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EXISTENCE AND UNIQUENESS OF SOLUTIONS OF THE DARBOUX PROBLEM FOR PARTIAL DIFFERENTIAL-FUNCTIONAL EQUATIONS

by

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The subject of this paper is the Darboux problem, for functional equations of hyperbolic type. Such problems were first studied by D.V.Ionescu [4] and many contributions are to be found in some recent papers [1], [2], [5] - [8].

We consider equations of the following type

(1) $u_{xy}(x,y) = f(x,y,u(x,y),u(g(x,y)))$

(2) $u_{xy}(x,y) = f(x,y, u(x,y),u_x(x,y),u_y(x,y),u(g_1(x,y)),u_x(g_2(x,y)), u_y(g_3(x,y)))$.

For $T_1, T_2 \geq 0$ and $\alpha, \beta > 0$ we denote

(3) $D = [0,\alpha] \times [0,\beta]$ and $D_0 = ([-T_1,0] \times [-T_2,\beta]) \cup ([0,\alpha] \times [-T_2,0])$

(4) $E = [0,\infty)^2$ and $E_0 = ([-T_1,0] \times [-T_2,\infty)) \cup ([0,\infty) \times [-T_2,0])$.

A. We study at first the case when the equation is considered on the compact D with boundary conditions on D_0 (not only on $([0,\alpha] \times \{0\}) \cup (\{0\} \times [0,\beta])$, which corresponds to $T_1 = T_2 = 0$) in order to admit also retarded arguments. The boundary condition is

(5) $u(x,y) = \varphi(x,y), (x,y) \in D_0$.

A solution of the problem (1) (or (2)) with the condition (5) is a function $U \in C(D \cup D_0)$ (or $U \in C^1(D \cup D_0)$) such that U_{xy} exists on D and U satisfies the equation (1) (or (2)) for any $(x,y) \in D$ and the condition (5) on D_0.

All the functions whose range is not mentioned have real values.

THEOREM 1. *If*

(a) $f \in C(D \times R^2)$ *satisfies a Lipschitz condition with respect to the last two variables*

$$|f(x,y,z_1,z_2) - f(x,y,t_1,t_2)| \leq L_1 |z_1 - t_1| + L_2 |z_2 - t_2|$$

(b) $\varphi \in C^1(D_0)$

(c) $g \in C(D)$, $g: D \to D \cup D_0$, $g = (h,k)$ *with* $h(x,y) + k(x,y) \leq x+y+a$

and $a < \sup\left\{ \frac{1}{t} \ln \frac{t^2 - L_1}{L_2} : t > \sqrt{L_1} \right\}$,

then the problem (1) with the condition (5) has in $C(D \cup D_0)$ *a unique solution.*

Proof. We consider the Banach space $C(D \cup D_0)$ with the Bielecki type norm

(6) $$\| u \| = \max \left\{ |u(x,y)| \exp(-t(x+y)) : (x,y) \in D \cup D_0 \right\}$$

and the operator $T : C(D \cup D_0) \to C(D \cup D_0)$ given by

(7) $$\begin{cases} Tu(x,y) = \int_0^x \int_0^y f(r,s,u(r,s),u(g(r,s)))drds + \psi(x,y), \text{ if } (x,y) \in D \\ Tu(x,y) = \varphi(x,y), \text{ for } (x,y) \in D_0, \end{cases}$$

where

(8) $$\psi(x,y) = \varphi(x,0) + \varphi(0,y) - \varphi(0,0), \quad (x,y) \in D.$$

The problem is equivalent to the equation $Tu = u$. We prove that T is a contraction and then the Banach fixed point theorem guarantees the existence and uniqueness of a fixed point for T.

Let $u, v \in C(D \cup D_0)$. If $(x,y) \in D_0$, we have $Tu(x,y) - Tv(x,y) = 0$. For $(x,y) \in D$,

$$|Tu(x,y) - Tv(x,y)| \leq \int_0^x \int_0^y (L_1 |u(r,s) - v(r,s)| +$$

$$+ L_2 |u(g(r,s)) - v(g(r,s))|)drds \leq \int_0^x \int_0^y (L_1 \| u-v \| \exp(t(r+s)) +$$

$$+ L_2 \| u-v \| \exp(t(h(r,s) + k(r,s)))drds \leq \int_0^x\int_0^y (L_1 \| u-v \| \exp(t(r+s)) +$$
$$+ L_2 \| u-v \| \exp(t(r+s+a)))drds \leq \frac{1}{t^2}(L_1 + L_2\exp(ta)) \| u-v \| \exp(t(x+y))$$

and therefore

$$\| Tu-Tv \| \leq \frac{1}{t^2}(L_1 + L_2\exp(ta) \| u-v \| .$$

The hypothesis on a makes $\frac{1}{t^2}(L_1 + L_2\exp(ta)) < 1$.

