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FIXED POINT THEOREMS FOR RETRACTIBLE MAPPINGS

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The fixed point theorems in this paper extend some results in /6/ ; if the retract is chosen to be the metric projection one obtains some known theorems.

We state firstly some definitions to be used in the following.

Let X be a nonvoid set and $\emptyset \neq A \subseteq X$. A map $r : X \longrightarrow A$ is a <u>retract</u> of X on A if the restriction of r to the set A is the identity map $id_A : A \longrightarrow A$. A map $f : A \longrightarrow X$ is <u>retractible</u> on A with respect to the retract r if $\mathrm{Fix}\ r \circ f = \mathrm{Fix}\ f$, where "Fix" denotes the set of the fixed points of a map /3,7,1/.

It is obvious that always $\mathrm{Fix}\ r \circ f \subseteq \mathrm{Fix}\ f$, so in the definition of the retractible map one can demand only $\mathrm{Fix}\ f \subseteq \mathrm{Fix}\ r \circ f$. Brown in /3/ has given the following necessary and sufficient condition for the map $f : A \longrightarrow X$ to be retractible on A with respect to the retract r :

$$x \in r(f(A) \setminus A) \text{ implies } f(x) = x \text{ or } f(x) \notin r^{-1}(x).$$

Let B be a bounded nonempty set in the metric space X. The <u>measure of noncompactness</u> in the sense of Kuratowsky, denoted $a(B)$, is the infimum of the numbers α such that the set B can be covered by a finite number of subsets of X of diameter less than or equal to α.

Let A be a nonvoid subset of the metric space X and $f : A \longrightarrow X$ a map. If there exists k, $o \leq k \leq 1$ such that for each nonempty subset B of A, B being bounded, we have

$$a(f(B)) \leq k\, a(B),$$

then f is called k-set-contractive.

Each nonexpansive map $f : A \longrightarrow X$ (i.e. $d(f(x), f(y)) \leq d(x,y)$ for each x,y in A) is obviously 1-set-contractive.

In the following we give a generalization of Theorem 1 in /6/, using in the proof the fixed point theorem of Darbo:

THEOREM 1 /5/. *Let X be a Banach space, $A \subseteq X$ a closed convex nonvoid set, $f : A \longrightarrow A$ continuous k-set-contractive with $o < k < 1$ such that $f(A)$ is a bounded set. Then* $\mathrm{Fix}\, f \neq \emptyset$.

In the initial form of Darbo's theorem A is bounded, but it suffices to require $f(A)$ bounded, taking $A_1 = \mathrm{cl\ co}\, f(A) \subseteq A$ and $f : A_1 \longrightarrow A_1$.

We need also the following

LEMMA 1. *Let X be a normed space, $C \subseteq X$ and $f : C \longrightarrow X$ such that $(I - f)(C)$ is a closed set. If there exists a sequence of maps $f_n : C \longrightarrow X$, $n \geq 1$, each of them having a fixed point x_n ($f_n(x_n) = x_n$), f_n converging uniformly to f, then f has also a fixed point.*

Proof. Denote g, $g_n : C \longrightarrow X$, $g = I - f$, $g_n = I - f_n$, $n \geq 1$. Each f_n having a fixed point, it follows that each g_n has a zero, hence $o \in g_n(C)$. Because $f_n \xrightarrow{n} f$ uniformly, $g_n \xrightarrow{n} g$ uniformly.

Let $\varepsilon > o$ be arbitrarily chosen; from the uniform convergence of g_n to g one obtains $n_\varepsilon \geq 1$ such that

$$\| g_n(x) - g(x) \| < \varepsilon \quad \text{for each } n \geq n_\varepsilon \text{ and } x \text{ in } C.$$

It follows that for each $n \geq n_\varepsilon$ we have

$$g_n(C) \subseteq g(C) + B(o, \varepsilon).$$

Therefore $o \in g(C) + B(o, \varepsilon)$ for each $\varepsilon > o$ and $o \in \mathrm{cl}\, g(C) = g(C)$. This means that there exists $x \in C$ such that $f(x) = x$.

