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ON MULTIVALUED MAPPINGS SATISFYING THE CONDITION $T(F_T) = F_T$

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Let X be a nonvoid set and $T : X \multimap X$ a multivalued mapping. Let

$F_T = \{x \in X : x \in T(x)\}$, respectively

$(SF)_T = \{x \in X : T(x) = \{x\}\}$

denote the fixed point set, respectively the strict fixed point set of T.

For the multivalued mapping $T : X \multimap X$ we consider the graph

$G(T) = \{(x,y) \in X \times X : y \in T(x)\}$

which may be regarded as a relation on X. The properties of the relations are exposed, for example, in [1].

It is obvious that any function $f : X \to X$ satisfies the equality $f(F_f) = F_f$. In the paper [2], I.A.Rus shows that there are multivalued mappings which have not this property, but any multivalued mapping having only strict fixed points ($F_T = (SF)_T$) verifies the condition $T(F_T) = F_T$ (Lemma 4.1). The Problem 4.1[2] asks what are the conditions under which the set F_T is fixed for a multivalued mapping T.

In the sequel we give sufficient conditions for the equality $T(F_T) = F_T$ holds and then we study some properties of a multivalued mapping $\tilde{T}$ induced by the given T.

For the set $X \neq \emptyset$ we denote

$I_X : X \multimap X$, $I_X(x) = \{x\}$ for any $x \in X$

$\Delta_X = G(I_X) = \{(x,x) \in X \times X : x \in X\}$,

and for the multivalued mapping $T : X \multimap X$

$\operatorname{dom} T = \{x \in X : T(x) \neq \emptyset\}$

$\operatorname{Im} T = \{y \in X :$ there is $x \in X$ such that $y \in T(x)\}$

$T^{-1} : X \multimap X$, $T^{-1}(y) = \{x \in X : y \in T(x)\}$.

For a multivalued mapping $T : X \multimap X$ the following theorem holds.

THEOREM 1. Let the below conditions be given

(a) $T(F_T) \subseteq F_T$

(a') $T(F_T) = F_T$

(a") $T(\operatorname{dom}(I_X \cap T)) \subseteq \operatorname{dom}(I_X \cap T)$

(b) for any $x \in F_T$, $T(x) \subseteq \bigcap_{y \in T(x)} T(y)$

(b') for any $x \in F_T$ and $z \in T(x)$, it follows $T(x) \subseteq T(z)$

(c) $G(T)$ is a symmetrical and transitive relation

(d) $G(T)$ is a reflexive relation

(e) $F_T = (SF)_T$.

The following implications are true

$$\begin{array}{l} (d) \Rightarrow (a) \Leftrightarrow (a') \Leftrightarrow (a") \\ \qquad\quad \Uparrow \\ (e) \Rightarrow (b) \Leftrightarrow (b') \\ \qquad\quad \Uparrow \\ \qquad\quad (c) \end{array}$$

Proof. (a) $\Leftrightarrow$ (a'). It is obvious that (a') $\Rightarrow$ (a); if (a) is true, we obtain (a') because from $x \in F_T$ we deduce $x \in T(x)$, so $x \in T(F_T)$.

(a) $\Leftrightarrow$ (a"). $\operatorname{dom}(I_X \cap T) = \{x \in X : I_X(x) \cap T(x) \neq \emptyset\} = \{x \in X : x \in T(x)\} = F_T$ and the equivalence holds.

(b) $\Rightarrow$ (b'). Let $x \in F_T$ and $z \in T(x)$. We prove that $T(x) \subseteq T(z)$. For any $y \in T(x)$, we obtain $y \in T(x) \subseteq \bigcap_{y \in T(x)} T(y) \subseteq T(z)$, so $T(x)$

$\subseteq T(z)$.

(b') $\Rightarrow$ (B). Let $x \in F_T$ and $z \in T(x)$; by (b') we have $T(x) \subseteq T(z)$, hence $T(x) \subseteq \bigcap_{z \in T(x)} T(z)$.

(b) $\Rightarrow$ (a). Let $y \in T(F_T)$, i.e. there exists $x \in F_T$ such that $y \in \in T(x)$. The condition (b) implies $T(x) \subseteq \bigcap_{z \in T(x)} T(z) \subseteq T(y)$; but $y \in \in T(x)$ and it follows $y \in T(y)$, hence $y \in F_T$.

(c) $\Rightarrow$ (B). Let $x \in F_T$ and $z \in T(x)$. We prove that for any $y \in X$ such that $y \in T(x)$ we have $y \in T(z)$. The symmetry of $G(T)$ implies $x \in T(y)$; but $z \in T(x)$ and from the transitivity of $G(T)$ we obtain $z \in T(y)$. Applying again the symmetry, we have $y \in T(z)$ and $T(x) \subseteq \subseteq \bigcap_{z \in T(x)} T(z)$.

(e) $\Rightarrow$ (b). Let $x \in F_T = (SF)_T$, so $T(x) = \{x\} = \bigcap_{y \in T(x)} T(y)$ and (b) holds.

