From Uniform to Statistical Convergence of Binomial-Type Operators

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Abstract Sequences of binomial operators introduced by using umbral calculus are investigated from the point of view of statistical convergence. This approach is based on a detailed presentation of delta operators and their associated basic polynomials. Bernstein–Sheffer linear positive operators are analyzed, and some particular cases are highlighted: Cheney–Sharma operators, Stancu operators, Lupaş operators.

Keywords Statistical convergence · Binomial sequence · Linear positive operator · Umbral calculus · Bernstein–Sheffer operator · Pincherle derivative

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1 Introduction

Let $(L_n)_{n\geq 1}$ be a sequence of linear positive operators acting on the space $C([a, b])$ of all real-valued and continuous functions defined on the interval $[a, b]$, equipped with the norm $\|\cdot\|$ of the uniform convergence, namely $\|h\| = \sup_{a\leq t\leq b} |h(t)|$. Bohman–Korovkin's theorem asserts: If the operators L_n, $n \in \mathbb{N}$, map $C([a, b])$ into itself such that

$$\lim_{n\to\infty} \|L_n e_j - e_j\| = 0 \text{ for } j \in \{0, 1, 2\}, \tag{1.1}$$

then one has

$$\lim_{n\to\infty} \|L_n f - f\| = 0 \text{ for every } f \in C([a, b]). \tag{1.2}$$

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In the above, e_j represents the monomial of jth degree, $e_0(x) = 1$ and $e_j(x) = x^j$, $j \geq 1$.

A current subject in approximation theory is the approximation of continuous functions by using the statistical convergence, the first research of this topic being done by Gadjiev and Orhan [5]. This approach models and improves the technique of signals' approximation in different function spaces.

On the other hand, sequences of polynomials of binomial type have been the subject of many mathematical studies, drawing to light their role in approximation theory. Practically, the theory of the approximation operators of binomial type is based on the technique of the *umbral calculus*. In its modern form, this is a strong tool for calculations with polynomials representing a successful combination between the finite differences calculus and certain chapters of probability theory. The topic discussed in this chapter is at the confluence of the two concepts mentioned above, statistical convergence and binomial-type operators, from the point of view of the approximation of some function classes. The material is structured in three sections.

First of all, we recall the variant of Bohman–Korovkin theorem via statistical convergence and we present elementary facts about polynomial sequences of binomial type. Further on, we deal with delta operators and their basic polynomials. In the last section, we will analyze the approximation properties of some binomial operators in terms of the statistical convergence.

We mention that at the first sight this work seems disproportionate, dominated by a lot of notions introduced and results already achieved. The goal was to be self-contained paper. It will be seen that for a clear understanding of the last paragraph, it was necessary to structure the article in this way.

2 Preliminaries

The concept of statistical convergence was first defined by Steinhaus [13] and Fast [4]. It is based on the notion of the asymptotic density of subsets of $\mathbb{N}$. The density of $S \subseteq \mathbb{N}$ denoted by $\delta(S)$ is given by

$$\delta(S) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \chi_S(j),$$

where χ_S stands for the characteristic function of the set S. Clearly, $0 \leq \delta(S) \leq \delta(\mathbb{N}) = 1$. A sequence $(x_n)_{n \geq 1}$ of real numbers is said to be statistically convergent to a real number l, if, for every $\varepsilon > 0$,

$$\delta(\{n \in \mathbb{N} : \ |x_n - l| \geq \varepsilon\}) = 0,$$

the limit being denoted by $st - \lim_{n \to \infty} x_n = l$. It is known that any convergent sequence is statistically convergent but the converse of this statement is not true. Even though this notion was introduced in 1951, its application to the study of sequences of positive linear operators was attempted only in 2002. We refer to the A.D. Gadjiev and C. Orhan [5] result, which reads as follows.