Now we can prove the following

THEOREM 2. *Let X be a Banach space, $A \subseteq X$ a nonvoid closed convex set, $f : A \longrightarrow X$ and $r : X \longrightarrow A$ such that $F = r \circ f : A \longrightarrow A$ is a continuous 1-set-contractive map having $F(A)$ bounded and $(I - f)(A)$ a closed set. Then* $\mathrm{Fix}\; r \circ f \neq \emptyset$.

Proof. Let $t \in (o,1)$, $x_o \in A$ and $F_t = tF + (1-t)x_o$. We show that F_t is t-set contractive.

Let $B \subseteq A$ a bounded subset ; then

$$a(F_t(B)) = a(tF(B) + (1-t)x_o) \leq a(tF(B)) = t a(F(B)) \leq t a(B).$$

Applying Darbo's theorem, each F_t has a fixed point x_t. Considering $t_n \longrightarrow 1$, $t_n < 1$ we obtain for each x in A

$$\| F_{t_n}(x) - F(x) \| = (1-t_n) \| F(x) - x_o \| \leq$$

$$\leq (1 - t_n)\, d(x_o, F(A)) \xrightarrow{t_n \to 1} o,$$

hence $F_{t_n} \longrightarrow F$ uniformly on A.

Now Lemma 1 applies and $\mathrm{Fix}\, F \neq \emptyset$.

In the above theorem, instead of $(I - F)(A)$ to be closed, one could require $(I - F)(\mathrm{cl\ co}\, F(A))$ to be closed in X, considering the restriction of F on $\mathrm{cl\ co}\, F(A)$, whose range is also in $\mathrm{cl\ co}\, F(A)$.

In the terms of F, Theorem 2 is exactely Lemma 1 in /6/, given there without proof.

A natural example of a map $r : X \longrightarrow A$ is the metric projection, which is well-defined if for example X is uniformly convex. In this case we obtain obviously

COROLLARY 1. Let X be a uniformly convex Banach space, $A \subseteq X$ a nonvoid closed convex set, $f : A \longrightarrow X$ and $p = P_A : X \longrightarrow A$ the metric projection. If $F = p \circ f : A \longrightarrow A$ is a continuous 1-set-contractive map with $F(A)$ bounded and $(I - F)(A)$ a closed set, then $\mathrm{Fix}\, p \circ f \neq \emptyset$.

The conclusion means exactely that there exists x in A such that $p \circ f(x) = x$, i.e. $\| x - f(x) \| = d(f(x), A)$; this result appears in the well-known theorem of Ky Fan (1969) given for a nonvoid compact convex subset K of a normed space X and a continuous map $f : X \longrightarrow K$.

COROLLARY 2 (Theorem 1 in /6/). Let X be a Hilbert space, $A \subseteq X$ a nonvoid closed convex set, $f : A \longrightarrow X$ a continuous 1-set-contractive map. We suppose that either $(I - p \circ f)(A)$ is closed or $(I - p \circ f)(\mathrm{cl\ co}\ p \circ f(A))$ is closed in X, where $p = P_A : X \longrightarrow A$ is the metric projection. If $f(A)$ is bounded, then there exists u in A such that

$$\| u - f(u) \| = d(f(u), A).$$

Proof. Because $f : A \longrightarrow X$ is continuous 1-set-contractive and $p : X \longrightarrow A$ is nonexpansive /4/, it follows that $F = p \circ f$ is a continuous 1-set-contractive map. The fact that $f(A)$ is bounded implies $F(A)$ bounded and Corollary 1 applies.

It is obvious that in Corollary 2, instead of $f(A)$ to be bounded it is enough to require $p \circ f(A)$ to be bounded.

In the paper /4/ there are given many results which follow from Corollary 2, among which theorems of Lin, Singh and Watson.

If in Theorem 1 we can choose $r : X \longrightarrow A$ to be a retract such that f is a retractible map with respect to r, we obtain a fixed point theorem for f.

THEOREM 3. If in the conditions in Theorem 1 $r : X \longrightarrow A$ is a retract and $f : A \longrightarrow X$ is retractible with respect to r, then $\mathrm{Fix}\, f \neq \emptyset$.