(d) $\Rightarrow$ (a). $G(T)$ being reflexive, we have $\Delta_X \subseteq G(T)$ and it follows $F_T = X$; it is obvious that $T(F_T) \subseteq F_T$.

We mention now some connections between the classes of multivalued mappings satisfying the conditions in Theorem 1.

THEOREM 2. <u>If the graph of the multivalued mapping</u> T <u>satisfying</u> (b) <u>is a reflexive relation, it is also symmetrical and transitive.</u>

<u>Proof</u>. $G(T)$ being reflexive, the condition (b) is satisfied for any $x \in X$. Let $x \in X$ be arbitrary and $y \in T(x)$. Using (b), $x \in T(x) \subseteq$ $\subseteq T(y)$, so $x \in T(y)$ and the symmetry is proved.

For the transitivity, we consider $x \in X$, $y \in T(x)$ and $z \in T(y)$. It follows by (b) that $x \in T(x) \subseteq T(y) \subseteq T(z)$, so $x \in T(z)$. Applying the symmetry of $G(T)$ we obtain $z \in T(x)$ and the proof is over.

REMARK 1. There is only one multivalued mapping which has only strict fixed points and a reflexive graph, namely I_X, whose graph Δ_X is also symmetrical and transitive.

We are able now to present the relative position of the classes of multivalued mappings involved in Theorem 1 using the diagram in Fig. 1; rectangles having the bases on the same line and the top vertexes marked with a letter stand for the classes denoted by that letter. All the regions marked by a number are nonvoid, as the following examples show.

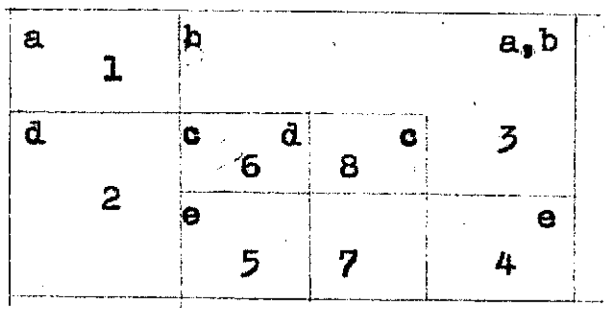

Fig. 1

EXAMPLES.

1. T satisfies (a), but none of (b) and (d).

$$X = R,\ T(x) = \begin{cases} \{o, x\}, & x \in [-1,1] \\ \{1\}, & x \in R \setminus [-1,1] \end{cases}$$

2. T satisfies (d), but it does not satisfy (b).

$$X = R,\ T(x) = \begin{cases} \{x\}, & x \neq o \\ \{o, 1\}, & x = o \end{cases}$$

3. T satisfies (b), but none of (c), (e) and (d).

$$X = R,\ T(x) = \begin{cases} \{2\}, & x = o \\ \{-x, x\}, & x \in [-1,1] \setminus \{o\} \\ \{o\}, & x \in R \setminus [-1,1] \end{cases}$$

4. T satisfies (e), but none of (c) and (d).

$X = R,\ T(x) = \{o, -x\}.$

5. T satisfies (c), (d) and (e).

By Remark 1, the only multivalued mapping satisfying these conditions is I_X.

6. T satisfies (c) and (d), but it does not satisfy (e).

$$X = R,\ T(x) = \begin{cases} \{x\}, & x \in R \setminus \{1,2\} \\ \{1,2\}, & x \in \{1,2\} \end{cases}$$

7. T satisfies (c) and (e), but it does not satisfy (d).

$$X = R \cup \{\infty\},\ T(x) = \begin{cases} \{x\}, & x \in R \\ \phi, & x = \infty \end{cases}$$

8. T satisfies (c), but none of (d) and (e).

$$X = R \cup \{\infty\},\ T(x) = \begin{cases} \{-x, x\}, & x \in R \\ \phi, & x = \infty \end{cases}$$

REMARK 2. If we consider only multivalued mappings $T : X \multimap X$ such that $X = \operatorname{dom} T$ (all the values of T are nonvoid), the condition (c) implies (d).

Indeed, for any $x \in X$ we have $T(x) \neq \phi$ and we can choose $y \in T(x)$; from the symmetry we obtain $x \in T(y)$ and the transitivity of G(T) implies $x \in T(x)$, i.e. G(T) is reflexive.

If this is the case, the regions denoted by 7 and 8 in Fig. 1 ar are void and the diagram looks like this

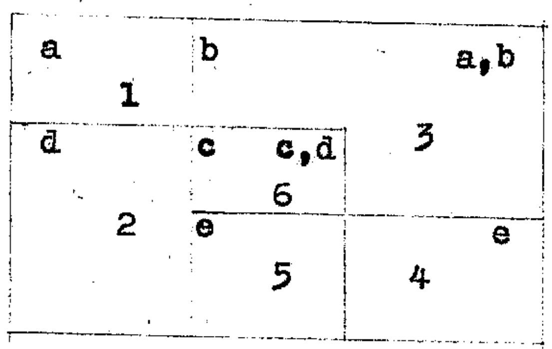

Fig. 2

It follows that a multivalued mapping $T : X \multimap X$ with $X = \operatorname{dom} X$ has a symmetrical and transitive graph (satisfies the condition (c)) if and only if T satisfies the condition (b) and G(T) is a reflexive relation.

