q^ix+\alpha[i]_q}, \tag{2.6}$$

$x\ge 0$, *where* $\Delta_q^k f_0$ *is defined as in (2.4).*

Proof. By using (2.2) in relation (1.4), we get

$$\begin{aligned}(L_{n,q}^{\langle\alpha\rangle}f)(x)&=\frac{1}{l_{n,q}^{\langle\alpha\rangle}(x)}\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q f_k\prod_{i=0}^{k-1}(q^ix+\alpha[i]_q)\\&\quad\times\sum_{r=0}^{n-k}(-1)^r q^{r(r-1)/2}\begin{bmatrix} n-k\\ r\end{bmatrix}_q\prod_{j=0}^{r-1}(q^jy+\alpha[j]_q)\prod_{s=r}^{n-k-1}(1+q^sy+\alpha[s]_q),\end{aligned} \tag{2.7}$$

where $y \in \mathbb{R}$. Substituting $y = q^k x + \alpha[k]_q$, we get

$$\prod_{j=0}^{r-1}(q^j y + \alpha[j]_q) = \prod_{j=k}^{r+k-1}(q^j x + \alpha[j]_q)$$

and

$$\prod_{s=r}^{n-k-1}(1 + q^s y + \alpha[s]_q) = \prod_{s=r+k}^{n-1}(1 + q^s x + \alpha[s]_q).$$

Returning to (2.7), we can write

$$(L_{n,q}^{\langle\alpha\rangle} f)(x) = \frac{1}{l_{n,q}^{\langle\alpha\rangle}(x)} \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix}_q f_r \sum_{k=0}^{n-r} (-1)^k q^{k(k-1)/2} \begin{bmatrix} n-r \\ k \end{bmatrix}_q \prod_{i=0}^{r+k-1} (q^i x + \alpha[i]_q) \prod_{s=r+k}^{n-1} (1 + q^s x + \alpha[s]_q).$$

In view of the equality

$$\begin{bmatrix} n \\ r \end{bmatrix}_q \begin{bmatrix} n-r \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k+r \end{bmatrix}_q \begin{bmatrix} k+r \\ r \end{bmatrix}_q,$$

we obtain

$$\begin{aligned}(L_{n,q}^{\langle\alpha\rangle} f)(x) &= \frac{1}{l_{n,q}^{\langle\alpha\rangle}(x)} \sum_{r=0}^{n} \sum_{k=r}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k \\ r \end{bmatrix}_q f_r (-1)^{k-r} q^{(k-r)(k-r-1)/2} \prod_{i=0}^{k-1}(q^i x + \alpha[i]_q) \prod_{s=k}^{n-1}(1 + q^s x + \alpha[s]_q) \\ &= \frac{1}{l_{n,q}^{\langle\alpha\rangle}(x)} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \prod_{i=0}^{k-1}(q^i x + \alpha[i]_q) \prod_{s=k}^{n-1}(1 + q^s x + \alpha[s]_q) \sum_{r=0}^{k} (-1)^r q^{r(r-1)/2} \begin{bmatrix} k \\ r \end{bmatrix}_q f_{k-r}.\end{aligned}$$

Knowing the expression of the polynomial $l_{n,q}^{\langle\alpha\rangle}$, condition (2.5) completes the proof of our theorem. □

Lemma 2.2. *Considering the functions φ^k, $k \in \mathbb{N}_0$, $\varphi : \mathbb{R}_+ \to [0, 1)$, $\varphi(t) = t/(t+1)$, the following identities*

$$\Delta_q^s \varphi_j^k = 0, \quad j \in \{0, 1, \ldots, n\}, \tag{2.8}$$

hold for each $s \geq k + 1$.

Proof. On the basis of (1.5), $\varphi_j^k = \varphi^k\left(\frac{[j]_q}{[n-j+1]_q q^j}\right) = \left(\frac{[j]_q}{[n+1]_q}\right)^k$. Taking into account (2.4) and following [15, p. 268], the statement is obvious. □

