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ON CARISTI'S THEOREM AND SUCCESSIVE APPROXIMATIONS

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Caristi's theorem [4] is an interesting and powerful generalization of the contraction principle. The first proofs [2,3,7] have shown the existence of a fixed point for the given function lacking the constructive aspects. J. Siegel [9] has considered the problem of the approximation of the fixed point in Caristi's theorem by countable iterations of some functions related to the initial one.

In the paper [10], M. R. Taskovič gives general conditions for each sequence of successive approximations of the function $T : X \to X$, where X is a topological space, to have a subsequence which converges to a fixed point for T. Then Caristi's theorem is derived, considering that each function T which satisfies the hypotheses in Caristi's one satisfies also those in Taskovič's theorem. In order to present this theorem, we mention firstly some definitions.

Let X be a topological space and $T : X \to X$ a function. Denote by $F_T = \{x \in X : Tx = x\}$ the *fixed point set* of the function T. The set $o(x) = \{x, Tx, T^2x, ...\}$ is called the *orbit of* x for each x in X. A function $B : X \to \mathbb{R}$ is said *T-orbitally lower semicontinuous* (*T-orbitally l.s.c.*) *at* p if from $\{x_n\}_{n\in\mathbb{N}}$ sequence in $o(x)$ and $x \xrightarrow{n} p$ it follows $B(p) \leq \liminf_n B(x_n)$. If B is T-orbitally l.s.c. at each p in X, it is called *T-orbitally l.s.c.*.

The topological space X is said to satisfy the *condition of TCS-convergence* if $B(T^n x) \xrightarrow{n} 0$ implies that $\{T^n x\}_{n\in\mathbb{N}}$ has a convergent subsequence.

Then Theorem 2 in [10], with the conclusions completed with some results which follow from its proof, is stated like this.

Theorem 1 . *Let X be a topological space, $T : X \to X$, $B : X \to [0,\infty)$ a T-orbitally l.s.c. function such that X satisfies the condition of TCS-convergence and $B(x) = 0$ implies $Tx = x$.*

Let $\gamma : [0,\infty) \to [0,\infty)$ be a function such that $\gamma(t) < t$ and $\limsup_{z\to t+0} \gamma(z) < t$ for each $t > 0$, the following condition being satisfied

$$B(Tx) \leq \gamma(B(x)) \text{ for each } x \text{ in } X \setminus F_T. \tag{1}$$

Then for each x in X there exists a subsequence $\{T^{n_j}x\}_{j\in\mathbb{N}}$ of successive approximations starting from x, which is convergent to a fixed point ξ of T.

For the sake of completeness we present the proof.

Proof of Theorem 1. Let x be an arbitrary element of X; if x is a fixed point of T (or if $T^n x$ is for some $n \geq 1$), the conclusion holds. Let now $T^{n+1}x \neq T^n x$ for each $n \geq 0$. The condition (1) gives then

$$B(T^{n+1}x) \leq \gamma(B(T^n x)) < B(T^n x), \text{ for each } n \geq 0.$$

The properties of γ assure the fact that $B(T^n x) \xrightarrow{n} 0$. The space X satisfying the condition of TCS-convergence, it follows that there is a subsequence $\{T^{n_j}x\}_{j\in\mathbb{N}}$ convergent to $\xi \in X$. The function B being T-orbitally l.s.c., we have

$$B(\xi) \leq \liminf_j B(T^{n_j}x) = \liminf_n B(T^n) = 0,$$

hence $B(\xi) = 0$ and $T\xi = \xi$, so the theorem is proved. ■

Then, in the paper [10], Caristi's theorem is presented as a consequence of this theorem; we show that this is not the case.

We recall Caristi's theorem.

Theorem 2 [3,4,6]. *Let (X, d) be a complete metric space and $T : X \to X$ a given function. Suppose that there is an lower semicontinuous (l.s.c.) function $G : X \to [0, \infty)$ such that*

$$d(x, Tx) \leq G(x) - G(Tx) \text{ for each } x \text{ in } X. \tag{2}$$

Then the function T has a fixed point in X.

A simple proof, based upon an idea of Brondsted, is given in the book [5, p.16] in the following way.

