The fixed point theorems in this paper extend some results in [6]; if the retract is chosen to be the metric projection one obtains some known theorems.

We state firstly some definitions to be used in the following.

Let \( X \) be a nonvoid set and \( \emptyset \neq A \subseteq X \). A map \( r : X \to A \) is a retract of \( X \) on \( A \) if the restriction of \( r \) to the set \( A \) is the identity map \( id_A : A \to A \). A map \( f : A \to X \) is retractable on \( A \) with respect to the retract \( r \) if \( \text{Fix } r \subseteq \text{Fix } f \), where "Fix" denotes the set of the fixed points of a map [3, 7, 11].

It is obvious that always \( \text{Fix } r \subseteq \text{Fix } f \), so in the definition of the retractable map one can demand only \( \text{Fix } f \subseteq \text{Fix } r \). Brown in [3] has given the following necessary and sufficient condition for the map \( f : A \to X \) to be retractable on \( A \) with respect to the retract \( r \):

\[ x \in r(f(A) \setminus A) \implies f(x) = x \text{ or } f(x) \notin r^{-1}(x). \]

Let \( B \) be a bounded nonempty set in the metric space \( X \). The measure of noncompactness in the sense of Kuratowski, denoted \( \alpha(B) \), is the infimum of the numbers \( \alpha \) such that the set \( B \) can be covered
by a finite number of subsets of $X$ of diameter less than or equal to $\alpha$.

Let $A$ be a nonvoid subset of the metric space $X$ and $f : A \to X$ a map. If there exists $k, 0 < k < 1$ such that for each nonempty subset $B$ of $A$, $B$ being bounded, we have

$$a(f(B)) \leq k \cdot a(B),$$

then $f$ is called $k$-set-contractive.

Each nonexpansive map $f : A \to X$ (i.e., $d(f(x), f(y)) \leq d(x, y)$ for each $x, y$ in $A$) is obviously 1-set-contractive.

In the following we give a generalization of Theorem 1 in /6/, using in the proof the fixed point theorem of Darbo:

**Theorem 1 /5/.** Let $X$ be a Banach space, $A \subseteq X$ a nonvoid closed convex set, $f : A \to A$ a continuous $k$-set-contractive with $0 < k < 1$ such that $f(A)$ is a bounded set. Then $Fix f \neq \emptyset$.

In the initial form of Darbo's theorem $A$ is bounded, but it suffices to require $f(A)$ bounded, taking $A_1 = cl \operatorname{co} f(A)$ $\subseteq A$ and $f : A_1 \to A_1$.

We need also the following

**Lemma 1.** Let $X$ be a normed space, $C \subseteq X$ and $f : C \to X$ such that $(I - f)(C)$ is a closed set. If there exists a sequence of maps $\{f_n : C \to X, n \geq 1\}$, each of them having a fixed point $x_n$ $(f_n(x_n) = x_n)$, $f_n$ converging uniformly to $f$, then $f$ has also a fixed point.

**Proof.** Denote $g = I - f, g_n = I - f_n, n \geq 1$. Each $f_n$ having a fixed point, it follows that each $g_n$ has a zero, hence $0 \in g_n(C)$. Because $f_{n+1} \to f$ uniformly, $g_n \to g$ uniformly.

Let $\varepsilon > 0$ be arbitrarily chosen; from the uniform convergence of $g_n$ to $g$ one obtains $n_0 \geq 1$ such that

$$\|g_n(x) - g(x)\| < \varepsilon$$

for each $n \geq n_0$ and $x$ in $C$.

It follows that for each $n \geq n_0$ we have

$$g_n(C) \subseteq g(C) + B(0, \varepsilon).$$

Therefore $0 \in g(C) + B(0, \varepsilon)$ for each $\varepsilon > 0$ and $0 \in cl g(C)$.

This means that there exists $x \in C$ such that $f(x) = x$.

Now we can prove the following

**Theorem 2.** Let $I$ be a Banach space, $A \subseteq X$ a nonvoid closed convex set, $f : A \to X$ and $r : X \to A$ such that $F = r \circ f : A \to A$ is a continuous 1-set-contractive map having $F(A)$ bounded and $(I - f)(A)$ a closed set. Then $Fix F \neq \emptyset$.

**Proof.** Let $t \in (0, 1), x_0 \in A$ and $F_t = tF + (1-t)x_0$. We show that $F_t$ is $t$-set contractive.

Let $B \subseteq A$ a bounded subset; then

$$a(F_t(B)) \leq a(tF(B)) + (1-t)a(x_0) \leq a(tF(B)) \leq ta(B).$$

Applying Darbo's theorem, each $F_t$ has a fixed point $x_t$. Considering $t_n \to 1$, $t_n < 1$ we obtain for each $x$ in $A$

$$\|F_{t_n}(x) - F(x)\| = (1-t_n)\|F(x) - x_0\| \leq$$

$$\leq (1-t_n) \cdot d(x_0, F(A)) \frac{\|\|}{\varepsilon},$$

hence $F_{t_n} \to F$ uniformly on $A$.