REMARK 1. The maximum of the expression that denotes a majorant for a is attained in the case when t is the unique solution greater than $\sqrt{L_1}$ of the equation

$$\frac{2x^2}{x^2-L_1} - \ln \frac{x^2-L_1}{L_2} = 0.$$

If the right member of the equation (1) contains only the function having modified argument, i.e. the equation becomes

(9) $u_{xy}(x,y) = f(x,y,u(g(x,y)))$,

then we have

COROLLARY 1. If

(a) $f \in C(D \times R)$ is Lipschitz with respect to the last variable

$|f(x,y,z) - f(x,y,t)| \leq L|z-t|$

(b) $\varphi \in C^1(D_0)$

(c) $g \in C(D)$, $g : D \to D \cup D_0$, $g = (h,k)$ with $h(x,y) + k(x,y) \leq$ $\leq x+y+a$, $a < \frac{2}{e\sqrt{L}}$, then the problem (9) with the condition (5) has in $C(D \cup D_0)$ a unique solution.

<u>Proof</u>. We may consider that f satisfies (a) from Theorem 1 with $L_1 = 0$ and $L_2 = L$. The equation in Remark 1 becomes

$2 - \ln \frac{x^2}{L} = 0$ with the positive solution $t = e\sqrt{L}$, for which

$\frac{1}{t}\ln \frac{t^2}{L} = \frac{2}{e\sqrt{L}}$ and all the conditions in Theorem 1 are satisfied. It follows that the problem (9) with the condition (5) has a unique solution in $C(D \cup D_0)$.

For the problem (2) with the condition (5) we obtain

THEOREM 2. If

(a) $f \in C(D \times R^6)$ satisfies a Lipschitz condition with respect to the last six variables

$$|f(x,y,z_1,\dots,z_6) - f(x,y,t_1,\dots,t_6)| \le L_1 \sum_{i=1}^{3} |z_i - t_i| + L_2 \sum_{i=4}^{6} |z_i - t_i|$$

(b) $\varphi \in C^1(D_0)$ and φ_{xy} exists, $f(0,y,z_1,\dots,z_6) = \varphi_{xy}(0,y)$, $f(x,0,z_1,\dots,z_6) = \varphi_{xy}(x,0)$, $(x,y) \in D$, $(z_1,\dots,z_6) \in R^6$

(c) $g_i \in C(D_0)$, $g_i : D \to D \cup D_0$, $g_i = (h_i,k_i)$ with $h_i(x,y) + k_i(x,y) \le$ $\le x + y + a$, $i = \overline{1,3}$ and

$$a < \sup \left\{ \frac{1}{t} \ln \frac{t^2 - 2L_1 t - L_1}{L_2(2t+1)} : t > L_1 + \sqrt{L_1^2 + L_1} \right\},$$

then the problem (2) with the condition (5) has in $C^1(D \cup D_0)$ a unique solution.

Proof. In the Banach space $C^1(D \cup D_0)$ endowed with the norm $\|u\|_1 = \|u\| + \|u_x\| + \|u_y\|$, with $\|\cdot\|$ given by (6) we consider the operator $T : C^1(D \cup D_0) \to C(D \cup D_0)$ given by

$$(10) \quad \begin{cases} Tu(x,y) = \int_0^x \int_0^y f(r,s,u(r,s),u_x(r,s),u_y(r,s),u(g_1(r,s)), \\ \qquad u_x(g_2(r,s)),u_y(g_3(r,s)))drds + \psi(x,y), \text{ for } (x,y) \in D \\ Tu(x,y) = \varphi(x,y), \text{ for } (x,y) \in D_0 \end{cases}$$

where ψ is defined by (8).

