"BABEŞ-BOLYAI" UNIVERSITY
Faculty of Mathematics and Physics
Research Seminars
Seminar on Mathematical Analysis
Preprint Nr. 7, 1988, pp. 87 – 96

FIXED POINTS OF RETRACTIBLE MAPPINGS WITH RESPECT TO THE METRIC PROJECTION

Mira-Cristiana Anisiu

In the paper one presents the notions of retract, retractible function and some properties following the papers /2,5/ in order to obtain generalizations of some fixed point theorems. The notions are extended to point-to-set mappings.

Let $X \neq \emptyset$ a set and $\emptyset \neq A \subseteq X$. A function $r : X \longrightarrow A$ is a retract of X onto A if $r|_A = id_A$. A function $f : A \longrightarrow X$ is retractible onto A with respect to the retract r if Fix $r \circ f$ = Fix f, where Fix f denotes the set of the fixed points of f.

Remark 1. It is obvious that Fix $f \subseteq$ Fix $r \circ f$, because $x = f(x) \in A$ implies $x = r(x) = r(f(x))$; it follows that in the definition of the retractible function one may demand only Fix $r \circ f \subseteq$ Fix f.

R.F.Brown gives in /2/ the following condition for f to be retractible onto A with respect to the retract r :

(1) $x \in r(f(A) \setminus A)$ implies $f(x) = x$ or $f(x) \notin r^{-1}(x)$.

Condition (1) may be reformulated as

(1') $r(f(A) \setminus A) \subseteq \{x \in A : f(x) = x \text{ or } f(x) \notin r^{-1}(x)\}$.

The next proposition is obvious.

Proposition 1. The following sets are equal

$M = \{x \in A : f(x) = x \text{ or } f(x) \notin r^{-1}(x)\}$

$N = \{ x \in A : f(x) \notin r^{-1}(x) \setminus \{x\} \}$

$P = \mathrm{Fix}\, f \cup C_A \mathrm{Fix}\, rof.$

Proposition 2. Condition (1) is equivalent to each of the following

(2) $\mathrm{Fix}\, rof \subseteq \mathrm{Fix}\, f$

(3) $\mathrm{Fix}\, rof \subseteq f^{-1}(A).$

Proof. (1) $\Rightarrow$ (2). Let $x \in \mathrm{Fix}\, rof$, hence $x \in A$ and $x = r(f(x))$. If $f(x) \notin A$, we have $y = f(x) \in f(A) \setminus A$, hence $x = r(y) \in \mathrm{Fix}\, f \cup C_A \mathrm{Fix}\, rof$. It follows $x \in \mathrm{Fix}\, f$, which contradicts $f(x) \notin A$. It remains that $f(x) \in A$ and $x = r(f(x)) = f(x)$, so $x \in \mathrm{Fix}\, f$ and (2) is proved.

(2) $\Rightarrow$ (3) is obvious, since $\mathrm{Fix}\, f \subseteq f^{-1}(A)$.

(3) $\Rightarrow$ (1), in fact (3) $\Rightarrow$ ($A \subseteq \mathrm{Fix}\, f \cup C_A \mathrm{Fix}\, rof$). Let $x \in A$. Suppose that $x \notin C_A \mathrm{Fix}\, rof$, hence $x \in \mathrm{Fix}\, rof \subseteq f^{-1}(A)$ and $f(x) = y \in A$. Then $x = r(f(x)) = f(x) \in A$ and $x \in \mathrm{Fix}\, f$, so $A \subseteq \mathrm{Fix}\, f \cup C_A \mathrm{Fix}\, rof$ and (1) is proved. The last inclusion is in fact an equality, the reverse inclusion being obvious.

It follows that if a function $f : A \longrightarrow X$ admits a retract r and $\mathrm{Fix}\, rof \neq \emptyset$, then $\mathrm{Fix}\, f \neq \emptyset$.

In the following we give a general form of some fixed point theorems, using as a retract the metric projection. We recall some of the properties of the metric projection in Hilbert spaces which are mentioned and used in /6/ to obtain fixed point theorems.

Let H be a Hilbert space and $\emptyset \neq C \subseteq H$ a closed convex set. Then for each x in H there exists a unique $y \in C$ such that

$\|x-y\| = d(x,C) = \inf \{ \|x-z\| : z \in C \}$.

In this case $P = P_C : H \longrightarrow C$, $P(x) = y$ is a function named metric projection.

