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ON SOME CONDITIONS FOR THE EXISTENCE OF THE FIXED POINTS IN HILBERT SPACES

by
MIRA-CRISTIANA ANISIU and VALERIU ANISIU
(Cluj-Napoca)

In the paper /3/, Ky Fan gave the following theorem:

Let K be a nonempty compact convex set in a normed linear space X. For any continuous map f of K into X, there exists a point $u \in K$ such that

$$\| u - f(u) \| = d(f(u),K).$$

In the case of Hilbert space, similar theorems were proved by Singh and Watson /7/ and Lin and Yen /4/ for closed convex sets and nonexpansive maps, respectively continuous 1-set-contractive maps subject to some supplementary conditions.

It is interesting to observe that in the case of Hilbert space with K closed convex set, the conclusion in Fan theorem means exactely $u = P_K \circ f(u)$, where P_K is the metric projection. Consequently, the theorems of this type prove the existence of fixed points for the map $P_K \circ f$. If such a fixed point is also a fixed point for the map f, we obtain a fixed point theorem for the given map.

In the paper /4/ one gives five conditions, the fourth of them being just Fix f = Fix $P_K \circ f$. The aim of this note is to study the relations between the classes of maps satisfying these conditions. All of them are sufficient conditions for $\mathrm{Fix}\, f = \mathrm{Fix}\, P_E \circ f$. Such problems were studied also by Reich /7/ and Williamson /8/.

We first prove two auxiliary lemmas.

Let X be a normed space, $A \subseteq X$ a nonvoid convex set and a_o in A. Denote $I_A(a_o) = \{a_o + t(a-a_o) : t > o,\ a \in A\}$ and

$d(x,A) = \inf \{ \| x-a \| : a \in A \}$ for any x in X.

<u>Lemma</u> 1. <u>For each</u> x <u>in</u> X, $\lim_{t\to o+} \frac{1}{t} d(a_o + t(x-a_o), A) =$

$= \inf_{t>o} \frac{1}{t} d(a_o + t(x-a_o), A) = d(x, I_A(a_o))$.

<u>Proof</u>. Because of the convexity of A, the map $d(\cdot,A) : X \longrightarrow$ $\longrightarrow R_+$ is convex. It follows that $\varphi : R_+ \longrightarrow R_+$, $\varphi(t) =$ $= d(a_o + t(x-a_o), A)$ is a convex map, hence $\psi : R_+ \setminus \{o\} \longrightarrow R_+$, $\psi(t) = \frac{1}{t} d(a_o + t(x-a_o), A)$ is an increasing map (we have $\psi(t) = \frac{1}{t}(\varphi(t) - \varphi(o))$). That is why the first equality in the Lemma holds.

It is obvious that

$$\frac{1}{t} d(a_o + t(x-a_o), A) = \frac{1}{t} d(t(x-a_o), A-a_o) = d(x-a_o, \frac{1}{t}(A-a_o)),$$

hence

$$\inf_{t>o} \frac{1}{t} d(a_o + t(x-a_o), A) = \inf_{t>o} d(x-a_o, t(A-a_o)) =$$

$$= \inf_{t>o} d(x, a_o + t(A-a_o)) = \inf_{t>o} \inf_{a\in A} \| x-a_o-t(a-a_o) \| =$$

$$= d(x, I_A(a_o)).$$

<u>Remark</u>. The monotony of ψ implies $d(x,I_A(a_o)) =$

$$= \inf_{o<t\leq 1} \frac{1}{t} d(a_o + t(x-a_o), A).$$

Let X be a prehilbertian space, $A \subseteq X$ a nonvoid complete convex set, a_o in A, x in X and $P : X \longrightarrow A$ the proximity map.

The next lemma gives a characterization of the fact that a_o is the point of best approximation in A for x, using the well-known one:

$Px = a_o$ iff $\mathrm{Re}\langle x-a_o, a-a_o\rangle \leq o$ for each a in A.