We have to show that the range of T is contained in $C^1(D \cup D_0)$. We obtain

$(Tu)_x(x,y) = \int_0^y f(x,s,u(x,s),\dots,u_y(g_3(x,s)))ds + \varphi_x(x,0)$, for $(x,y) \in (0,\alpha] \times (0,\beta]$ and

$(Tu)_x(x,y) = \varphi_x(x,y)$ for $(x,y) \in D_0 \setminus \{(x,y) \in D : x=0 \text{ or } y=0\}$.

The limits in the points of the form $(x_0,0)$ or $(0,y_0)$ are the same for both expressions, in view of the condition (b). We have proved that $(Tu)_x \in C(D \cup D_0)$ and a similar argument shows that $(Tu)_y \in$ $\in C(D \cup D_0)$, so $Tu \in C^1(D \cup D_0)$.

Let $u,v \in C^1(D \cup D_0)$. For $(x,y) \in D_0$ we have $Tu(x,y) - Tv(x,y) = 0$. For $(x,y) \in D$ we obtain $|Tu(x,y) - Tv(x,y)| \leq$

$\leq (L_1 + L_2\exp(ta)) \| u-v \|_1 \int_0^x\int_0^y \exp(t(r+s))drds$ and it follows that

$$\| Tu - Tv \| \leq \frac{1}{t^2} (L_1 + \exp(ta)L_2) \| u-v \|_1 .$$

We obtain similarly

$$\| (Tu)_x - (Tv)_x \| \leq \frac{1}{t} (L_1 + \exp(ta)L_2) \| u-v \|_1$$

$$\| (Tu)_y - (Tv)_y \| \leq \frac{1}{t} (L_1 + \exp(ta)L_2) \| u-v \|_1 .$$

The last three inequalities leed us to

$$\| Tu - Tv \|_1 \leq \left(\frac{1}{t^2} + \frac{2}{t} \right) (L_1 + L_2\exp(ta)) \| u-v \|_1 .$$

The hypothesis on a implies that $\left(\frac{1}{t^2} + \frac{2}{t} \right) (L_1 + L_2\exp(ta)) < 1$, hence T is a contraction and the considered problem has a unique solution.

REMARK 2. The maximum of the majorant of a is attained then this the unique solution greater than $L_1 + \sqrt{L_1^2 + L_1}$ of the equation

$$\frac{2x^2(x+1)}{(2x+1)(x^2-2L_1x-L_1)} - \ln \frac{x^2-2L_1x-L_1}{L_2(2x+1)} = 0.$$

In the case of the equation

(11) $\quad u_{xy}(x,y) = f(x,y,u(g_1(x,y)),u_x(g_2(x,y)),u_y(g_3(x,y)))$

with the condition (5) we obtain

COROLLARY 2. If

(a) $f \in C(D \times R^3)$ satisfies a Lipschitz condition with respect to the last three variables

$$|f(x,y,z_1,z_2,z_3) - f(x,y,t_1,t_2,t_3)| \leq L\sum_{i=1}^{3}|z_i-t_i|$$

(b) $\varphi \in C^1(D_0)$ such that φ_{xy} exists and $f(0,y,z_1,z_2,z_3) = \varphi_{xy}(0,y)$ $f(0,x,z_1,z_2,z_3) = \varphi_{xy}(x,0)$, for any $(x,y) \in D$, $(z_1,z_2,z_3) \in R^3$

(c) $g_i \in C(D)$, $g_i : D \to D \cup D_0$, $g_i = (h_i, k_i)$ with $h_i(x,y) + k_i(x,y) \leq$ $\leq x+y+a$, $i = \overline{1,3}$ and

$$a < \sup \left\{ \frac{1}{t} \ln \frac{t^2}{L(2t+1)} : t > 0 \right\},$$

then the problem (11) with the condition (5) has in $C^1(D \cup D_0)$ a unique solution. The maximum of the expression which denotes a majorant for a is attained in the case when t is the unique positive solution of the equation $\frac{2(x+1)}{2x+1} - \ln \frac{x^2}{L(2x+1)} = 0$.

Proof. One applies Theorem 2 and Remark 2 considering f Lipschitz with $L_1 = 0$ and $L_2 = L$.