The function $P : H \longrightarrow C$ is a retract of H on C, because $P|_C = id_C$; it satisfies the well-known relations

(4) $\mathrm{Re}\,(x - Px, Px - y) \geq 0, \ \forall\, x,y \in C$

(5) $\|Px - Py\| \leq \|x-y\|, \ \forall\, x,y \in H$ (P is nonexpansive).

Proposition 3. Let H be a Hilbert space, $\emptyset \neq C \subseteq H$ a closed convex set and $f : C \longrightarrow H$ a given function. If for each $x \in P(f(C) \setminus C)$ which is not a fixed point for f it follows that there exists $y \in C$ such that

(6) $\mathrm{Re}(f(x) - x, x-y) < 0$,

then f is retractible on C with respect to the retraction P.

Proof. Let $x \in P(f(C) \setminus C)$ and $x \notin \mathrm{Fix}\, f$. If $P(f(x)) = x$, then for each $y \in C$ $\mathrm{Re}\,(f(x) - x, x-y) \geq 0$, contradiction. It follows $f(x) \notin P^{-1}(x)$, hence the condition (1) takes place and f is retractible onto C with respect to the retract P.

Now we can prove

THEOREM 1. Let H be a Hilbert space, $\emptyset \neq C \subseteq H$ a closed bounded convex set. Let $f : C \longrightarrow H$ such that for each $x \in P(f(C) \setminus C)$, $x \notin \mathrm{Fix}\, f$ there exists $y \in C$ such that $\mathrm{Re}\,(f(x) - x, x-y) < 0$ and $Pof : C \longrightarrow C$ is nonexpansive. Then f has in C at least a fixed point.

Proof. Accordingly to Proposition 3, f is retractible on C with respect to the retract P and $\mathrm{Fix}\, Pof = \mathrm{Fix}\, f$. But Pof being nonexpansive, the theorem of Browder /1/ implies $\mathrm{Fix}\, Pof \neq \emptyset$.

Corollary 1 /6/. Let H be a Hilbert space, $\emptyset \neq C \subseteq H$ a closed bounded convex set. Let $f : C \longrightarrow H$ a nonexpansive mapping such that for each $x \in \partial C$ there exists $y \in C$ such that

(7) $\|f(x) - y\| \leq \|x-y\|$.

Then f has at least a fixed point.

Proof. Let $x \in P(f(C) \setminus C) \subseteq \partial C$, $x \notin \mathrm{Fix}\, f$. There exists $y \in C$ such that (7) takes place and

$0 \geq \|f(x) - y\|^2 - \|x-y\|^2 = (f(x) - x + x - y, f(x) - x + x - y) - \|x-y\|^2 = \|f(x) - x\|^2 + 2\,\mathrm{Re}(f(x) - x, x-y)$, hence Re $(f(x) - x, x-y) < 0$.

The hypotheses of Theorem 1 are satisfied, since f and P are nonexpansive.

Remark 2. There are functions $f : C \longrightarrow H$ which are not nonexpansive and fulfil the condition in Theorem 1, but not those in Corollary 1.

Let $f : [0,1) \longrightarrow R$,

$$f(x) = \begin{cases} -4x+3, & x \in [0,1/2) \\ 3/2-x, & x \in [1/2,1] \end{cases}$$

a function which verifies the hypotheses of Theorem 1.

Let $x = 0 \in \partial C$; if y verifies (7), it follows $3-y \le y$, hence $y \ge 3/2$, contradiction to $y \in C$.

Applying Browder's theorem in uniformly convex spaces one obtains

THEOREM 2. Let X be a uniformly convex Banach space, $\emptyset \ne C \subseteq X$ a closed bounded convex set. If $f : C \longrightarrow X$ is retractible onto C with respect to the metric projection $P = P_C$ and $P \circ f$ is nonexpansive, then f has in C at least a fixed point.

Proof. Because f is retractible onto C with respect to P, we have Fix f = Fix $P \circ f$. But $P \circ f$ is nonexpansive, hence it has a fixed point by the Browder's theorem.

Remark 3. In the conditions of Theorem 2, $f : C \longrightarrow X$ is retractible onto C with respect to the metric projection P if and only if for each $x \in P(f(C) \setminus C)$, $f(x) \ne x$ there exists $y \in C$ such that $\| f(x) - x \| > \| f(x) - y \|$.

Indeed, let $x \in P(f(C) \setminus C) \subseteq C$, $f(x) \ne x$; in this case we have $f(x) \notin P^{-1}(x) \Longleftrightarrow P(f(x)) \ne x \Longleftrightarrow \exists y \in C, \; \| f(x) - x \| > \| f(x) - y \|$.