The condition (b) leeds us to the definition of a multivalued mapping attached to T.

Let X be a nonvoid set and $T : X \multimap X$ a multivalued mapping. We define $\tilde{T} : X \multimap X$ given by

$$\tilde{T}(x) = \begin{cases} \bigcap_{y \in T(x)} T(y), & \text{for } x \in \operatorname{dom} T \\ X, & \text{for } x \in X \setminus \operatorname{dom} T \end{cases}$$

In the terms of $\tilde{T}$, the condition (b) becomes

$T|_{F_T} \subseteq \tilde{T}|_{F_T}$.

If we consider a new condition

(f) $T \subseteq \tilde{T}$

we obtain obviously that (f) implies (b).

If $T = g : X \to X$ is a function, we have $\tilde{g}(x) = (g \circ g)(x)$ for any $x \in X$; the condition (f) is equivalent to $g(x) = (g \circ g)(x)$ for an any $x \in X$, i.e. to $\operatorname{Im} g = F_g$.

REMARK 3. If T is a multivalued mapping, we have only $\tilde{T} \subseteq T \circ T$ on dom T, the inclusion being generally strict, as the following example shows. Let $T : R \multimap R$ be given by $T(x) = \{0, x\}$; then $T \circ T(x) = \{0, x\} \supsetneq \tilde{T}(x) = \{0\}$, for any $x \neq 0$.

THEOREM 3. *If $T : X \multimap X$ satisfies the condition (f) we have $\operatorname{Im} T = F_T$.*

Proof. We have obviously $F_T \subseteq \operatorname{Im} T$. Let $y \in \operatorname{Im} T$ and $x \in X$ such that $y \in T(x)$. Applying (f), we get $y \in T(x) \subseteq \tilde{T}(x) \subseteq T(y)$ and $y \in F_T$. It follows that $\operatorname{Im} T \subseteq F_T$, so $\operatorname{Im} T = F_T$ holds.

REMARK 4. The reverse implication is not true. For the multivalued mapping T from Example 1 we have $\operatorname{Im} T = F_T = [-1,1]$, but

$$\tilde{T}(x) = \begin{cases} \{0\}, & x \in [-1,1] \\ \{0,1\}, & x \in R \setminus [-1,1] \end{cases}, \text{ so } T \not\subseteq \tilde{T}.$$

The next theorem gives a condition for a point $x \in X$ be a fixed point for $\tilde{T}$.

THEOREM 4. *The element $x \in X$ is a fixed point for $\tilde{T}$ if and only if $T(x) \subseteq T^{-1}(x)$.*

Proof. Let $x \in X$ be a fixed point for $\tilde{T}$; if $x \notin \operatorname{dom} T$, we have $T(x) = \emptyset \subseteq T^{-1}(x)$. In the case that $x \in \operatorname{dom} T$, $T(x)$ is a nonvoid set; let $y \in T(x)$. It follows that $x \in \tilde{T}(x) \subseteq T(y)$, i.e. $y \in T^{-1}(x)$ and we obtain again $T(x) \subseteq T^{-1}(x)$.

Let now $x \in X$ be a point such that $T(x) \subseteq T^{-1}(x)$. If $T(x) = \emptyset$, we have $\tilde{T}(x) = X$ and x is obviously a fixed point for $\tilde{T}$. But y was arbitrary in $T(x)$, so $x \in \bigcap_{y \in T(x)} T(y) = T(x)$, hence $x \in F_{\tilde{T}}$.

COROLLARY. The fixed point set for $\tilde{T}$ is the largest subset of X on which $G(T)$ is symmetrical.

Proof. $F_{\tilde{T}} = \{x \in X : T(x) \subseteq T^{-1}(x)\}$.

REMARK 5. Any strict fixed point for T is also a strict fixed point for $\tilde{T}$. The reverse implication is not true. Indeed, for $T : R \to R$ given by

$$T(x) = \begin{cases} \{-1,1\}, & x = o \\ \{o,-2\}, & x = -1 \\ \{o,2\}, & x = 1 \\ \{x\}, & x \in R \setminus \{o,-1,1\} \end{cases}$$

we have $\tilde{T}(o) = T(-1) \cap T(1) = \{o\}$, so $o \in (SF)_{\tilde{T}}$, but $o \notin (SF)_T$.

REFERENCES

1. Purdea, I., Pic, Gh., Treatise of modern algebra, Vol. 1, ED. Acad R.S.R. 1977 (in Romanian)

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