Theorem 2.2. *Let $L_{n,q}^{\langle\alpha\rangle}$, $n \in \mathbb{N}$, be defined by (1.4). For each $n \in \mathbb{N}$ and $x \geq 0$, the following relations hold.*

$$(L_{n,q}^{\langle\alpha\rangle} \mathbf{1})(x) = 1, \tag{2.9}$$

$$(L_{n,q}^{\langle\alpha\rangle} \varphi)(x) = \left(1 - \frac{q^n}{[n+1]_q}\right) \frac{x}{1+x}, \tag{2.10}$$

$$(L_{n,q}^{\langle\alpha\rangle} \varphi^2)(x) = \frac{x}{1+x} \frac{qx+\alpha}{1+qx+\alpha} + \frac{1}{[n+1]_q} \cdot \frac{x}{1+x} K(n, \alpha, x), \tag{2.11}$$

where

$$K(n, \alpha, x) = \frac{[n]_q}{[n+1]_q} - \frac{qx+\alpha}{1+qx+\alpha} \left(1 + [2]_q q^n \frac{[n]_q}{[n+1]+q}\right). \tag{2.12}$$

If $q \in (0, 1]$, then

$$|K(n, \alpha, x)| \leq 4, \quad \textit{for each } \alpha \geq 0. \tag{2.13}$$

Here **1** *represents the function on $\mathbb{R}_+$ of constant value 1 and φ is defined as in Lemma 2.2.*

Proof. Identity (2.9) is implied by the relations (2.6) and (2.8). Further on, taking the advantage of (2.5), we deduce $\Delta_q^0 \varphi_0 = \varphi_0 = 0$, $\Delta_q^1 \varphi_0 = \varphi_1 - \varphi_0 = \frac{1}{[n+1]_q}$ and $\Delta_q^s \varphi_0 = 0$ for $s \geq 2$, see (2.8). By using Theorem 2.1 and knowing $[n]_q = [n+1]_q - q^n$, we arrive at (2.10).

On the basis of (2.5), for φ^2 we have

$$\Delta_q^0\varphi_0^2 = 0, \qquad \Delta_q^1\varphi_0^2 = \frac{1}{[n+1]_q^2}, \qquad \Delta_q^2\varphi_0^2 = \frac{q[2]_q}{[n+1]_q^2}.$$

Applying again both Theorem 2.1 and Lemma 2.2, we can write

$$\begin{aligned}(L_{n,q}^{\langle\alpha\rangle}\varphi^2)(x) &= \frac{[n]_q}{[n+1]_q^2}\frac{x}{1+x} + \frac{q[n]_q[n-1]_q}{[n+1]_q^2}\frac{x}{1+x}\frac{qx+\alpha}{1+qx+\alpha} \\ &= \frac{[n+1]_q - q^n}{[n+1]_q^2}\frac{x}{1+x} + \frac{([n+1]_q-1)([n+1]_q - q^{n-1}[2]_q)}{[n+1]_q^2}\frac{x}{1+x}\frac{qx+\alpha}{1+qx+\alpha} \\ &= \frac{x}{1+x}\frac{qx+\alpha}{1+qx+\alpha} + \frac{1}{[n+1]_q}\frac{x}{1+x} \\ &\quad\times\left(\frac{1}{[n+1]_q}\left(q^{n-1}[2]_q\frac{qx+\alpha}{1+qx+\alpha} - q^n\right) + 1 - (1+q^{n-1}[2]_q)\frac{qx+\alpha}{1+qx+\alpha}\right).\end{aligned}$$

Taking into account (2.12), by a straightforward calculation, we obtain (2.11).

In view of (2.12), under the hypothesis $q \in (0, 1]$, the statement (2.13) is evident. □

3. Approximation properties

Our aim is to establish when $(L_{n,q}^{\langle\alpha\rangle})_n$ is an approximation process in certain spaces of functions.

$C_B(\mathbb{R}_+)$ stands for the space of the continuous bounded real valued functions defined on $\mathbb{R}_+$ endowed with the usual sup-norm $\|\cdot\|_\infty$.

Also, let $C^*(\mathbb{R}_+)$ be the Banach lattice of all continuous functions f on $\mathbb{R}_+$ such that $\lim_{x\to\infty} f(x)$ exists and is finite endowed with the same norm $\|\cdot\|_\infty$. For any $f \in C(\mathbb{R}_+)$ and $0 \le a < b$, set $\|f\|_{[a,b]} = \sup_{a\le x\le b}|f(x)|$. One observes $\|f\|_{[0,1]} = \|f\circ\varphi\|_\infty$ for any $f \in C(\mathbb{R}_+)$, where φ was introduced in Lemma 2.2. We also mention that (2.9) implies the fact that $L_{n,q}^{\langle\alpha\rangle}$ is a non-expansive operator, this means $\|L_{n,q}^{\langle\alpha\rangle} f\|_\infty \le \|f\|_\infty$, for any $f \in C_B(\mathbb{R}_+)$. Since $L_{n,q}^{\langle\alpha\rangle} f$ is a rational function on $\mathbb{R}_+$ satisfying $\lim_{x\to\infty}(L_{n,q}^{\langle\alpha\rangle} f)(x) = f([n]_q q^{-n})$, the relation (1.4) implies

$$L_{n,q}^{\langle\alpha\rangle}(C^*(\mathbb{R}_+)) \subset C^*(\mathbb{R}_+). \tag{3.1}$$