Proof of Theorem 2. Define a multifunction $C : X \to 2^X \setminus \{\emptyset\}$ by $Cx = \{y \in X : G(y) - d(x, y) \leq G(x)\}$. The function G being l.s.c., Cx is closed for each x in X. Let x_0 be an arbitrary element of X. We construct a sequence $\{x_n\}_{n \in \mathbb{N}}$ choosing $x_1 \in Cx_0$ such that $G(x_1) \leq 1 + \inf G|_{Cx_0}$; after obtaining $x_1, x_2, \ldots, x_{n-1}$, we take $x_n \in Cx_{n-1}$ such that

$$G(x_n) \leq 1/n + \inf G|_{Cx_{n-1}}.$$

The sequence $Cx_0 \supseteq Cx_1 \supseteq \ldots$ is nonincreasing. For each x in Cx_n, $n \geq 1$, we have $x \in Cx_n \subseteq Cx_{n-1}$, hence $G(x) \geq \inf G|_{Cx_{n-1}} \geq G(x_{n)} - 1/n$. Because $x \in Cx_n$, $d(x_n, x) \leq G(x_n) - G(x) \leq 1/n$. It follows that diam $Cx_n \leq 2/n$ for each $n \geq 1$ and applying Cantor's theorem we obtain $x^* \in X$ such that $\bigcap_{n=0}^{\infty} Cx_n = \{x^*\}$. The condition (2) implies $d(x^*, Tx^*) \leq G(x^*) - G(Tx^*)$; but $x^* \in Cx_n$, hence $d(x^*, x_n) \leq G(x_n) - G(x^*)$ for each n in $\mathbb{N}$. It follows that $d(Tx^*, x_n) \leq G(x_n) - G(Tx^*)$, i.e. $Tx^* \in \bigcap_{n=0}^{\infty} Cx_n = \{x^*\}$ and x^* is a fixed point for T. ■

Remark 1 *In this proof of Caristi's theorem, the fixed points of T are obtained as limits of sequences of successive approximations for the multifunction C defined above.*

As it was shown in [2], the proof of Theorem 2 is obvious when T is continuous; in this case the sequence of successive approximations of the given function T starting from each x in X converges to a fixed point of T, even in the absence of the lower semicontinuity of G.

We formulate the mentioned result in [2] in two propositions.

Proposition 1 *Let (X, d) be a complete metric space, $T : X \to X$ and $G : X \to [0, \infty)$ arbitrary functions that satisfy the condition (2). Then the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges for each x in X.*

Proof. Let x be a given element in X. Applying (2) for x, Tx,..., $T^n x$ we obtain

$$\begin{aligned}
&d(x, Tx) \le G(x) - G(Tx)\\
&d(Tx, T^2x) \le G(Tx) - G(T^2x)\\
&\dots\\
&d(T^n x, T^{n+1}x) \le G(T^n x) - G(T^{n+1}x).
\end{aligned}$$

Summing up these inequalities we have

$$\sum_{k=0}^{n} d(T^k x, T^{k+1}x) \le G(x) - G(T^{n+1}x) \le G(x),$$

hence $\{T^n x\}_{n\in\mathbb{N}}$ is a Cauchy sequence; the completeness of X guarantees the convergence of $\{T^n x\}_{n\in\mathbb{N}}$. ■

The next proposition shows that if T is continuous (or even *orbitally continuous* in the sense that $T^n x \overset{n}{\to} \xi$ implies $T^{n+1}x \overset{n}{\to} T\xi$ for each ξ in X), the limit of the sequence of successive approximations of T starting from every point x in X is a fixed point for T.

Proposition 2 *In the hypotheses of Proposition 1, if T is (orbitally) continuous, the sequence of successive approximations of T starting from every point x in X converges to a fixed point of T.*

Proof. Let x be an arbitrary point in X; by Proposition 1, $\{T^n x\}_{n\in\mathbb{N}}$ converges to $\xi \in X$; T being (orbitally) continuous, $T^{n+1}x \xrightarrow{n} T\xi$. But $\{T^{n+1}x\}_{n\in\mathbb{N}}$ converges also to ξ, hence ξ is a fixed point of T. ■

In the absence of the (orbitally) continuity of T, Proposition 2 is no more true (even if G is continuous), as the following example shows.