Theorem 2.1 *If the sequence of positive linear operators* $L_n : C([a,b]) \to B([a,b])$ *satisfies the condition*

$$st - \lim_{n\to\infty} \|L_n e_j - e_j\| = 0, \ j \in \{0, 1, 2\}, \tag{2.1}$$

then one has

$$st - \lim_{n\to\infty} \|L_n f - f\| = 0 \textit{ for every function } f \in C([a,b]). \tag{2.2}$$

As usual, $B([a,b])$ stands for the space of all real-valued bounded functions defined on $[a,b]$, endowed with the sup norm. The identities (2.1) and (2.2) generalize, respectively, relations (1.1), (1.2). From this moment, the statistical convergence of positive linear operators represented a new direction in the study of so-called KAT—Korovkin-type approximation theory.

Set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For any $n \in \mathbb{N}_0$, we denote by Π_n the linear space of polynomials of degree no greater than n and by Π_n^* the set of all polynomials of degree n. We also set

$$\Pi := \bigcup_{n\geq 0} \Pi_n,$$

representing the commutative algebra of polynomials with coefficients in $\mathbb{K}$, this symbol standing either for the field $\mathbb{R}$ or for the field $\mathbb{C}$.

A sequence $p = (p_n)_{n\geq 0}$ such that $p_n \in \Pi_n^*$ for every $n \in \mathbb{N}_0$ is called a *polynomial sequence*.

Definition 2.2 A polynomial sequence $b = (b_n)_{n\geq 0}$ is called of binomial type if for any $(x, y) \in \mathbb{K} \times \mathbb{K}$ the following identities hold

$$b_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} b_k(x) b_{n-k}(y), n \in \mathbb{N}_0. \tag{2.3}$$

Remark 2.3 Knowing that $\deg(b_0) = 0$, we get $b_0(x) = 1$ for any $x \in \mathbb{K}$ and by induction we easily obtain $b_n(0) = 0$ for any $n \in \mathbb{N}$.

The most common example of binomial sequence is $e = (e_n)_{n\geq 0}$ (the monomials). Some nontrivial examples are given below.

(a) *The generalized factorial power with the step a:* $p = (p_n)_{n\geq 0}$,

$$p_0(x) = x^{[0,a]} := 1 \text{ and } p_n(x) = x^{[n,a]} := x(x-a)\cdots(x-(n-1)a), \ n \in \mathbb{N}.$$

The Vandermonde formula, i.e.,

$$(x+y)^{[n,a]} = \sum_{k=0}^{n} \binom{n}{k} x^{[k,a]} y^{[n-k,a]},$$

guarantees that this is a binomial-type sequence. There are two particular cases: For $a = 1$, we obtain the *lower factorials* which, usually, are denoted by $\langle x \rangle_n$; for $a = -1$, we obtain the *upper factorials* denoted by Pochhammer's symbol $(x)_n$. By convention, we consider

$$x^{[-n,a]} := 1/(x+na)^{[n,a]}.$$

(b) *Abel polynomials:* $\widetilde{a} = (a_n^{\langle a \rangle})_{n \ge 0}$,

$$a_0^{\langle a \rangle} = 1,\ a_n^{\langle a \rangle}(x) = x(x-na)^{n-1},\ n \in \mathbb{N},\ a \neq 0.$$

Rewriting the identity (2.3) for these polynomials, we obtain the Abel-Jensen (1902) combinatorial formula

$$(x+y)(x+y+na)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} xy(x+ka)^{k-1}(y+(n-k)a)^{n-1-k},\ n \in \mathbb{N}.$$

(c) *Gould polynomials:* $g = (g_n^{\langle a,b \rangle})_{n \ge 0}$,

$$g_0^{\langle a,b \rangle} = 1,\ g_n^{\langle a,b \rangle}(x) = \frac{x}{x-an} \left\langle \frac{x-an}{b} \right\rangle_n,\ n \in \mathbb{N},\ ab \neq 0.$$

The space of all linear operators $T : \Pi \to \Pi$ will be denoted by $\mathscr{L}$. Among these operators, an important role will be played by the *shift operator*, named E^a. For every $a \in \mathbb{K}$, E^a is defined by

$$(E^a p)(x) = p(x+a),\ \text{where } p \in \Pi.$$

An operator $T \in \mathscr{L}$ which switches with all shift operators, that is

$$TE^a = E^a T \text{ for every } a \in \mathbb{K},$$

is called a *shift-invariant operator*, and the set of these operators is denoted by $\mathscr{L}_s$.

