Caristi’s theorem [4] is an interesting and powerful generalization of the contraction principle. The first proofs [2,3,7] have shown the existence of a fixed point for the given function lacking the constructive aspects. J. Siegel [9] has considered the problem of the approximation of the fixed point in Caristi’s theorem by countable iterations of some functions related to the initial one.

In the paper [10], M. R. Tasković gives general conditions for each sequence of successive approximations of the function $T : X \rightarrow X$, where $X$ is a topological space, to have a subsequence which converges to a fixed point for $T$. Then Caristi’s theorem is derived, considering that each function $T$ which satisfies the hypotheses in Caristi’s one satisfies also those in Tasković’s theorem. In order to present this theorem, we mention firstly some definitions.

Let $X$ be a topological space and $T : X \rightarrow X$ a function. Denote by $F_T = \{x \in X : Tx = x\}$ the fixed point set of the function $T$. The set $o(x) = \{x, Tx, T^2x, ...\}$ is called the orbit of $x$ for each $x$ in $X$. A function $B : X \rightarrow \mathbb{R}$ is said $T$-orbitally lower semicontinuous (T-orbitally l.s.c.) at $p$ if from $\{x_n\}_{n \in \mathbb{N}}$ sequence in $o(x)$ and $x_n \rightarrow p$ it follows $B(p) \leq \lim \inf B(x_n)$. If $B$ is $T$-orbitally l.s.c. at each $p$ in $X$, it is called $T$-orbitally l.s.c.
The topological space $X$ is said to satisfy the condition of TCS-convergence if $B(T^n x) \xrightarrow{n} 0$ implies that $\{T^n x\}_{n \in \mathbb{N}}$ has a convergent subsequence.

Then Theorem 2 in [10], with the conclusions completed with some results which follow from its proof, is stated like this.

**Theorem 1.** Let $X$ be a topological space, $T : X \to X$, $B : X \to [0, \infty)$ a $T$-orbitally l.s.c. function such that $X$ satisfies the condition of TCS-convergence and $B(x) = 0$ implies $Tx = x$.

Let $\gamma : [0, \infty) \to [0, \infty)$ be a function such that $\gamma(t) < t$ and $\limsup_{z \to t+0} \gamma(z) < t$ for each $t > 0$, the following condition being satisfied

\begin{equation}
B(Tx) \leq \gamma(B(x)) \text{ for each } x \text{ in } X \setminus F_T.
\end{equation}

Then for each $x$ in $X$ there exists a subsequence $\{T^{n_j} x\}_{j \in \mathbb{N}}$ of successive approximations starting from $x$, which is convergent to a fixed point $\xi$ of $T$.

For the sake of completeness we present the proof.

**Proof of Theorem 1.** Let $x$ be an arbitrary element of $X$; if $x$ is a fixed point of $T$ (or if $T^n x$ is for some $n \geq 1$), the conclusion holds. Let now $T^{n+1} x \neq T^n x$ for each $n \geq 0$. The condition (1) gives then

$$B(T^{n+1} x) \leq \gamma(B(T^n x)) < B(T^n x), \text{ for each } n \geq 0.$$ 

The properties of $\gamma$ assure the fact that $B(T^n x) \xrightarrow{n} 0$. The space $X$ satisfying the condition of TCS-convergence, it follows that there is a subsequence $\{T^{n_j} x\}_{j \in \mathbb{N}}$ convergent to $\xi \in X$. The function $B$ being $T$-orbitally l.s.c., we have

$$B(\xi) \leq \liminf_{j} B(T^{n_j} x) = \liminf_{n} B(T^n) = 0,$$

hence $B(\xi) = 0$ and $T\xi = \xi$, so the theorem is proved. \QED
Then, in the paper [10], Caristi’s theorem is presented as a consequence of this theorem; we show that this is not the case.

We recall Caristi’s theorem.

**Theorem 2** [3,4,6]. Let \((X,d)\) be a complete metric space and \(T : X \to X\) a given function. Suppose that there is an lower semicontinuous (l.s.c.) function \(G : X \to [0, \infty)\) such that

\[
(2) \quad d(x, Tx) \leq G(x) - G(Tx) \quad \text{for each } x \text{ in } X.
\]

Then the function \(T\) has a fixed point in \(X\).

A simple proof, based upon an idea of Brondsted, is given in the book [5, p.16] in the following way.