Gadjiev and Çakar [16] presented a Korovkin type theorem on uniform approximation of a certain subspace of $C_B(\mathbb{R}_+)$. Their result involved the test functions φ^ν, $\nu = 0, 1, 2$, where φ was indicated by us in Lemma 2.2. The authors also gave a simple proof of the corresponding statement for BBH-operators. In the framework of weighted approximation, a more general theorem was obtained by Bustamante and Morales de la Cruz [17]. In this comprehensive survey, a particular case of Theorem 4.3 will be read as follows.

Theorem 3.1 (*[17]*). *A sequence $(L_n)_n$ of positive linear operators $L_n : C^*(\mathbb{R}_+) \to C^*(\mathbb{R}_+)$ is an approximation process if and only if*

$$\lim_n \|L_n\varphi^i - \varphi^i\|_\infty = 0, \quad \textit{for } i = 0, 1, 2.$$

Thanks to relation (3.1), we can apply this theorem to our operators. We also point out the following. As Gadjiev and Çakar proved [16, Theorem 3.3], the above result does not hold in the whole space $C_B(\mathbb{R}_+)$. Other strict Korovkin subspaces of $C(\mathbb{R}_+)$ can be found in the monograph of Altomare and Campiti [18, Section 4.2].

Examining (2.11) we deduce $\lim_n L_{n,q}^{\langle\alpha\rangle}\varphi^2 \ne \varphi^2$ for any fixed $q \in (0, 1)$ and $\alpha \ge 0$. Consequently, our operators $L_{n,q}^{\langle\alpha\rangle}$, $n \in \mathbb{N}$, still do not form an approximation process on $C^*(\mathbb{R}_+)$. To enjoy this property, for each $n \in \mathbb{N}$, the constants q and α will be replaced by the numbers $q_n \in (0, 1)$ and $\alpha_n \in \mathbb{R}_+$.

Theorem 3.2. *Let $(q_n)_{n\ge1}$, $(\alpha_n)_{n\ge1}$ be real sequences such that $0 < q_n < 1$ and $\alpha_n \ge 0$. Let $L_{n,q_n}^{\langle\alpha_n\rangle}$, $n \in \mathbb{N}$, be defined as in (1.4). If $\lim_n q_n = 1$ and $\lim_n \alpha_n = 0$, for each $f \in C^*(\mathbb{R}_+)$ one has*

$$\lim_n (L_{n,q_n}^{\langle\alpha_n\rangle} f)(x) = f(x), \quad \textit{uniformly in } x \in \mathbb{R}_+. \tag{3.2}$$

Proof. We check the three conditions of Theorem 3.1. In view of (2.9), the first of them is evident. From (2.10) we get

$$|(L_{n,q_n}^{\langle\alpha_n\rangle}\varphi)(x) - \varphi(x)| \le \frac{q_n^n}{[n+1]_{q_n}} \le \frac{1}{[n+1]_{q_n}}, \quad x \in \mathbb{R}_+. \tag{3.3}$$

Further on, from (2.10) and (2.13) we obtain

$$|(L_{n,q_n}^{\langle\alpha_n\rangle}\varphi^2)(x)-\varphi^2(x)| \le \frac{x}{(x+1)^2}\frac{(1-q_n)x+\alpha_n}{1+q_nx+\alpha_n}+\frac{1}{[n+1]_{q_n}}\frac{4x}{x+1}$$
$$\le \max\{1-q_n,\alpha_n\}+\frac{4}{[n+1]_{q_n}},\quad x\in\mathbb{R}_+. \tag{3.4}$$

The requirements imposed on $(q_n)_n$ and $(\alpha_n)_n$ ensure $\lim_n \|L_{n,q_n}^{\langle\alpha_n\rangle}\varphi^s-\varphi^s\|_\infty = 0$ for $s = 1$ and $s = 2$. This ends the proof. □

Remark 3.1. For a given $q \in (0, 1)$ and $\alpha \ge 0$, for each $x \in \mathbb{R}_+$ one has

$$L_{n,q}^{\langle\alpha\rangle}((\varphi-\varphi(x))^2,x)\le \max\{1-q,\alpha\}+\frac{6}{[n+1]_q}:=\beta. \tag{3.5}$$

The statement is a consequence of Theorem 3.2.