3 On Delta Operators

Definition 3.1 An operator $Q : \Pi \to \Pi$ is called delta operator if $Q \in \mathscr{L}_s$ and Qe_1 is a nonzero constant.

Let $\mathscr{L}_\delta$ denote the set of all delta operators. For a better understanding, we present some examples of delta operators. In the following, the symbol I stands for the identity operator on the space Π.
(a) The *derivative operator*, denoted by D.
(b) The operators used in calculus of divided differences. Let h be a fixed number belonging to the field $\mathbb{K}$. We set

$$\Delta_h := E^h - I, \text{ the } \textit{forward difference operator},$$
$$\nabla_h := I - E^{-h}, \text{ the } \textit{backward difference operator},$$
$$\delta_h := E^{h/2} - E^{-h/2}, \text{ the } \textit{central difference operator}.$$

It is evident that $\nabla_h = \Delta_h E^{-h}$, $\delta_n = \Delta_h E^{-h/2} = \nabla_h E^{h/2}$. The properties of these operators as well as their usefulness can be found in [6].
(c) *Abel operator*, $A_a := DE^a$. For any $p \in \Pi$, $(A_a p)(x) = \dfrac{dp}{dx}(x+a)$.
Writing (symbolically) Taylor's series in the following manner

$$E^h = \sum_{\nu=0}^{\infty} \frac{h^\nu D^\nu}{\nu!} = e^{hD}, \tag{3.1}$$

we can also get $A_a = D(e^{aD})$.
(d) *Gould operator*, $G_{a,b} := \Delta_b E^a = E^{a+b} - E^a$, $ab \neq 0$.

Definition 3.2 Let Q be a delta operator. A polynomial sequence $p = (p_n)_{n\geq 0}$ is called the sequence of basic polynomials associated with Q if

(i) $p_0(x) = 1$ for any $x \in \mathbb{K}$.
(ii) $p_n(0) = 0$ for any $n \in \mathbb{N}$.
(iii) $(Qp_n)(x) = np_{n-1}(x)$ for any $n \in \mathbb{N}$ and $x \in \mathbb{K}$.

Remark 3.3 If $p = (p_n)_{n\geq 0}$ is a sequence of basic polynomials associated with Q, then $\{p_0, p_1, \dots, p_{n-1}, e_n\}$ is a basis of the linear space Π_n. Taking this fact into account, by induction it can be proved that every delta operator has a unique sequence of basic polynomials; see [9, *Proposition 3*].

Here are some examples. The basic polynomials associated with the operators $Q = D$, $Q = \Delta_h$, and $Q = \nabla_h$, are, respectively, $(e_n)_{n\geq 0}$, $(x^{[n,h]})_{n\geq 0}$, and $((x + (n-1)h)^{[n,h]})_{n\geq 0}$. Also, we can easily prove that $\widetilde{a} = (a_n^{\langle a\rangle})_{n\geq 0}$ respectively $g = (g_n^{\langle a,b\rangle})_{n\geq 0}$ is the sequence of basic polynomials associated with Abel operator A_a, respectively Gould operator $G_{a,b}$.

The connection between the delta operator and the binomial-type sequences is given by the following result [9, *Theorem 1*].

Theorem 3.4 *Let $p = (p_n)_{n\geq 0}$ be a sequence of polynomials. It is a sequence of binomial type if and only if it is a basic sequence for some delta operator.*

The following statement generalizes the Taylor expansion theorem to delta operators and their basic polynomials.