Lemma 2. <u>In a prehilbertian space</u> X, <u>for a nonvoid complete convex set</u> $A \subseteq X$ <u>and</u> a_o <u>in</u> A, x <u>in</u> X <u>the following assertions are equivalent</u>:

1^o $Px = a_o$

2^o $\| x-a_o \| = d(x,I_A(a_o))$

3^o $\| x-a_o \| \leq d(x,I_A(a_o))$.

<u>Proof</u>.

$1^o \Longrightarrow 2^o$. Let $a_o = Px$. Denote $A_o = A-a_o$ and $x_o = x-a_o$. Then using Lemma 1 and the remark after, one has

$$d(x,I_A(a_o))^2 = \inf_{o<t\leq 1} \frac{1}{t^2} d(a_o + t(x-a_o), A)^2 =$$

$$= \inf_{o<t\leq 1} \frac{1}{t^2} d(tx_o, A_o)^2 = \inf_{o<t\leq 1} d(x_o, \tfrac{1}{t}A_o)^2 =$$

$$= \inf_{t\geq 1} d(x_o, tA_o)^2 = \inf_{t\geq 1} \inf_{a\in A_o} \| x_o-ta \|^2 =$$

$$= \inf_{t\geq 1} \inf_{a\in A_o} (t^2 \| a \|^2 - 2t\mathrm{Re}\langle x_o,a\rangle + \| x_o \|^2) =$$

$$= \inf_{a\in A_o} \inf_{t\geq 1} (t^2 \| a \|^2 - 2t\mathrm{Re}\langle x_o,a\rangle + \| x_o \|^2).$$

But $\mathrm{Re}\langle x_o,a\rangle = \mathrm{Re}\langle x-a_o, a+a_o-a_o\rangle \leq o$ and the function to be minimized for $t \geq 1$ is increasing on the interval $[\mathrm{Re}\langle x_o,a\rangle, \infty)$ which includes $[o, \infty)$, hence the infimum is attained in $t = 1$.

It follows

$$d(x,I_A(a_o))^2 = \inf_{a\in A_o} \| x_o-a \|^2 = d(x_o, A_o)^2 = d(x, A)^2 =$$

$$= \| x-a_o \|^2$$

and $d(x,I_A(a_o)) = \| x-a_o \|$.

$2^0 \Rightarrow 3^0$ being obvious, we have to prove

$3^0 \Rightarrow 1^0$. But $\| x-a_0 \| \leq d(x, I_A(a_0)) \leq d(x,A) \leq \| x-a_0 \|$, because $A \subseteq I_A(a_0)$ and the lemma is proved.

Let X be a prehilbertian space, $A \subseteq X$ a nonvoid complete convex set and $f : A \to X$.

The following conditions appear in /4/, where are named also the authors to which they belong:

(1) For each a in A, there is a number λ (real or complex, depending on whether the vector space X is real or complex) such that $|\lambda| < 1$ and $a + (1-\lambda)f(a) \in A$.

(2) For each $a \in A$ with $a \neq f(a)$, there exists b in $I_A(a)$ such that $\| b-f(a) \| < \| a-f(a) \|$.

(3) For each $a \in A$, $f(a) \in \mathrm{cl}\, I_A(a)$, i.e. f is weakly inward.

(4) For each a in the boundary ∂A of A with $a = P \circ f(a)$, a is a fixed point of f.

(5) For each a in ∂A, $\| f(a)-a' \| \leq \| a-a' \|$ for some a' in A.

<u>Remark</u>. Fix f = Fix P∘f iff (4) holds.

If Fix f = Fix P∘f, (4) is obvious.

We have always Fix f $\subseteq$ Fix P∘f. Let $a \in$ Fix P∘f, hence $a = P(f(a))$; $a \in \partial A$; (4) implies $a \in$ Fix f and Fix P∘f = Fix f.

We mention that Fix f = Fix P∘f is exactely the condition given by Brown in /2/ for f to be retractible on A with respect to P (see also /6/); fixed point theorems for such maps are obtained in /1/.