Indeed, since $L_{n,q}$ is a linear, positive operator and reproduces the constants, we can write

$$L_{n,q}^{\langle\alpha\rangle}((\varphi-\varphi(x))^2,x) = L_{n,q}^{\langle\alpha\rangle}(\varphi^2-\varphi^2(x),x)+2\varphi(x)(\varphi(x)-(L_n\varphi)(x))$$
$$\le |(L_{n,q}^{\langle\alpha\rangle}\varphi^2)(x)-\varphi^2(x)|+|(L_n\varphi)(x)-\varphi(x)|.$$

Combining (3.3) and (3.4), $q_n = q$, $\alpha_n = \alpha$, we arrive at (3.5).

Starting from Theorem 3.2 we can obtain the uniform convergence of $L_{n,q_n}^{\langle\alpha_n\rangle} f$ to f for any $f \in C(\mathbb{R}_+)$. The disadvantage: the uniform convergence is not valid on the whole positive semiaxis but only on compact intervals included in $\mathbb{R}_+$. The advantage: the uniform convergence is valid for unbounded continuous functions.

Theorem 3.3. *Let $(q_n)_{n\ge1}$, $(\alpha_n)_{n\ge1}$ be real sequences such that $0 < q_n < 1$ and $\alpha_n \ge 0$. Let $L_{n,q_n}^{\langle\alpha_n\rangle}$, $n \in \mathbb{N}$, be defined as in (1.4). If $\lim_n q_n = 1$ and $\lim_n \alpha_n = 0$, for each $f \in C(\mathbb{R}_+)$ and any $[a, b] \subset \mathbb{R}_+$ one has*

$$\lim_{n\to\infty}\|L_{n,q_n}^{\langle\alpha_n\rangle}f-f\|_{[a,b]}=0. \tag{3.6}$$

Proof. For a given compact $[a, b]$, $0 \le a < b$, and an arbitrary function $f \in C(\mathbb{R}_+)$, we define the maps

$$\theta_{1,f}(t)=\begin{cases}(1+t-a)f(t), & t\in[0,a],\\ f(t), & t\in(a,b),\\ (-t+1+b)f(t), & t\in[b,b+1),\\ 0, & t\ge b+1,\end{cases}$$
$$\theta_{2,f}(t)=\begin{cases}(-t+a)f(t), & t\in[0,a],\\ 0, & t\in(a,b),\\ (t-b)f(t), & t\in[b,b+1),\\ f(t), & t\ge b+1.\end{cases}$$

Clearly, $\theta_{1,f} \in C_B(\mathbb{R}_+)$, $\theta_{2,f} \in C(\mathbb{R}_+)$ and $f = \theta_{1,f} + \theta_{2,f}$. We can write

$$\|L_{n,q_n}^{\langle\alpha_n\rangle}f-f\|_{[a,b]} \le \|L_{n,q_n}^{\langle\alpha_n\rangle}\theta_{1,f}-\theta_{1,f}\|_{[a,b]}+\|L_{n,q_n}^{\langle\alpha_n\rangle}\theta_{2,f}-\theta_{2,f}\|_{[a,b]}$$
$$= \|L_{n,q_n}^{\langle\alpha_n\rangle}\theta_{1,f}-\theta_{1,f}\|_{[a,b]} \le \|L_{n,q_n}^{\langle\alpha_n\rangle}\theta_{1,f}-\theta_{1,f}\|_{\infty}.$$

Since $\theta_{1,f} \in C^*(\mathbb{R}_+)$, in view of Theorem 3.2, relation (3.6) follows. □

Theorem 3.4. *Let the operators $L_{n,q}^{\langle\alpha\rangle}$, $n \in \mathbb{N}$, be defined by (1.4). There exists $\overline{q} \in (0, 1)$ such that for any $q \in (\overline{q}, 1)$ the following inequality*

$$|L_{n,q}^{\langle\alpha\rangle}(f\circ\varphi,x)-(f\circ\varphi)(x)|\le\frac{\|D_qf\circ\varphi\|_\infty}{[n+1]_q}+\sqrt{\beta}(\sqrt{\beta}+1)\|D_q^2f\circ\varphi\|_\infty \tag{3.7}$$

holds, where φ was introduced at Lemma 2.2 and β is given by (3.5).