Theorem 3.5 *Let T be a shift-invariant operator, and let Q be a delta operator with its basic sequence $(p_n)_{n\geq 0}$. Then, the following identity holds*

$$T = \sum_{k\geq 0} \frac{(Tp_k)(0)}{k!} Q^k. \tag{3.2}$$

Let Q be a delta operator, and let $(\mathcal{F}, +, \cdot)$ be the ring of the formal power series in the variable t over the same field. Here, the product means the Cauchy product between two series. Further, let $(\mathcal{L}_s, +, \cdot)$ be the ring of shift-invariant operators, the product being defined as usually: For any $P_1, P_2 \in \mathcal{L}_s$, we have $P_1 P_2 : \Pi \to \Pi$, $(P_1 P_2)(q) = P_1(P_2(q))$, $q \in \Pi$. Then, there exists an isomorphism ψ from $\mathcal{F}$ onto $\mathcal{L}_s$ such that

$$\psi(f(t)) = T, \text{ where } f(t) = \sum_{k\geq 0} \frac{a_k}{k!} t^k \text{ and } T = \sum_{k\geq 0} \frac{a_k}{k!} Q^k. \tag{3.3}$$

This isomorphism allows us to conclude: A shift-invariant operator T is invertible if and only if $Te_0 \neq 0$. Since for every $Q \in \mathcal{L}_\delta$ we have $Qe_0 = 0$, we deduce that any delta operator is not invertible. Also, we can write $T = \phi(D)$, where $T \in \mathcal{L}_s$ and $\phi(t)$ is a formal power series, to indicate that the operator T corresponds to the series $\phi(t)$ under the isomorphism defined by (3.3).

Remark 3.6 In relation (3.1), we choose $T = E^x$ and expand E^x in terms of Q. Due to the identity $(E^x p_k)(0) = p_k(x)$ and the relation (3.2), one obtains

$$e^{xD} = \sum_{k\geq 0} \frac{p_k(x)}{k!} \phi^k(D).$$

Substituting D by u, the series terms lead us to the following result [9, *Corollary 3*].

Theorem 3.7 *Let Q be a delta operator with $p = (p_n)_{n\geq 0}$ its sequence of basic polynomials. Let $\phi(D) = Q$ and $\varphi(t)$ be the inverse formal power series of $\phi(u)$. Then,*

$$e^{x\varphi(t)} = \sum_{n\geq 0} \frac{p_n(x)}{n!} t^n, \tag{3.4}$$

where $\varphi(t)$ has the form $c_1 t + c_2 t^2 + \cdots$ $(c_1 \neq 0)$.

Another characterization of delta operators was included in [9] without proof. For this reason, we prove the following statement.

Theorem 3.8 *$Q \in \mathcal{L}_s$ is a delta operator if and only if $Q = DP$ for some shift-invariant operator P, where the inverse operator P^{-1} exists.*

Proof If in (3.3) we substitute T by a delta operator Q, then we get $a_0 = Q(e_0) = 0$ and $a_1 = Q(e_1) = c \neq 0$. Consequently, we can write

$$Q = \sum_{k \geq 1} \frac{a_k}{k!} D^k. \tag{3.5}$$

Denoting $\sum_{k \geq 1} \frac{a_k}{k!} D^{k-1}$ by P, we have $P \in \mathscr{L}_s$ and $P(e_0) = a_1 \neq 0$; thus, P is invertible; see the conclusion that emerges from (3.3). So, Q can be written as DP.

Reciprocally, for every $P \in \mathscr{L}_s$ such that P is invertible, DP is a shift-invariant operator, $E^a(DP) = (DP)E^a$, and

$$(DP)(e_1) = P(D(e_1)) = P(e_0) = c \neq 0$$

thus $DP \in \mathscr{L}_\delta$. □

Now we are ready to analyze some binomial operators investigating their statistical convergence to the identity operator.

4 Classes of Binomial Operators

We consider a delta operator Q and its sequence of basic polynomials $p = (p_n)_{n \geq 0}$, under the assumption that $p_n(1) \neq 0$ for every $n \in \mathbb{N}$. Also, according to Theorem 3.7, we shall keep the same meaning of the functions ϕ and φ. For every $n \geq 1$, we consider $L_n^Q : C([0, 1]) \to C([0, 1])$ defined as follows

$$(L_n^Q f)(x) = \frac{1}{p_n(1)} \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(1 - x) f\left(\frac{k}{n}\right). \tag{4.1}$$

They are called by P. Sablonnière [10] *Bernstein–Sheffer operators*. As D.D. Stancu and M.R. Occorsio motivated in [12], these operators can be named *Popoviciu operators*. T. Popoviciu [8] indicated the construction (4.1) in front of the sum appearing the factor d_n^{-1} from the identities

$$(1 + d_1 t + d_2 t^2 + \cdots)^x = e^{x\varphi(t)} = \sum_{n=0}^{\infty} p_n(x) \frac{t^n}{n!},$$

see (3.4). If we choose $x = 1$, it becomes obvious that $d_n = p_n(1)/n!$.