THEOREM. The conditions above are related by the following implications :

$$\begin{array}{ccccccc} (1) & \Longrightarrow & (3) & \Longrightarrow & (4) & \Longleftarrow & (5), \text{ where in (1) } \lambda \in \mathbb{R}. \\ & & & & \Uparrow & & \\ & & & & (2) & & \end{array}$$

$$\begin{array}{ccccc} & & (1') & & \\ & & \Downarrow & & \\ (3) & \Longrightarrow & (4) & \Longleftarrow & (5), \text{ where in (1') } \lambda \in \mathbb{C}. \\ & & \Updownarrow & & \\ & & (2) & & \end{array}$$

<u>Proof</u>.

(1) $\Rightarrow$ (3). Let $a \in A$ and $a' = \lambda a + (1-\lambda)f(a) \in A$, $\lambda \in (-1,1)$. It follows $f(a) = \frac{1}{1-\lambda}(a' - \lambda a) = a + \frac{1}{1-\lambda}(a'-a) \in I_A(a) \subseteq \mathrm{cl}\, I_A(a)$.

(3) $\Rightarrow$ (4). Let $a \in \partial A$, $a = P \circ f(a)$. Using the implication $1^0 \Rightarrow 2^0$ in Lemma 2, one has $\| f(a)-a \| = d(f(a), I_A(a))$. But $f(a) \in \mathrm{cl}\, I_A(a)$, hence $\| f(a) - a \| = 0$ and $f(a) = a$.

(5) $\Rightarrow$ (4). Let $a \in \partial A$, $a \neq f(a)$. Applying (5), one obtains a point a' in A such that $\| f(a)-a' \| \leq \| a-a' \|$. Then $0 \geq \| a-a' \|^2 - \| f(a)-a' \|^2 = \| a-f(a)+f(a)-a' \|^2 - \| f(a)-a' \|^2 = \| a-f(a) \|^2 + 2\mathrm{Re}\langle a-f(a), f(a)-a' \rangle$. For this a' one has

$$2\mathrm{Re}\langle a-f(a), a'-f(a) \rangle \leq -\| a-f(a) \|^2 < 0$$

and using the characterization of the metric projection it follows $a \neq P \circ f(a)$.

(4) $\Rightarrow$ (2). Let $a \in A$, $a \neq f(a)$. For $a \in \mathrm{int}\, A$ and $t \in (0,1)$ sufficiently small, $b = a + t(f(a)-a) \in A$ and $\| b-f(a) \| = (1-t) \| a-f(a) \| < \| a-f(a) \|$.

For $a \in \partial A$, let us suppose $\| b-f(a) \| \geq \| a-f(a) \|$ for each b in $I_A(a)$. Then $\| a-f(a) \| \leq d(f(a), I_A(a))$ and applying $3^0 \Rightarrow 1^0$ in Lemma 2 it follows $P \circ f(a) = a$, contradicting (4). It follows that (2) is true in this case too.

(2) $\Rightarrow$ (4). Let $a \in \partial A$, $a \neq f(a)$. Condition (2) implies $d(f(a), I_A(a)) < \| a-f(a) \|$; using now the implication $1^0 \Rightarrow 3^0$ in Lemma 2, $P \circ f(a) \neq a$ and (4) is proved.

All the implications which do not contain condition (1) are true in the complex case too. It remains to prove only that in this case $(1') \Longrightarrow (4)$.

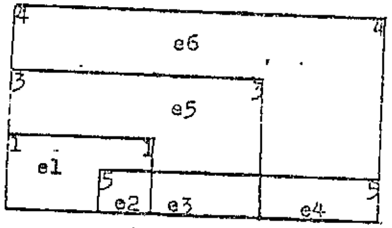
Let $a \in \partial A$, $a \neq f(a)$. By $(1')$, there is $\lambda \in C$, $|\lambda| < 1$ such that $a' = \lambda a + (1-\lambda)f(a) \in A$.

Then $\| a-f(a)\| > |\lambda| \| a-f(a)\| = \| \lambda a - \lambda f(a) \| =$

$= \| \lambda a + (1-\lambda)f(a) - f(a)\| = \| a'-f(a)\| \geq d(f(a), A)$, hence

$d(f(a), A) < \| a-f(a)\|$ and $a \neq P_\circ f(a)$.

One may present the five classes of functions in the following Venn diagram (2 and 4 coincide):



It follows a list of six examples e1-e6 showing that every set which appears in the diagram is nonvoid.