Proof. From Theorem 1 it follows $\mathrm{Fix}\, r \circ f \neq \emptyset$ and f being retractible with respect to r we have $\mathrm{Fix}\, f = \mathrm{Fix}\, r \circ f \neq \emptyset$.

COROLLARY 3 (Theorem 5 in /6/). Let X be a Hilbert space, A a nonvoid closed convex set, $f : A \longrightarrow X$ a continuous 1-set-contractive map. We suppose that either $(I - p \circ f)(A)$ is closed in X or $(I - p \circ f)(\mathrm{cl\ co}\ p \circ f(A))$ is closed in X, where $p = P_A$ is the metric projection. If $f(A)$ is bounded and f satisfies one of the following conditions :

(1) For each x in A, there is a number λ (real or complex, depending on whether the vector space X is real or complex) such that $|\lambda| < 1$ and $\lambda x + (1-\lambda) f(x) \in A$.

(2) For each x in A with $x \neq f(x)$, there exists y in $I_A(x) = \{x + c(z - x) : z \in A,\ c > 0\}$ such that

$\| y - f(x) \| < \| x - f(x) \|$.

(3) f is weakly inward (i.e. $f(x) \in \mathrm{cl}\ I_A(x)$ for each x in A).

(4) For each u in the boundary of A with $u = p \circ f(u)$, u is a fixed point of f.

(5) For each x in the boundary of A, $\| f(x) - y \| \leq \| x - y \|$ for some y in A.

Then f has a fixed point in A.

Proof. Corollary 2 applies, so $\mathrm{Fix}\, p \circ f \neq \emptyset$. Each of the five conditions implies (4) /2/, which is in fact exactly $\mathrm{Fix}\, r \circ f = \mathrm{Fix}\, f$.

This means that $f : A \longrightarrow X$ is retractible with respect to p and applying Theorem 3 one has $\mathrm{Fix}\, f \neq \emptyset$.

We mention that the condition of the retractibility $\mathrm{Fix}\, p\circ f = \mathrm{Fix}\, f$ takes in Hilbert spaces the following equivalent forms :

(i) For each u in the boundary of A which is not a fixed point for f, there exists y in A such that

$$\mathrm{Re}\,(f(u) - u,\, u - y) < 0.$$

(ii) For each u in the boundary of A which is not a fixed point for f, there exists y in A such that

$$\| y - f(u) \| < \| u - f(u) \| .$$

(iii) For each u in the boundary of A which is not a fixed point for f,

$$\liminf_{t \to 0_+} \frac{1}{t}\, d((1-t)u + tf(u),\, A) < \| f(u) - u \| .$$

The equivalence follows easily from the next lemma, which relies on the fact that in a Hilbert space for a closed convex set A and x in X one has $p(x) = a$ iff $\mathrm{Re}\,(x - a,\, y - a) \leq 0$ for each y in A ($p = P_A$ being the metric projection).

LEMMA 2. <u>Let X be a Hilbert space, $A \subseteq X$ a nonvoid closed convex set, a in A, x in X. The following assertions are equivalent</u> :

1^0 $p(x) = a$, i.e. $\| x - a \| \leq \| x - y \|$ <u>for each y in A</u>.

2^0 $$\liminf_{t \to 0_+} \frac{1}{t}\, d((1-t)a + tx,\, A) = \lim_{t \to 0_+} \frac{1}{t}\, d((1-t)a + tx,\, A) = \inf_{1 \geq t > 0} \frac{1}{t}\, d((1-t)a + tx,\, A) = \| x - a \| .$$

3^0 $$\inf_{1 \geq t > 0} \frac{1}{t}\, d((1-t)a + tx,\, A) \geq \| x - a \| .$$

<u>Proof</u>. $1^0 \Longrightarrow 2^0$. The first equalities in 2^0 are true because

the map $\psi : \mathbb{R}_+ \setminus \{0\} \longrightarrow \mathbb{R}_+$, $\psi(t) = \frac{1}{t}\, d(a + t(x-a),\, A)$ is increasing and

$$\inf_{1 \geq t > 0} \psi(t) = \inf_{t > 0} \psi(t).$$

Now
$$\inf_{1 \geq t > 0} \psi^2(t) = \inf_{1 \geq t > 0} \frac{1}{t}\, d^2(t(x-a),\, A-a) = \inf_{1 \geq t > 0} d^2(x-a,\, \tfrac{1}{t}(A-a)) = \inf_{t \geq 1} d^2(x-a,\, t(A-a)) = \inf_{t \geq 1} \inf_{y \in A-a} \| x-a-ty \|^2 = \inf_{y \in A-a} \inf_{t \geq 1} (t^2 \| y \|^2 - 2t\, \mathrm{Re}\,(x-a,\, y) + \| x-a \|^2).$$