In the particular case $Q = D$, L_n^D becomes genuine Bernstein operator of degree n. An integral generalization of L_n^Q in Kantorovich sense was introduced and studied in [1].

The operators $L_n^Q, n \in \mathbb{N}$, are linear and reproduce the constants. Indeed, choosing in (2.3) $y := 1 - x$, we obtain $L_n^Q e_0 = e_0$. The positivity of these operators is given by the sign of the coefficients of the series $\varphi(t) = c_1 + c_2 t + \cdots$ $(c_1 \neq 0)$. More precisely, in [8, 10], the authors established the following.

Lemma 4.1 *L_n^Q is a positive operator on $C([0,1])$ for every $n \geq 1$ if and only if $c_1 > 0$ and $c_n \geq 0$ for all $n \geq 2$.*

Moreover, if L_n^Q satisfies the above conditions, then one has

$$L_n^Q e_1 = e_1,\ n \in \mathbb{N},\ \text{and}\ L_n^Q e_2 = e_2 + a_n(e_1 - e_2),\ n \geq 2, \tag{4.2}$$

where $a_n = \dfrac{1}{n}\left(1 + (n-1)\dfrac{r_{n-2}(1)}{p_n(1)}\right)$, the sequence $(r_n(x))_{n\geq 0}$ being generated by

$$\varphi''(t)\exp(x\varphi(t)) = \sum_{n\geq 0} r_n(x)\frac{t^n}{n!}.$$

Theorem 4.2 *Let the operators L_n^Q, $n \in \mathbb{N}$, be defined by (4.1) such that the hypothesis of Lemma 4.1 takes place.*

If $st - \lim\limits_{n\to\infty} \dfrac{r_{n-2}(1)}{p_n(1)} = 0$, *then*

$$st - \lim_{n\to\infty} \|L_n^Q f - f\| = 0,\ \ f \in C([0,1]). \tag{4.3}$$

Proof We apply Theorem 2.1. Based on algebraic operations with statistically convergent sequences of real numbers, our hypothesis guarantees the identity (4.3). For a profound documentation of operations with such sequences [2, *Theorem 3.1*] can be consulted. □

Further, choosing particular delta operators Q, we reobtain some classical linear positive operator of discrete type.

Example 4.3 If $Q = A_a$ with its basic sequence $\widetilde{a}$ and assuming that the parameter a depends on n, $a := t_n$, one obtains the *Cheney–Sharma operators* [3]. The corresponding operators Q_n , $n \in \mathbb{N}$, are defined by the equation

$$(Q_n f)(x) := (1 + nt_n)^{1-n} \sum_{k=0}^{n} \binom{n}{k} x(x + kt_n)^{k-1}(1-x)[1 - x + (n-k)t_n]^{n-1-k}.$$

Clearly, $Q_n e_0 = e_0$. To compute $Q_n e_j$, $j \in \{1, 2\}$, we follow the same path as in [3, *Section 3*]. We can deduce: If the sequence $(nt_n)_{n\geq 1}$ is statistically convergent to zero, then (4.3) takes place.

Example 4.4 If $Q = \dfrac{1}{\alpha}\nabla_\alpha$, $\alpha \neq 0$, with its basic polynomials

$$p_n(x) = (x + (n-1)\alpha)^{[n,\alpha]},$$

L_n^Q becomes *Stancu operator* [11] denoted by $P_n^{[\alpha]}$,

$$(P_n^{[\alpha]} f)(x) = \sum_{k=0}^{n} w_{n,k}(x;\alpha) f\left(\frac{k}{n}\right),$$
$$w_{n,k}(x;\alpha) = \binom{n}{k} \frac{x^{[k,-\alpha]}(1-x)^{[n-k,-\alpha]}}{1^{[n,-\alpha]}},$$

α being a parameter which may depend on a natural number n. One has

$$P_n^{[\alpha]} e_j = e_j, \ \ j \in \{0,1\}, \text{ and } (P_n^{[\alpha]} e_2)(x) = \frac{1}{1+\alpha}\left(\frac{x(1-x)}{n} + x(x+\alpha)\right),$$

in accordance with [11, *Lemma 4.1*].