REMARK 3. In the given theorems, the condition imposed to the functions that modify the variables is less restrictive than in [2] and determines a larger class of functions which satisfy the integral condition from [8]. It is said in [1] that one has the result in Theorem 2 even if one renounces to the last part of the conditions (b) and (c), but it is not true. The following examples, which satisfy all the conditions in Theorem 2 but the last part of the conditions (b) and (c), show that the uniqueness or even the existence of the solution is not guaranteed any more.

EXAMPLE 1. Let $D = [0,1]^2$, $T_1 = T_2 = 0$, $g : D \to D$, $g(x,y) = (1,y)$. The equation

(12) $u_{xy}(x,y) = u_y(1,y)$, $(x,y) \in D$

with the boundary conditions

$$u(x,0) = u(0,y) = 0, \quad (x,y) \in D$$

has infinitely many solutions of the form $u(x,y) = xF(y)$, $F : [0,1] \to R$ being differentiable with $F(0) = 0$.

EXAMPLE 2. If in Example 1 we consider instead of (12) the equation $u_{xy}(x,y) = u_y(1,y) + 2y$, we obtain a problem which has no solutions in $C^1(D)$.

Indeed, if the problem has a solution U, this will verify

$$U(x,y) = \int_0^x\int_0^y (U_y(1,s) + 2s)drds,$$

therefore $U_y(x,y) = x(2y + U_y(1,y))$. For $x = 1$ we get a contradiction.

B. We analyse now the existence of the global solutions of the equations (1) and (2) on E, the boundary condition being

(13) $u(x,y) = \varphi(x,y)$, $(x,y) \in E_0$,

where E and E_0 are given by (4).

We extend some results of [3] to the case of functional equations, admitting also delays, using the methods given by Bielecki. We need the following estimations.

LEMMA 1. Let $t>0$, $L\in C(E)$, $L(x,y) \geq 0$ on E and $K(x,y) = \int_0^x\int_0^y L(p,q)dpdq$, $(x,y)\in E$. Then the following inequality holds

$$\int_0^x\int_0^y L(r,s)\exp(tK(r,s))drds \leq \frac{1}{t}\exp(tK(x,y)), (x,y) \in E.$$

<u>Proof</u>.

Let $U(x,y) = \frac{1}{t}\exp(tK(x,y)) - \int_0^x\int_0^y L(r,s)\exp(tK(r,s))drds$. It follows that $U_x(x,y) = \int_0^y L(x,q)dq.\exp(tK(x,y)) - \int_0^y L(x,s)\exp(tK(x,s))ds$. Because $U_{xy}(x,y) \geq 0$ for any $(x,y)\in E$, we have $U_x(x,y) \geq U_x(x,0) = 0$ and $U(x,y) \geq U(0,y) \geq 0$, hence the inequality is proved.

The proof is similar for

LEMMA 2. Let $t > 0$, $L \in C^1(E)$, $L(x,y) \geq 0$, $L_x(x,y) \geq 0$ and $L_y(x,y) \geq 0$ on E; $K(x,y) = \int_0^x\int_0^y L(p,q)dpdq + \int_0^x L(p,y)dp + \int_0^y L(x,q)dq$, $(x,y) \in E$. We have then for any $(x,y) \in E$

$$\int_0^x\int_0^y L(r,s)\exp(tK(r,s))drds \leq \frac{1}{t}\exp(tK(x,y))$$

$$\int_0^y L(x,s)\exp(tK(x,s))ds \leq \frac{1}{t}\exp(tK(x,y))$$

$$\int_0^x L(r,y)\exp(tK(r,y))dr \leq \frac{1}{t}\exp(tK(x,y)).$$

We prove now

THEOREM 3. *If*

(a) $f \in C(E\times R^2)$ *is Lipschitz with respect to the last two variables such that*

$|f(x,y,z_1,z_2) - f(x,y,t_1,t_2)| \le L(x,y)(|z_1-t_1| + |z_2-t_2|)$ *and*

$|f(x,y,0,0)| \le L(x,y)$, *where* $L \in C(E)$ *is nonnegative*

(b) $\varphi \in C^1(E_0)$ *and* $\sup\{|\varphi(x,y)| : (x,y) \in E_0\} = S < \infty$

(c) $g \in C(E)$, $g : E \to E\cup E_0$, $g = (h,k)$ *with* $h(x,y) \le x$, $k(x,y) \le y$ *for any* $(x,y) \in E$,

then the problem (1) with the condition (13) has in the space X defined below a unique solution.