**Proof of Theorem 2.** Define a multifunction \(C : X \to 2^X \setminus \{\emptyset\}\) by \(Cx = \{y \in X : G(y) - d(x, y) \leq G(x)\}\). The function \(G\) being l.s.c., \(Cx\) is closed for each \(x\) in \(X\). Let \(x_0\) be an arbitrary element of \(X\). We construct a sequence \(\{x_n\}_{n \in \mathbb{N}}\) choosing \(x_1 \in Cx_0\) such that \(G(x_1) \leq 1 + \inf G|_{Cx_0}\); after obtaining \(x_1, x_2, ..., x_{n-1}\), we take \(x_n \in Cx_{n-1}\) such that

\[
G(x_n) \leq 1/n + \inf G|_{Cx_{n-1}}.
\]

The sequence \(Cx_0 \supseteq Cx_1 \supseteq ...\) is nonincreasing. For each \(x\) in \(Cx_n\), \(n \geq 1\), we have \(x \in Cx_n \subseteq Cx_{n-1}\), hence \(G(x) \geq \inf G|_{Cx_{n-1}} \geq G(x_n) - 1/n\). Because \(x \in Cx_n\), \(d(x_n, x) \leq G(x_n) - G(x) \leq 1/n\).

It follows that \(\text{diam } Cx_n \leq 2/n\) for each \(n \geq 1\) and applying Cantor’s theorem we obtain \(x^* \in X\) such that \(\bigcap_{n=0}^{\infty} Cx_n = \{x^*\}\). The condition (2) implies \(d(x^*, Tx^*) \leq G(x^*) - G(Tx^*)\); but \(x^* \in Cx_n\), hence \(d(x^*, x_n) \leq G(x_n) - G(x^*)\) for each \(n\) in \(\mathbb{N}\). It follows that \(d(Tx^*, x_n) \leq G(x_n) - G(Tx^*)\), i.e. \(Tx^* \in \bigcap_{n=0}^{\infty} Cx_n = \{x^*\}\) and \(x^*\) is a fixed point for \(T\). ■
Remark 1 In this proof of Caristi’s theorem, the fixed points of $T$ are obtained as limits of sequences of successive approximations for the multifunction $C$ defined above.

As it was shown in [2], the proof of Theorem 2 is obvious when $T$ is continuous; in this case the sequence of successive approximations of the given function $T$ starting from each $x$ in $X$ converges to a fixed point of $T$, even in the absence of the lower semicontinuity of $G$.

We formulate the mentioned result in [2] in two propositions.

Proposition 1 Let $(X,d)$ be a complete metric space, $T : X \to X$ and $G : X \to [0, \infty)$ arbitrary functions that satisfy the condition (2). Then the sequence $\{T^n x\}_{n \in \mathbb{N}}$ converges for each $x$ in $X$.

Proof. Let $x$ be a given element in $X$. Applying (2) for $x, Tx, \ldots, T^nx$ we obtain

\[
d(x, Tx) \leq G(x) - G(Tx)
\]
\[
d(Tx, T^2x) \leq G(Tx) - G(T^2x)
\]
\[\ldots\]
\[
d(T^nx, T^{n+1}x) \leq G(T^nx) - G(T^{n+1}x).
\]

Summing up these inequalities we have

\[
\sum_{k=0}^{n} d(T^k x, T^{k+1}x) \leq G(x) - G(T^{n+1}x) \leq G(x),
\]

hence $\{T^n x\}_{n \in \mathbb{N}}$ is a Cauchy sequence; the completeness of $X$ guarantees the convergence of $\{T^n x\}_{n \in \mathbb{N}}$. 

The next proposition shows that if $T$ is continuous (or even orbitally continuous in the sense that $T^n x \xrightarrow{n} \xi$ implies $T^{n+1} x \xrightarrow{n} T\xi$ for each $\xi$ in $X$), the limit of the sequence of successive approximations of $T$ starting from every point $x$ in $X$ is a fixed point for $T$. 


Proposition 2 In the hypotheses of Proposition 1, if $T$ is (orbitally) continuous, the sequence of successive approximations of $T$ starting from every point $x$ in $X$ converges to a fixed point of $T$.

Proof. Let $x$ be an arbitrary point in $X$; by Proposition 1, $\{T^n x\}_{n\in \mathbb{N}}$ converges to $\xi \in X$; $T$ being (orbitally) continuous, $T^{n+1} x \to T\xi$. But $\{T^{n+1} x\}_{n\in \mathbb{N}}$ converges also to $\xi$, hence $\xi$ is a fixed point of $T$. 