We obtain

Corollary 2. In the conditions of Theorem 2, if $f : C \longrightarrow X$ has the property that for each $x \in P(f(C) \setminus C) \subseteq \partial C$, $x \ne f(x)$ there

exists $y \in C$ such that

(8) $\quad \| f(x) - x \| > \| f(x) - y \|$

and $P \circ f$ is nonexpansive, then f has in C at least a fixed point.

Using the condition (7) one obtains

Corollary 3. In the hypotheses of Theorem 2, if $f : C \longrightarrow X$ has the property that for each $x \in P(f(C) \setminus C) \subseteq \partial C$ there exists $y \in C$ such that (7) takes place and $P \circ f$ is nonexpansive, then f has in C at least a fixed point.

Proof. Let $x \in P(f(C) \setminus C)$; if $x = f(x)$ the theorem is proved. If $x \ne f(x)$, it follows that there exists $y \in C$ such that $\| f(x) - y \| \le \| x-y \|$. If we suppose $x = P \circ f(x)$, then $\| f(x) - x \| = \inf \{ \| f(x) - y \| : y \in C \} \le \inf \{ \| x-y \| : y \in C \} = 0$, which is a contradiction. It follows that f is retractible onto C with respect to P, hence Fix $f \ne \emptyset$.

Now we shall generalize the notion of retractible function to point-to-set mappings (shortly, mappings).

Let $X \ne \emptyset$ a set, $\emptyset \ne A \subseteq X$. A mapping $R : X \longrightarrow 2^A \setminus \{\emptyset\}$ is a retract of X onto A if $R|_A = id_A$. Therefore R restricted on the set A is a function which coincides to the identical function. A mapping $F : A \longrightarrow 2^X \setminus \{\emptyset\}$ is retractible onto A with respect to the retract R if Fix $R \circ F$ = Fix F, where Fix $F = \{ x \in A : x \in F(x) \}$.

The analogous of condition (1) is

(9) $\quad x \in R(F(A) \setminus A)$ implies $x \in F(x)$ or $F(x) \cap R^{-1}(x) = \emptyset$,

where $R^{-1}(x) = \{ z \in X : x \in R(z) \}$. The condition (9) may be reformulate as

(9') $\quad R(F(A) \setminus A) \subseteq \{ x \in A : x \in F(x) \text{ or } F(x) \cap R^{-1}(x) = \emptyset \}$.

We obtain some results analogous to those for functions.

Proposition 4. The next two sets are equal

$U = \{ x \in A : x \in F(x) \text{ or } F(x) \cap R^{-1}(x) = \emptyset \}$

Proposition 5. The condition (9) is equivalent to

(1o) $\quad \mathrm{Fix}\ R\circ F \subseteq \mathrm{Fix}\ F$.

Remark 4. It is obvious that $\mathrm{Fix}\ F \subseteq \mathrm{Fix}\ R\circ F$, since $x \in \mathrm{Fix}\ F$ implies $x \in F(x)$ and $x \in A$, hence $x = R(x) \subseteq R(F(x))$ and $x \in \mathrm{Fix}\ R\circ F$. In fact, (1o) means that $\mathrm{Fix}\ R\circ F = \mathrm{Fix}\ F$.

Proof of Proposition 5.

(9') $\Longrightarrow$ (1o). Let $x \in \mathrm{Fix}\ R\circ F$, so $x \in R\circ F(x)$; there exists $y \in F(x)$ such that $x \in R(y)$. If $y \in A$, $R(y) = \{y\}$ and $x = y$, hence $x \in \mathrm{Fix}\ F$. If $y \notin A$, $x \in R(y) \subseteq R(F(A) \setminus A) \subseteq \mathrm{Fix}\ F \cup C_A \mathrm{Fix}\ R\circ F$. But $x \in R\circ F(x)$ and again $x \in \mathrm{Fix}\ F$.

Conversely, we show that (1o) implies $A \subseteq \mathrm{Fix}\ F \cup C_A\mathrm{Fix}\ R\circ F$, the inclusion meaning in fact equality. Indeed, $\mathrm{Fix}\ R\circ F \subseteq \mathrm{Fix}\ F$ implies $C_A\mathrm{Fix}\ R\circ F \supseteq C_A\mathrm{Fix}\ F$, hence $A = \mathrm{Fix}\ F \cup C_A\mathrm{Fix}\ R\circ F$.