Proof. Let $x \in \mathbb{R}_+$ be arbitrarily fixed. Using q-Taylor's formula with the Cauchy remainder [14, Eq. (20.4)], for each $t \in \mathbb{R}_+$ we can write

$$f\left(\frac{t}{t+1}\right)=f\left(\frac{x}{x+1}\right)+D_qf\left(\frac{x}{x+1}\right)\left(\frac{t}{t+1}-\frac{x}{x+1}\right)+R_{f,q,x}(t), \tag{3.8}$$

where

$$R_{f,q,x}(t) = \int_{x/(x+1)}^{t/(t+1)} \left(\frac{t}{t+1} - q\nu\right) D_q^2 f(\nu) \mathrm{d}_q \nu.$$

On the basis of an estimation formula for the remainder term [19, Theorem 5.3], there exists $\overline{q} \in (0, 1)$ such that for all $q \in (\overline{q}, 1)$ one has

$$R_{f,q,x}(t) = \frac{D_q^2 f(\xi_{t,x})}{[2]_q!} \left(\frac{t}{t+1} - \frac{x}{x+1}\right) \left(\frac{t}{t+1} - q\frac{x}{x+1}\right), \tag{3.9}$$

where $\xi_{t,x} \in (u, v) \subset [0, 1]$, $u = \min\{\varphi(x), \varphi(t)\}$, $v = \max\{\varphi(x), \varphi(t)\}$.

Applying $L_{n,q}^{\langle\alpha\rangle}$ to relation (3.8) and using (2.9), we get

$$L_{n,q}^{\langle\alpha\rangle}(f \circ \varphi, x) - (f \circ \varphi)(x) = D_q f \circ \varphi(x) L_{n,q}^{\langle\alpha\rangle}(\varphi - \varphi(x), x) + L_{n,q}^{\langle\alpha\rangle}(R_{f,q,x}, x). \tag{3.10}$$

On the other hand, from (3.9) we obtain

$$\begin{aligned} L_{n,q}^{\langle\alpha\rangle}(R_{f,q,x}, x) &\le \frac{\|D_q^2 f\|_{[0,1]}}{[2]_q!} |L_{n,q}^{\langle\alpha\rangle}((\varphi - \varphi(x))(\varphi - q\varphi(x)), x)| \\ &= \frac{\|D_q^2 f\|_{[0,1]}}{[2]_q!} |L_{n,q}^{\langle\alpha\rangle}((\varphi - \varphi(x))^2 + (1-q)\varphi(x)(\varphi - \varphi(x)), x)| \\ &\le \|D_q^2 f\|_{[0,1]} (L_{n,q}^{\langle\alpha\rangle}((\varphi - \varphi(x))^2, x) + |L_{n,q}^{\langle\alpha\rangle}(\varphi - \varphi(x), x)|) \\ &\le \|D_q^2 f \circ \varphi\|_\infty (L_{n,q}^{\langle\alpha\rangle}((\varphi - \varphi(x))^2, x) + \sqrt{L_{n,q}^{\langle\alpha\rangle}((\varphi - \varphi(x))^2, x)}). \end{aligned}$$

The last increase was obtained by applying Schwarz's inequality and knowing (2.9). We return at (3.10) and, in view of (3.5) and (3.3), the conclusion (3.7) follows. □

Remark 3.2. Under the additional conditions which guarantee that our sequence is an approximation process, see Theorem 3.3, we deduce that $\beta_n = \max\{1 - q_n, \alpha_n\} + 6/[n+1]_{q_n}$ tends to zero for n tending to infinity. Consequently, (3.7) is an appropriate upper bound of the error of approximation.

The next result should be of interest in its own right.

Lemma 3.1. *If $g \in C([0, 1])$, then there exists $\overline{q} \in (0, 1)$ such that for any $q \in (\overline{q}, 1)$ the following inequality*

$$\|D_q g\|_{[0,1]} \le 2(\|g\|_{[0,1]} + \|D_q^2 g\|_{[0,1]}) \tag{3.11}$$

holds.