In the absence of the (orbitally) continuity of $T$, Proposition 2 is no more true (even if $G$ is continuous), as the following example shows.

Example 1 Let $X$ be the complete metric space $X = \{0, -1\} \cup \{1/n : n \in \mathbb{N}\}$ with the usual metric on $\mathbb{R}$, $T : X \to X$ given by

$$T x = \begin{cases} 
1/(n + 1), & x = 1/n \\
-1, & x \in \{0, -1\}
\end{cases}$$

and $G : X \to [0, \infty)$, $G(x) = x + 1$ for each $x$ in $X$.

$G$ is continuous on $X$ and the condition (2) is satisfied, so Caristi’s theorem applies; but the sequence of successive approximations of $T$ starting from all the points of the form $x = 1/n$, $n \in \mathbb{N}$, converges to 0 which is not a fixed point for $T$. There are only two points, namely 0 and $-1$, having the property that the sequence of successive approximations converges to the fixed point of $T$. In this example the conclusion of Theorem 1 does not hold, hence we cannot find any functions $B$ and $\gamma$ to fulfill the conditions in Theorem 1. It is clear then that Caristi’s theorem is not a corollary of Theorem 1.

There are two questions related to the above considerations.

QUESTION 1. Does there exist a complete metric space $X$ and the functions $T : X \to X$, $T \neq id_X$ and $G : X \to [0, \infty)$, satisfying (2) such that $\{T^n x\}_{n\in \mathbb{N}}$ converges to a fixed point of $T$ iff $x$ is a fixed point?
QUESTION 2. In the conditions in Question 1, what can one say about the convergence of \( \{T^n x\}_{n \in \mathbb{N}} \) to a fixed point of \( T \) if \( X \) is also connected?

The problem of the approximation of the fixed points in the case of multifunctions satisfying conditions of Caristi type was considered in papers like [1,8].

For a multifunction \( A : X \to 2^X \), a fixed point is an element \( x \) in \( X \) such that \( x \in Ax \); a strict fixed point is an element \( y \) in \( X \) such that \( Ay = \{ y \} \).

In the following we present the analogous of Propositions 1 and 2 for multifunctions, the sequences of successive approximations converging to some fixed point of \( A \).

**Proposition 3** Let \((X, d)\) be a complete metric space, \( A : X \to 2^X \setminus \{\emptyset\}, G : X \to [0, \infty) \) arbitrary functions which satisfy the condition (3)

for each \( x \) in \( X \) there exists \( y \) in \( Ax \) such that \( d(x, y) \leq G(x) - G(y) \).

Then for each \( x \) in \( X \) there exists a sequence \( \{x_n\}_{n \in \mathbb{N}} \), with \( x_1 = x \) and \( x_{n+1} \in Ax_n \) for each \( n \) in \( \mathbb{N} \), which is a convergent one.

**Proof.** Using the condition (3), we obtain for each \( x \) in \( X \) an element \( y = Tx \in Ax \) such that \( d(x, Tx) \leq G(x) - G(Tx) \). Applying Proposition 1, the sequence \( \{T^n x\}_{n \in \mathbb{N}} \) is convergent and it is obviously a sequence of successive approximation for the multifunction \( A \).

As it was shown in the paper of J.P. Aubin and J. Siegel [1, T2.4], if \( A : X \to 2^X \setminus \{\emptyset\} \) is closed (in the sense that \( GrA = \{(x, y) \in X \times X : y \in Ax\} \) is a closed set in the space \( X \times X \)), the following Proposition similar to Proposition 2 can be easily proved. \( \blacksquare \)

**Proposition 4** In the hypotheses of Proposition 3, if \( A \) is a closed multifunction, \( A \) has fixed points and for each \( x \) in \( X \) there exists a sequence of successive approximations for \( A \) which converges to a fixed point of \( A \).
Proof. Using Proposition 3, we obtain a convergent sequence \( \{x_n\}_{n \in \mathbb{N}} \), such that \( \lim_{n} x_n = z \in X \). But \( (x_n, x_{n+1}) \in \text{Gr}A \), hence \( (z, z) \in \overline{\text{Gr}A} = \text{Gr}A \) and \( x \in Ax \). ■

It is also possible to obtain strict fixed points as limits of successive approximations, imposing suitable conditions on the multifunction \( A \).