In the following we obtain for point-to-set mappings some results which are analogous to those in the first part of this paper. We shall use again as a retract the metric projection on closed convex sets in uniformly convex spaces, which is in fact a function.

In uniformly convex spaces, the metric projection is no more a nonexpansive mapping, but it is a continuous one.

Indeed, let X be a uniformly convex Banach space, $\emptyset \neq C \subseteq X$ a closed convex set. We prove that $P_C = P$ is continuous.

Let $x_n \xrightarrow{n} x$. We have

$$d(x,C) \le \|x - Px_n\| \le \|x-x_n\| + \|x_n - Px_n\| = \|x-x_n\| + d(x_n,C) \le 2\|x-x_n\| + d(x,C).$$

It follows that $\|x - Px_n\| \xrightarrow{n} d(x,C)$, hence $(Px_n)_{n\in N}$ is a minimizing sequence. If $x \in C$, this means precisely $\|Px - Px_n\| \xrightarrow{n} 0$ and the continuity of P is proved.

If $x \notin C$, then $d(x,C) > 0$. The set C being convex it follows that $(Px_n + Px_m)/2 \in C$ and

$$2d(x,C) \le 2\|x - (Px_n + Px_m)/2\| \le \|x - Px_n\| + \|x - Px_m\| \xrightarrow{m,n} 2d(x,C),$$ hence $\|(x - Px_n)/d(x,C) + (y - Px_m)/d(x,C)\| \xrightarrow{m,n} 2.$

Denoting $z_n = (x - Px_n)/d(x,C)$, we have $\|z_n\| \xrightarrow{n} 1$, $\|(z_n + z_m)/2\| \xrightarrow{m,n} 1$ and using the uniformly convexity of X it follows that $(z_n)_{n\in N}$ is a Cauchy sequence /8, L1o.2.2, p. 379/, hence a convergent one. Let $y = \lim_n Px_n$.

Using $d(x,C) \le \|x - Px_n\| \le 2\|x - x_n\| + d(x,C)$ we obtain $\|x - y\| = d(x,C)$, hence $y = Px$ and the continuity of P is proved in this case too.

Now we prove a theorem which extends to mappings whose range is not necessarily in C a theorem of Lim /4/. We denote by $\mathcal{P}_c(C)$ the family of the compact nonvoid subsets of C.

THEOREM 3 /4/. *Let X be a uniformly convex Banach space, $\emptyset \neq C \subseteq X$ a closed bounded convex set, $F : C \longrightarrow \mathcal{P}_c(C)$ a nonexpansive mapping (in $\mathcal{P}_c(C)$ one considers the Hausdorff-Pompeiu metric). Then there exists $x \in C$ such that $x \in F(x)$.*

THEOREM 4. *Let X be a uniformly convex Banach space, $\emptyset \neq C \subseteq X$ a closed bounded convex set, $F : C \longrightarrow \mathcal{P}_c(X)$ a mapping which is retractible onto C with respect to the metric projection $P = P_C$ such that $P\circ F$ is nonexpansive. Then the mapping F has in C at least a fixed point.*

Proof. Because $F(x)$ is a compact set for each x in C and P is a continuous function, we have $P\circ F : C \longrightarrow \mathcal{P}_c(C)$. The mapping F being retractible on C with respect to P, it follows $\mathrm{Fix}\ F = \mathrm{Fix}\ P\circ F$ and using Theorem 4 one obtains $\mathrm{Fix}\ F \neq \emptyset$.

One remarks that F is retractible onto C with respect to P if and only if for each $x \in F(F(C)\setminus C)$ which is not a fixed point for F and for each $z \in F(x)$ there exists $y \in C$ such that $\|z-x\| > \|z-y\|$. Equivalently, for each $z \in F(x)$ we have $\|z-x\| > d(z,C)$.

We obtain

Corollary 4. Let X and C be as in Theorem 4; if $F : C \longrightarrow \mathcal{P}_c(X)$ has theproperty that for each $x \in P(F(C) \setminus C)$ which is not a fixed point for F and for each $z \in F(x)$ there exists $y \in C$ such that $\| z-x \| > \| z-y \|$ and PoF is nonexpansive, then F has in C at least a fixed point.

Theorem 3 is extended in several papers to mappings whose range is not contained in the convex set C, but $F(x) \subseteq J_C(x) = \{(1-a)x + ay : y \; C, \operatorname{Re} a > 1/2\}$ for each x in C.

Remark 5. If X is a real Banach space, then $J_C(x) = \{(1-b)x + by : y \in C, b \geq o\}$ for x in C.