Proof. Let $h \in C([0, 2])$, $h(x) = g(x)$ for $x \in [0, 1]$ and $h(x) = g(1)$ for $x \in (1, 2]$. Using the q-Taylor formula we write

$$h(x+1) = h(x) + D_q h(x) + \int_x^{x+1} (x + 1 - q\nu) D_q^2 g(\nu) \mathrm{d}_q \nu. \tag{3.12}$$

Using again [19, Theorem 5.3], one finds $\overline{q} \in (0, 1)$ such that for all $q \in (\overline{q}, 1)$ one has

$$\int_x^{x+1} (x + 1 - q\nu) \mathrm{d}_q^2 g(\nu) \mathrm{d}_q \nu = \frac{D_q^2 h(\xi_x)}{[2]_q!} (1 + x(1-q)),$$

where $x \le \xi_x \le x + 1$. Relation (3.12) implies

$$\|D_q h\|_{[0,1]} \le \|h\|_{[0,1]} + \|h\|_{[1,2]} + 2\|D_q^2 h\|_{[0,2]}.$$

Since $\|h\|_{[1,2]} \le \|h\|_{[0,1]}$, $\|D_q^2 h\|_{[1,2]} = 0$ and $h = g$ on $[0, 1]$, from the previous inequality we obtain (3.11). □

Taking into account this result, after some calculations, Theorem 3.4 can be reformulated as follows.

Theorem 3.5. *Let $L_{n,q}^{\langle\alpha\rangle}$, $n \in \mathbb{N}$, be the operators defined by (1.4). There exists $\overline{q} \in (0, 1)$ such that for any $q \in (\overline{q}, 1)$ the following inequality*

$$\left|(L_{n,q}^{\langle\alpha\rangle} f)\left(\frac{x}{x+1}\right) - f\left(\frac{x}{x+1}\right)\right| \le \frac{2}{[n+1]_q} \|f\|_{[0,1]} + \left(\frac{2}{[n+1]_q} + \sqrt{\beta}(\sqrt{\beta} + 1)\right) \|D_q^2 f\|_{[0,1]},$$

holds, where β is given by (3.5).

Definition 3.1. A function $\Omega : \mathbb{R}_+ \to \mathbb{R}_+$ is called a modulus of continuity if it satisfies the following properties: $\lim_{\delta\to 0^+} \Omega(\delta) = 0$, Ω is non-decreasing, Ω is subadditive.

For a given modulus Ω, following [16], let H_Ω be the space of all real-valued functions f defined in $\mathbb{R}_+$ and satisfying the condition

$$|f(t) - f(x)| \le \Omega\left(\left|\frac{t}{t+1} - \frac{x}{x+1}\right|\right), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}_+. \tag{3.13}$$

It was proved that $H_\Omega \subset C_B(\mathbb{R}_+)$.

Theorem 3.6. *Let $L_{n,q}^{\langle\alpha\rangle}$, $n \in \mathbb{N}$, be defined by (1.4). For any function $f \in H_\Omega$, the following inequality*

$$|(L_{n,q}^{\langle\alpha\rangle} f)(x) - f(x)| \le (1 + \beta^\tau)\Omega(\beta^{1/2-\tau})$$

holds, where τ arbitrarily belongs to $[0, 1/2)$ and β is given in (3.5).

Proof. Let f belong to H_Ω. If Ω is a modulus of continuity, then $\Omega(\lambda\delta) \le (1 + \lambda)\Omega(\delta)$, $\lambda \ge 0$. This inequality and relation (3.13) imply

$$|f(t) - f(x)| \le \left(1 + \frac{1}{\delta}\left|\frac{t}{t+1} - \frac{x}{x+1}\right|\right)\Omega(\delta), \quad \delta > 0. \tag{3.14}$$

Since $L_{n,q}^{\langle\alpha\rangle}$ is a linear positive operator reproducing the constants, taking advantage of (3.14), we can write

$$\begin{aligned} |(L_{n,q}^{\langle\alpha\rangle} f)(x) - f(x)| &\le L_{n,q}^{\langle\alpha\rangle}(|f - f(x)|; x) \\ &\le \left(1 + \frac{1}{\delta} L_{n,q}^{\langle\alpha\rangle}(|\varphi - \varphi(x)|, x)\right)\Omega(\delta) \\ &\le \left(1 + \frac{1}{\delta}\sqrt{L_{n,q}^{\langle\alpha\rangle}((\varphi - \varphi(x))^2, x)}\right)\Omega(\delta) \\ &\le \left(1 + \frac{\sqrt{\beta}}{\delta}\right)\Omega(\delta). \end{aligned}$$

We also used Schwarz's inequality and relation (3.5). Choosing $\delta = \beta^{1/2-\tau}$, one gets the claimed result. □

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