Example 1 *Let X be the complete metric space $X = \{0, -1\} \cup \{1/n : n \in \mathbb{N}\}$ with the usual metric on $\mathbb{R}$, $T : X \to X$ given by*

$$Tx = \begin{cases} 1/(n+1), \ x = 1/n \\ -1, x \in \{0, -1\} \end{cases}$$

and $G : X \to [0,\infty)$, $G(x) = x + 1$ for each x in X.

G is continuous on X and the condition (2) is satisfied, so Caristi's theorem applies; but the sequence of successive approximations of T starting from all the points of the form $x = 1/n$, $n \in \mathbb{N}$, converges to 0 which is not a fixed point for T. There are only two points, namely 0 and -1, having the property that the sequence of successive approximations converges to the fixed point of T. In this example the conclusion of Theorem 1 does not hold, hence we cannot find any functions B and γ to fulfil the conditions in Theorem 1. It is clear then that Caristi's theorem is not a corollary of Theorem 1.

There are two questions related to the above considerations.

QUESTION 1. Does there exist a complete metric space X and the functions $T : X \to X$, $T \neq id_X$ and $G : X \to [0, \infty)$, satisfying (2) such that $\{T^n x\}_{n\in\mathbb{N}}$ converges to a fixed point of T iff x is a fixed point?

QUESTION 2. In the conditions in Question 1, what can one say about the convergence of $\{T^n x\}_{n\in\mathbb{N}}$ to a fixed point of T if X is also connected?

The problem of the approximation of the fixed points in the case of multifunctions satisfying conditions of Caristi type was considered in papers like [1,8].

For a multifunction $A : X \to 2^X$, a *fixed point* is an element x in X such that $x \in Ax$; a *strict fixed point* is an element y in X such that $Ay = \{y\}$.

In the following we present the analogous of Propositions 1 and 2 for multifunctions, the sequences of successive approximations converging to some fixed point of A.

Proposition 3 *Let (X, d) be a complete metric space, $A : X \to 2^X \setminus \{\emptyset\}$, $G : X \to [0, \infty)$ arbitrary functions which satisfy the condition*
(3)
for each x in X there exists y in Ax such that $d(x, y) \leq G(x) - G(y)$.

Then for each x in X there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$, with $x_1 = x$ and $x_{n+1} \in Ax_n$ for each n in $\mathbb{N}$, which is a convergent one.

Proof. Using the condition (3), we obtain for each x in X an element $y = Tx \in Ax$ such that $d(x, Tx) \leq G(x) - G(Tx)$. Applying Proposition 1, the sequence $\{T^n x\}_{n\in\mathbb{N}}$ is convergent and it is obviously a sequence of successive approximation for the multifunction A.

As it was shown in the paper of J.P. Aubin and J. Siegel [1,T2.4], if $A : X \to 2^X \setminus \{\emptyset\}$ is closed (in the sense that $GrA = \{(x, y) \in X \times X : y \in Ax\}$ is a closed set in the space $X \times X$), the following Proposition similar to Proposition 2 can be easily proved. ■

Proposition 4 *In the hypotheses of Proposition 3, if A is a closed multifunction, A has fixed points and for each x in X there exists a sequence of successive approximations for A which converges to a fixed point of A.*

Proof. Using Proposition 3, we obtain a convergent sequence $\{x_n\}_{n\in\mathbb{N}}$, such that $\lim_n x_n = z \in X$. But $(x_n, x_{n+1}) \in GrA$, hence $(z,z) \in \overline{GrA} = GrA$ and $x \in Ax$. ■

It is also possible to obtain strict fixed points as limits of successive approximations, imposing suitable conditions on the multifunction A.