If $0 \le \alpha := \alpha(n)$ and $st - \lim\limits_{n\to\infty} \alpha(n) = 0$, then (4.3) takes place.

Indeed, it is enough to prove (2.1) for $j = 2$. We get

$$|(P_n^{[\alpha]} e_2)(x) - x^2| = \frac{1}{1+\alpha}\left|\frac{x(1-x)}{n} - \alpha x^2 + x\alpha\right|$$
$$\le \frac{1}{1+\alpha}\left(\frac{1}{4n} + 2\alpha\right) \le \frac{1}{4n} + 2\alpha$$

and the conclusion follows.

At this moment, we take a break in order to illustrate some further properties of the binomial sequences. Keeping the notations $Q \in \mathscr{L}_\delta$, $p = (p_n)_{n\ge 0}$, $Q = \phi(D)$, $\phi^{-1} = \varphi$, we assume that the conditions of Lemma 4.1 are fulfilled.

In [7], A. Lupaş proved new inequalities between the terms of the binomial sequences p. For any $x > 0$ and $n \ge 2$, one has

$$\begin{cases} 0 < c_1 \dfrac{p_{n-1}(x)}{x} \le (Q'^{-2} p_{n-2})(x) \le \dfrac{p_n(x)}{x^2}; \\ \dfrac{1}{n} \le \rho_n(Q) < 1, \ \text{where } \rho_n(Q) := 1 - \dfrac{n(n-1)}{p_n(n)}(Q'^{-2} p_{n-2})(x). \end{cases} \tag{4.4}$$

In the above, Q' represents Pincherle derivative of Q. The concept is detailed further.

Knowing that the operator $X : \Pi \to \Pi$, $(Xp)(x) = xp(x)$ is called *multiplication operator*, we recall that the Pincherle derivative of an operator $U \in \mathscr{L}$ is defined by the formula

$$U' = UX - XU.$$

For example, we get $I' = 0$, $D' = I$, $(D^k)' = kD^{k-1}$, $k \geq 1$.

Example 4.5 Following Lupaş, we can define the operators $\widetilde{L}_n^Q : C([0,1]) \to C([0,1])$, $n \in \mathbb{N}$,

$$(\widetilde{L}_n^Q f)(x) = \frac{1}{p_n(n)} \sum_{k=0}^{n} \binom{n}{k} p_k(nx) p_{n-k}(n - nx) f\left(\frac{k}{n}\right).$$

It is proved that

$$\widetilde{L}_n^Q e_i = e_i,\ \ i \in \{0, 1\},\ \text{and}\ \widetilde{L}_n^Q e_2 = e_2 + (e_1 - e_2)\rho_n(Q),$$

see (4.4). Consequently, if

$$st - \lim_{n\to\infty} \frac{n(n-1)}{p_n(n)} (Q'^{-2} p_{n-2})(x) = 1,$$

then the operators $\widetilde{L}_n^Q$, $n \in \mathbb{N}$, satisfy identity (4.3).

Concluding remarks. The paper reintroduces some linear positive operators of discrete type by using umbral calculus. Relative to these operators have been studied approximation properties in Banach space $(C[0,1], \|\cdot\|)$. The approach was based on Bohman–Korovkin theorem via statistical convergence. The usefulness of this type of convergence can be summarized as follows: The statistical convergence of a sequence is that the majority, in a certain sense, of its elements converges and we are not interested in what happens to the remaining elements. The advantage of replacing the uniform convergence by statistical convergence consists in the fact that the second convergence is efficient in summing divergent sequences which may have unbounded subsequences. In short, it is more lax.

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