Proof. Let $u \in C(E\cup E_0)$. We define $K : E\cup E_0 \to R$ by

$K(x,y) = \int_0^x\int_0^y L(p,q)dpdq$ for $(x,y) \in E$ and $K(x,y) = 0$ for $(x,y) \in E_0$,

and consider

(14) $\|u\|_2 = \sup\{|u(x,y)|\exp(-tK(x,y)) : (x,y) \in E\cup E_0\}$

with $t > 2$. We denote $X = \{u \in C(E\cup E_0) : \|u\|_2 < \infty\}$ and observe that $(X, \|\cdot\|_2)$ is a Banach space.

For $u \in X$, we define Tu as in (7) respectively on E and E_0 and prove that the range of T is contained in X.

Let $(x,y) \in E_0$. Then $|Tu(x,y)| = |\varphi(x,y)| \le S$. For $(x,y) \in E$,

$$|Tu(x,y)| \le \int_0^x\int_0^y L(r,s)(|u(r,s)| + |u(g(r,s))| + 1)drds \le$$

$\le (2\|u\|_2 + 1)\int_0^x\int_0^y L(r,s)\exp(tK(r,s))drds$. Applying Lemma 1 we obtain

$|Tu(x,y)|\exp(-tK(x,y)) \le \frac{1}{t}(\|u\|_2 + 1)$, hence $\|Tu\|_2 \le$

$\le \max\{S, \frac{1}{t}(\|u\|_2 + 1)\} < \infty$.

We prove that T is Lipschitz with a constant which is less than 1. Let $u,v \in X$. For $(x,y) \in E_0$, we have $|Tu(x,y) - Tv(x,y)| = 0$. For

$(x,y) \in E$, we obtain

$$|Tu(x,y) - Tv(x,y)| \leq 2\|u\|_2 \int_0^x\int_0^y L(r,s)\exp(tK(r,s))drds \leq$$

$$\leq \frac{2}{t}\|u\|_2 \exp(tK(x,y)).$$

It follows that $\|Tu - Tv\|_2 \leq \frac{2}{t}\|u-v\|_2$ and for $t > 2$ we have a unique solution in X.

THEOREM 4. *If*

(a) $f \in C(E \times R^6)$ *is Lipschitz with respect to the last six variables such that*

$$|f(x,y,z_1,\dots,z_6) - f(x,y,t_1,\dots,t_6)| \leq L(x,y)\sum_{i=1}^{6}|z_i - t_i| \text{ and}$$

$|f(x,y,0,\dots,0)| \leq L(x,y)$ *for any* $(x,y) \in E$, *with* L, L_x *and* L_y *in* $C(E)$ *and nonnegative on* E

(b) $\varphi \in C^1(E_0)$ *with* $|\varphi|$, $|\varphi_x|$, $|\varphi_y| \leq S < \infty$, *where* $|\varphi| = \sup\{|\varphi(x,y)| : (x,y) \in E_0\}$, *and* φ_{xy} *exists such that* $f(0,y,z_1,\dots,z_6) = \varphi_{xy}(0,y)$, $f(x,0,z_1,\dots,z_6) = \varphi_{xy}(x,0)$, *for any* $(x,y) \in E$, $(z_1,\dots,z_6) \in R^6$

(c) $g_i \in C(E)$, $g_i : E \to E \cup E_0$, $g_i = (h_i,k_i)$ *with* $h_i(x,y) \leq x$, $k_i(x,y) \leq y$ *for any* $(x,y) \in E$, $i = \overline{1,3}$,

then the problem (2) with the condition (13) has in the space Y defined bellow a unique solution.

Proof. Let $u \in C^1(E \cup E_0)$. We define $K : E \cup E_0 \to R$ by

$$K(x,y) = \int_0^x\int_0^y L(p,q)dpdq + \int_0^x L(p,y)dp + \int_0^y L(x,q)dq, \quad (x,y) \in E \text{ and}$$

$K(x,y) = 0$, $(x,y) \in E_0$. We consider also

$\|u\|_3 = \|u\|_2 + \|u_x\|_2 + \|u_y\|_2$, where $\|\cdot\|_2$ is given by (14) with $t > 6$.