Proposition 5 *Let (X,d) be a complete metric space, $A : X \to 2^X \setminus \{\emptyset\}$, $G : X \to [0,\infty)$ such that*

(4) $\quad d(x,y) \le G(x) - G(y)$ *for each x in X and for each y in Ax*

$$A^2x \subseteq Ax \text{ for each } x \text{ in } X \tag{5}$$

(6) *Ax is closed for each x in X or A is l.s.c. (i.e. from $y \in Ax$ and $x_n \xrightarrow{n} x$ it follows that there exists $y_n \in Ax_n$, $y_n \xrightarrow{n} y$ for each x in X).*

Then the multifunction A has at least one strict fixed point and for each x in X there is a sequence of successive approximations for A which converges to a strict fixed point.

Proof. Let $x_0 = x$ be an arbitrary element of X : we obtain a sequence considering an element $x_1 \in Ax_0$ such that $G(x_1) \le 1 + \inf G|_{Ax_0}$, and, when x_1, ..., x_{n-1} are known, choosing $x_n \in Ax_{n-1}$ such that $G(x_n) \le 1/n + \inf G|_{Ax_{n-1}}$.

Using (4), we have

$$\begin{aligned} \operatorname{diam} Ax_n &\le \sup\{d(y,x_n) + d(x_n,z) : y,z \in Ax_n\} \\ &= 2\sup\{d(x_n,z) : z \in Ax_n\} \le 2[G(x_n) - \inf G|_{Ax_n}]. \end{aligned}$$

But from (5) it follows that $Ax_n \subseteq Ax_{n-1}$ for each n in $\mathbb{N}$, hence

$$\text{diam } Ax_n \leq 2[G(x_n) - \inf G|_{Ax_{n-1}}] \leq 2/n.$$

It follows that $\bigcap_n \overline{Ax_n} = \{x^*\}$. Because $x_{n+p}, x_{n+1} \in Ax_n$, we have $d(x_{n+p}, x_{n+1}) \leq 2/n$ and $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete metric space X, hence it is a convergent one. But $d(x_{n+1}, x^*) \leq \text{diam } \overline{Ax_n} \leq 2/n$ and it follows that $x^* is$ the limit of $\{x_n\}_{n\in\mathbb{N}}$.

Using the condition (6) we show that x^* is in fact a strict fixed point for A.

a) If Ax is closed for each x in X, we have $\bigcap_n Ax_n = \{x^*\}$. Since $x^* \in Ax_n$ for each n in $\mathbb{N}$, the condition (5) implies $Ax^* \subseteq Ax_n$ for each n in $\mathbb{N}$, i.e. $Ax^* \subseteq \bigcap_n Ax_n = \{x^*\}$. But $Ax^* \neq \emptyset$, hence $Ax^* = \{x^*\}$.

b) Let now A be l.s.c.. We have $x_n \xrightarrow{n} x^*$; then for each $y$in Ax^*there exists an element $y_n \in Ax_n (n \in \mathbb{N})$such that $y_n \xrightarrow{n} y$. But

$$d(y_n, x^*) \leq d(y_n, x_{n+1}) + d(x_{n+1}, x^*) \leq \text{diam } Ax_n + d(x_{n+1}, x^*),$$

so $y_n \xrightarrow{n} x^*$. It follows that $y = x^*$, hence we obtain again $Ax^* = \{x^*\}$.
■

Remark 2 *It is easily seen that Caristi's theorem for multifunctions is a consequence of Proposition 5 (therefore Theorem 2 is a consequence too). Indeed, suppose that* $G : X \to [0, \infty)$ *is l.s.c. and* A *satisfies* (4)*; then the multifunction* $C : X \to 2^X \setminus \{\emptyset\}$ *defined in the proof of theorem 2 has closed values and it obviously verifies the condition* (5)*. It follows that there exists an element* x^* *in* X *such that* $Cx^* = \{x^*\}$*. But the condition* (4) *implies* $Ax \subseteq Cx$ *for each* x *in* X*, and because of* $Ax^* \neq \emptyset$ *we have* $Ax^* = \{x^*\}$*. In this case, the strict fixed point is also obtained as a limit of successive approximations, but these are considered for the multifunction* C*, as it was pointed out in the case of the initial theorem of Caristi.*

To obtain the fixed points as limits of successive approximations for the given (multi)function, it is necessary to impose supplementary conditions on the functions $T : X \to X$, respectively $A : X \to 2^X \setminus \{\emptyset\}$.

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