We obtain now

Corollary 5. Let X and C be as in Theorem 4; if $F : C \rightarrow \mathcal{P}_c(X)$ has the property that $F(x) \subseteq J_C(x) = \{(1-a)x + ay : y \; C, \operatorname{Re} a > 1/2\}$ for each $x \in P(F(C) \setminus C)$ which is not a fixed point for F and PoF is nonexpansive, then Fix $F \neq \emptyset$.

Proof. Let $x \in P(F(C) \setminus C)$, $x \notin$ Fix F and $z \in F(x) \subseteq J_C(x)$. It follows that there exists $y \in C$ and $a \in C$, Re $a > 1/2$ such that $z = (1-a)x + ay$.

We have $y \neq x$, since $y = x$ implies $z = x \; F(x)$, contradiction. Then $\| z-x \| = |a| \, \| y-x \|$ and $\| z-y \| = |1-a| \, \| y-x \|$. But Re $a > 1/2$ implies $|a| > |1-a|$, hence $\| z-x \| > \| z-y \|$ and Corollary 4 applies.

Theorem 4 has as a Corollary Theorem 2.3 /3/, where one imposes another condition of inwardness than $F(x) \subseteq J_C(x)$.

Corollary 6 /3/. Let H be a Hilbert space, $\emptyset \neq C \subseteq H$ a closed bounded convex set, $F : C \longrightarrow \mathcal{P}_c(H)$ nonexpansive and $A : C \longrightarrow [o,1)$ an arbitrary function. In addition one suppose that for each $x \in C$ and $y \in F(x)$ one has

(11) $$\liminf_{h \to o_+} h^{-1} d((1-h)x + hy, K) \leq A(x) d(x, Tx).$$

Then F has in C at least a fixed point.

Proof. In the conditions of the corollary, PoF is nonexpansive. We have also Fix PoF $\subseteq$ Fix F. Indeed, let $x \in PoF(x)$, hence there exists $y \in F(x)$ such that $x = P(y)$. We show that for $h \in (o,1)$ one has

(12) $$d((1-h)x + hy, C) = \| (1-h)x + hy - x \|.$$

We suppose that there exists $h \in (o,1)$ such that $z \in C$, $z \neq x$ and

$$\| (1-h)x + hy - z \| < \| (1-h)x + hy - x \|.$$

But $\| y - z \| \leq \| (1-h)x + hy - z \| + \| (1-h)(x - y) \| <$
$< \| (1-h)x + hy - x \| + \| (1-h)(x - y) \| = \| x - y \|$,

which contradicts $x \in P(y)$. It follows that the relation (12) holds, and consequently,

$$d(x, F(x)) \leq \| x - y \| = \liminf h^{-1} d((1-h)x + hy, C) \leq$$
$$\leq A(x) \, d(x, F(x)).$$

But $A(x) < 1$, hence $d(x,F(x)) = o$ and $x \in F(x)$.

Applying Theorem 4 one obtains the conclusion.

R E F E R E N C E S

1. F.E.BROWDER, Fixed points theorems for noncompact mappings in Hilbert space, Proc. Nat. Acad. Sci. 53(1965), 1272-1276
2. R.F.BROWN, Retraction methods in Nielsen fixed point theory, Pacific J. Math. 115(1984), 277-316
3. D.J.DOWNING, W.O.RAY, Some remarks on set-valued mappings, Nonlinear Analysis, T.M.A. 5(12)(1981), 1367-1377

4. T.C.LIM, A fixed point theorem for multivalued nonexpansive mappings in a uniformly convex Banach space, Bull. Amer. Math. Soc. 80(1974), 1123-1126

5. I.A.RUS, The fixed point structures and the retraction mapping principle, "Babeş-Bolyai" Univ., Fac. of Math., Research Sem., Preprint nr. 3, 1986, 175-184

6. R.SCHONEBERG, Some fixed point theorems for mappings of nonexpansive type, Comm. Math. Univ. Carolinae 17 (1976), 399-411

7. S.P.SINGH, S.MASSA, D.ROUX, Approximation technique in fixed point theory, Rendiconti Sem. Math. Fis. Milano VIII(1983), 165-172

8. Gh.SIRETCHI, Analiză funcţională, Univ. din Bucureşti, 1982 (mimeographed)

Institutul de Calcul
Oficiul Poştal 1
C.P. 68
3400 Cluj-Napoca
ROMANIA

This paper is in final form and no version of it is or will be submitted for publication elsewhere.