**Proposition 5** Let \((X, d)\) be a complete metric space, \( A : X \to 2^X \setminus \{\emptyset\} \), \( G : X \to [0, \infty) \) such that

\[(4) \quad d(x, y) \leq G(x) - G(y) \text{ for each } x \text{ in } X \text{ and for each } y \text{ in } Ax \]

\[(5) \quad A^2x \subseteq Ax \text{ for each } x \text{ in } X \]

\( Ax \) is closed for each \( x \text{ in } X \) or \( A \) is l.s.c. (i.e. from \( y \in Ax \) and \( x_n \to x \) it follows that there exists \( y_n \in Ax_n, \ y_n \to y \) for each \( x \text{ in } X \)).

Then the multifunction \( A \) has at least one strict fixed point and for each \( x \text{ in } X \) there is a sequence of successive approximations for \( A \) which converges to a strict fixed point.

**Proof.** Let \( x_0 = x \) be an arbitrary element of \( X \) : we obtain a sequence considering an element \( x_1 \in Ax_0 \) such that \( G(x_1) \leq 1 + \inf G|_{Ax_0} \), and, when \( x_1, \ldots, x_{n-1} \) are known, choosing \( x_n \in Ax_{n-1} \) such that \( G(x_n) \leq 1/n + \inf G|_{Ax_{n-1}} \).

Using \((4)\), we have

\[
\text{diam } Ax_n \leq \sup \{d(y, x_n) + d(x_n, z) : y, z \in Ax_n\} \\
= 2 \sup \{d(x_n, z) : z \in Ax_n\} \leq 2[G(x_n) - \inf G|_{Ax_n}].
\]
But from (5) it follows that $Ax_n \subseteq Ax_{n-1}$ for each $n$ in $\mathbb{N}$, hence

$$\text{diam } Ax_n \leq 2[G(x_n) - \inf G|_{Ax_n}] \leq 2/n.$$ 

It follows that $\bigcap_n \overline{Ax_n} = \{x^*\}$. Because $x_{n+p}, x_{n+1} \in Ax_n$, we have $d(x_{n+p}, x_{n+1}) \leq 2/n$ and $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in the complete metric space $X$, hence it is a convergent one. But $d(x_{n+1}, x^*) \leq \text{diam } \overline{Ax_n} \leq 2/n$ and it follows that $x^*$ is the limit of $\{x_n\}_{n \in \mathbb{N}}$.

Using the condition (6) we show that $x^*$ is in fact a strict fixed point for $A$.

a) If $Ax$ is closed for each $x$ in $X$, we have $\bigcap Ax_n = \{x^*\}$. Since $x^* \in Ax_n$ for each $n$ in $\mathbb{N}$, the condition (5) implies $Ax^* \subseteq Ax_n$ for each $n$ in $\mathbb{N}$, i.e. $Ax^* \subseteq \bigcap Ax_n = \{x^*\}$. But $Ax^* \neq \emptyset$, hence $Ax^* = \{x^*\}$.

b) Let now $A$ be l.s.c. We have $x_n \xrightarrow{n} x^*$; then for each $y$ in $Ax^*$ there exists an element $y_n \in Ax_n (n \in \mathbb{N})$ such that $y_n \xrightarrow{n} y$. But

$$d(y_n, x^*) \leq d(y_n, x_{n+1}) + d(x_{n+1}, x^*) \leq \text{diam } Ax_n + d(x_{n+1}, x^*),$$

so $y_n \xrightarrow{n} x^*$. It follows that $y = x^*$, hence we obtain again $Ax^* = \{x^*\}$.

\[ \blacksquare \]

**Remark 2** It is easily seen that Caristi’s theorem for multifunctions is a consequence of Proposition 5 (therefore Theorem 2 is a consequence too). Indeed, suppose that $G : X \to [0, \infty)$ is l.s.c. and $A$ satisfies (4); then the multifunction $C : X \to 2^X \setminus \{\emptyset\}$ defined in the proof of theorem 2 has closed values and it obviously verifies the condition (5). It follows that there exists an element $x^*$ in $X$ such that $Cx^* = \{x^*\}$. But the condition (4) implies $Ax \subseteq Cx$ for each $x$ in $X$, and because of $Ax^* \neq \emptyset$ we have $Ax^* = \{x^*\}$. In this case, the strict fixed point is also obtained as a limit of successive approximations, but these are considered for the multifunction $C$, as it was pointed out in the case of the initial theorem of Caristi.
To obtain the fixed points as limits of successive approximations for the given (multi)function, it is necessary to impose supplementary conditions on the functions $T : X \to X$, respectively $A : X \to 2^X \setminus \{\emptyset\}$.

References