Let $Y = \{u \in C^1(E \cup E_0) : \|u\|_3 < \infty\}$ which is a Banach space with respect to the norm $\|\cdot\|_3$. For $u \in Y$, we define Tu by (10), respectively on E and E_0.

We prove that the range of T is contained in Y.

We have $Tu \in C^1(E \cup E_0)$ because of the condition (b). Let $(x,y) \in \in E_0$; then $|Tu(x,y)| = |\varphi(x,y)| \leq S$. For $(x,y) \in E$ we have

$$|Tu(x,y)| \leq (2\|u\|_3 + 1)\int\int L(r,s)\exp(tK(r,s))drds.$$

Following Lemma 2 we obtain $|Tu(x,y)|\exp(-tK(x,y)) \leq \frac{1}{t}(2\|u\|_3 + 1)$, $(x,y) \in E$ and

$$\|Tu\|_2 \leq \max\left\{S, \frac{1}{t}(2\|u\|_3 + 1)\right\}.$$

Similarly

$$\|(Tu)_x\|_2 \leq \max\left\{S, \frac{1}{t}(2\|u\|_3 + 1)\right\}$$

$$\|(Tu)_y\|_2 \leq \max\left\{S, \frac{1}{t}(2\|u\|_3 + 1)\right\}, \text{ hence}$$

$$\|Tu\|_3 \leq 3\max\left\{S, \frac{1}{t}(2\|u\|_3 + 1)\right\} < \infty \text{ and } Tu \in Y.$$

Using again Lemma 2 we prove that T is a Lipschitz operator. Let $u, v \in Y$; for $(x,y) \in E_0$, we have $|Tu(x,y) - Tv(x,y)| = 0$. For $(x,y) \in E$

$$|Tu(x,y) - Tv(x,y)| \leq 2\|u-v\|_3 \int\int L(r,s)\exp(tK(r,s))drds \leq$$

$$\leq \frac{2}{t}\|u-v\|_3\exp(tK(x,y)).$$

It follows that

$$\|Tu - Tv\|_2 \leq \frac{2}{t}\|u-v\|_3$$

and similarly

$$\|(Tu)_x - (Tv)_x\|_2 \leq \frac{2}{t}\|u-v\|_3$$

$$\|(Tu)_y - (Tv)_y\|_2 \leq \frac{2}{t}\|u-v\|_3$$

From the last three inequalities we obtain

$$\|Tu - Tv\|_3 \leq \frac{6}{t}\|u-v\|_3,$$

and for $t > 6$ the problem has in Y a unique solution.

REMARK 4. The results of Theorem 3 and 4 remain true imposing suitable conditions on f in the case of the equations (9) and (11).

REMARK 5. Because all the theorems were proved using the Banach fixed point theorem, the solutions may be approximated by the method of succesive approximations.

REFERENCES

1. AGARWAL, R.P., THANDAPANI, E., Existence and Uniqueness of Solutions of Hyperbolic Delay Differential Equations, Math. Sem. Notes 7(1979), 531-541

2. ALICU, M.C., The Darboux Problem for Partial Differential Functional Equations of Hyperbolic Type, Mathematica 19(42), 2(1977), 117-122

3. BIELECKI, A., Une remarque sur l'application de la méthode de Banach-Caccioppoli-Tikhonov dans la théorie de l'équation s = f(x,y,z,p,q), Bull. Acad. Polon. Sci. 4(1956), 265-268

4. IONESCU, D.V., Sur une classe d'équations fonctionnelles, Thèse, Paris 1927

5. KWAPISZ, M., TURO, J., On the Existence and Uniqueness of Solutions of Darboux Problem for Partial Differential-Functional Equations, Colloq. Math. 29(1974), 279-302

6. KWAPISZ, M., TURO, J., Some Integral Functional Equations, Funk. Ekv. 18(1975), 107-162

7. RADULESCU, D., Sur un problème de D.V.Ionescu, Studia Univ. Babeş-Bolyai, Math.26(2)(1981), 51-55

8. KIS, A.I., On the Problem of Darboux-Ionescu, "Babeş-Bolyai" Univ Faculty of Math. Research Sem., Preprint nr. 1, 1981, 1-32