<u>Example e1</u> satisfies (1), but not (5).

Let $X = R$, $A = [0,1]$, $f(x) = 3-3x$; $\bar{x}_0 = 3/4$ is a fixed point. For each a in A and $\lambda = 2/3$ one has $\lambda a + (1-\lambda)f(a) = 1-a/3 \in A$ and (1) holds. But for $a = 0$ one has $f(a) = 3$ and $|3-a'| > a'$ for each a' in A.

<u>Example e2</u> satisfies (1) and (5).

Let X be a normed space, $A \subseteq X$ nonvoid and $f : A \to X$, $f(x) = x$ (or any $f : A \longrightarrow A$, for A convex set).

<u>Example e3</u> satisfies (3) and (5), but not (1).

Let $X = R^2$, $A = \{ (a_1,a_2) \in R^2 : a_1 \geq 0,\ a_2 \geq 0,\ a_1^2 + a_2^2 \leq 1 \}$. The function $f(a) = \left(\frac{a_1}{1+\sqrt{1-a_1^2}}, 1 \right)$, $a = (a_1,a_2)$ has $(0,1)$ as a fixed point.

For $a = (1,0)$ one has $\lambda a + (1-\lambda)f(a) = (1,1-\lambda) \notin A$, for $|\lambda| < 1$.

<u>Example e4</u> satisfies (5), but not (3).

Let $X = R^2$, A the same as in Example e3 and $f : A \longrightarrow X$ be given by $f(a) = \left(\frac{a_1}{2} \left(1 + \frac{1}{1+\sqrt{1-a_1^2}} \right), 1 \right)$ for $a = (a_1,a_2) \in A$, which has $(0,1)$ as a fixed point.

For $a = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, one has $f(a) \notin \mathrm{cl}\, I_A(a)$.

<u>Example e5</u> satisfies (3), but not (1) and (5).

Let $X = R^2$, $\alpha = \arcsin 1/3$, $A = \{ (a_1,a_2) : a_1 \geq 0,\ a_2 \geq a_1 \mathrm{tg}\, \alpha,\ a_1^2 + a_2^2 \leq 1 \}$ and $f : A \longrightarrow X$ be given by $f(a) = \left(0, \frac{1}{\sqrt{1-a_1^2}} \right)$, for which $(0,1)$ is a fixed point.

For $a = (\cos\alpha, \sin\alpha)$, $f(a) = (0,3)$ and (1) and (5) do not hold.

<u>Example e6</u> satisfies (4), but not (3) and (5).

Let $X = R^2$, $A = \{ (a_1,a_2) : a_1^2 + a_2^2 \leq 1 \}$ and $f : A \longrightarrow X$, $f(a) = \sqrt{2}\,(a_1+a_2, a_2-a_1)$ for $a = (a_1,a_2)$, which has $(0,0)$ as a fixed point.

For $a = (\sqrt{2}/2, \sqrt{2}/2)$, (3) and (5) do not hold.

All the maps in these examples have fixed points, as a consequence of the results in /4/, for example of Corollary 4. There are maps with $\mathrm{Fix}\, f \neq \emptyset$ which do not satisfy condition (4), as $f(x) = 2x$, $f: [0,1] \longrightarrow R$.

<u>Remark</u>. From the Venn diagram one can see that in the real case all the implications in the theorem are irreversible. This is also true in the complex case, because the examples related to the conditions (3) and (4), respectively (5) and (4) can be considered in the space C with the scalar field C. It remains to give an example for $(1') \Longrightarrow (4)$.

<u>Example e7</u> satisfies (4), but not $(1')$.

Let $X = C^2$, $A = \{(a_1,a_2) \in C^2 : |a_1|^2 + |a_2|^2 \leq 1\}$, $f :$ $: A \to X$, $f(a) = (a_1+a_2,a_2)$, for $a = (a_1,a_2) \in C^2$, which has $a =$ $= (a_1,o) \in A$ as fixed points.

The point $a = (o,1)$ does not satisfy condition (1').

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This paper is in final form and no version of it will be submitted for publication elsewhere.