But $\mathrm{Re}\,(x-a,\, y) = \mathrm{Re}\,(x-a,\, y+a-a) \leq 0$, because $y+a \in A$ and $p(x) = a$.

It follows that the map to be minimized is increasing on $[\mathrm{Re}\,(x-a,\, y),\, +\infty)$, hence also on $[1, +\infty)$. The infimum will be attained on $t = 1$ and

$$\inf_{1 \geq t > 0} \psi^2(t) = \inf_{y \in A-a} \| x-a-y \|^2 = \inf_{y \in A} \| x-y \|^2 = \| x-a \|^2$$

and the implication is proved.

$2^0 \Longrightarrow 3^0$ is obvious.

$3^0 \Longrightarrow 1^0$. We have $\| x-a \| \leq \inf_{t > 0} \frac{1}{t}\, ((1-t)a + tx,\, A) \leq$

$$\leq \psi(1) = d(x,\, A) \leq \| x-y \|$$

for each y in A.

<u>Remark</u>. The equivalence in Lemma 2 is also true if X is a prehilbertian space and A a complete convex set.

The assertions (i) - (iii) are all equivalent to the condition (4) in Corollary 3.

For (i) one uses the fact that $p \circ f(u) \neq u$ is equivalent to the existence of y in A such that $\mathrm{Re}\,(u - f(u),\ y - u) > 0$, using the characterization of the metric projection mentioned before Lemma 2 was given.

For (ii) one applies just the definition of $p(f(u))$ and obtains the inequality in (ii).

For (iii) we use the equivalence $1^\circ \Longleftrightarrow 3^\circ$ which was proved in Lemma 2.

So the assertions (i) - (iii) are just reformulations of the fact that the map $f : A \longrightarrow X$ is retractible on A with respect to the retraction p.

The equivalence of (i) - (iii) was in fact proved in /9/, where (i) is called the Leray-Schauder condition, (ii) the Browder-Petryshyn condition and (iii) the Cramer-Ray condition. Here we emphazised the rôle of the properties of the metric projection in this equivalence.

We finish the paper giving a method of approximation of fixed points for maps with $p \circ f$ nonexpansive in Hilbert spaces by a procedure similar to that in the proof of Theorem 1.

THEOREM 4. Let X be a Hilbert space, A a nonvoid closed convex subset of X, $f : A \longrightarrow X$ a map retractible on A with respect to the metric projection such that $F = p \circ f : A \longrightarrow A$ is nonexpansive and $F(A)$ bounded.

Consider $F_k : A \longrightarrow A$, $F_k(x) = kF(x) + (1-k)x_o$, $o < k < 1$, $k \longrightarrow 1$, $x_o \in C$ and x_k the fixed point of the contraction F_k.

Then $x_k \xrightarrow{k \to 1} y_o$, where y_o is the fixed point of f which closest to x_o.

Proof.

The fixed point set Fix F is nonvoid; let y_o be the fixed point of F which is closest to x_o. For F one applies the approximation result in /8/; for $k_n \xrightarrow{n} 1$, $x_n \in (o,1)$, $n \in N$, one obtains firstly that $\{x_{k_n}\}_{n \in N}$ is a bounded sequence. Then a subsequence of $\{x_{k_n}\}_{n \in N}$ will converge weakly to a point x. Using the demiclosedness of $I - F$ one obtains $x = y_o$ and then the strong convergence of $\{x_{k_n}\}_{n \in N}$ to y_o.

But Fix F = Fix f and the theorem is proved.

R E F E R E N C E S

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