Now Lemma 1 applies and $Fix F \neq \emptyset$.

In the above theorem, instead of $(I - f)(A)$ to be closed, one could require $(I - F)(cl \operatorname{co} F(A))$ to be closed in $X$, considering the restriction of $F$ on $cl \operatorname{co} F(A)$, whose range is also in $cl \operatorname{co} F(A)$.

The terms of $F$, Theorem 2 is exactly Lemma 1 in /6/, given there without proof.
A natural example of a map \( r : X \rightarrow A \) is the metric projection, which is well-defined if for example \( X \) is uniformly convex. In this case we obtain obviously

**COROLLARY 2.** Let \( X \) be a uniformly convex Banach space, \( A \subseteq X \) a nonvoid closed convex set, \( f : A \rightarrow X \) and \( p = P_A : X \rightarrow A \) the metric projection. If \( F = \text{p} \circ f : A \rightarrow A \) is a continuous \( 1 \)-set-contractive map with \( F(A) \) bounded and \((1 - \text{p})(A) \) is a closed set, then \( \text{Fix } p \circ f \neq \emptyset \).

The conclusion means exactly that there exists \( x \in A \) such that \( p \circ f(x) = x \), i.e. \( \| x - f(x) \| = d(f(x), A) \); this result appears in the well-known theorem of Ky Fan (1959) given for a nonvoid compact convex subset \( K \) of a normed space \( X \) and a continuous map \( f : K \rightarrow K \).

**COROLLARY 2 (Theorem 1 in /6/).** Let \( X \) be a Hilbert space, \( A \subseteq X \) a nonvoid closed convex set, \( f : A \rightarrow X \) a continuous \( 1 \)-set-contractive map. We suppose that either \((1 - \text{p})(A) \) is closed or \((1 - \text{p})(\text{cl } p(f(A))) \) is closed in \( X \), where \( p = P_A : X \rightarrow A \) is the metric projection. If \( F(A) \) is bounded, then there exists \( u \in A \) such that

\[
\| u - f(x) \| = d(f(x), A).
\]

**Proof.** Because \( f : A \rightarrow X \) is continuous \( 1 \)-set-contractive and \( p : X \rightarrow A \) is nonexpansive \(/4/\), it follows that \( F = \text{p} \circ f \) is a continuous \( 1 \)-set-contractive map. The fact that \( F(A) \) is bounded implies \( F(A) \) bounded and Corollary 2 applies.

It is obvious that in Corollary 2, instead of \( \text{Fix } F(A) \) to be bounded, it is enough to require \( \text{Fix } p \circ f \) to be bounded.

In the paper /6/ there are given many results which follow from Corollary 2, among which theorems of Lin, Singh and Watson.

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If in Theorem 1 we can choose \( r : X \rightarrow A \) to be a retract such that \( f \) is a retractable map with respect to \( r \), we obtain a fixed point theorem for \( f \).

**Theorem 3.** If in the conditions in Theorem 1 \( r : X \rightarrow A \) is a retract and \( f : A \rightarrow X \) is retractable with respect to \( r \), then \( \text{Fix } f \neq \emptyset \).

**Proof.** From Theorem 1 it follows \( \text{Fix } f \neq \emptyset \) and \( f \) being retractable with respect to \( r \) we have \( \text{Fix } f = \text{Fix } r \neq \emptyset \).

**Corollary 3 (Theorem 5 in /6/).** Let \( X \) be a Hilbert space, \( A \subseteq X \) a nonvoid closed convex set, \( f : A \rightarrow X \) a continuous \( 1 \)-set-contractive map. We suppose that either \((1 - \text{p})(A) \) is closed in \( X \) or \((1 - \text{p})(\text{cl } p(f(A))) \) is closed in \( X \), where \( p = P_A : X \rightarrow A \) is the metric projection. If \( f(A) \) is bounded and \( f \) satisfies one of the following conditions:

1. For each \( x \in A \), there is a number \( \lambda \) (real or complex, depending on whether the vector space \( X \) is real or complex) such that \( |\lambda| < 1 \) and \( \lambda x + (1 - \lambda) f(x) \in A \).
2. For each \( x \in A \) with \( x \in f(A) \), there exists \( y \in I_A(x) = \{ x + \alpha(s - x) : s \in A, \alpha > 0 \} \) such that

\[
\| y - f(x) \| < \| x - f(x) \|.
\]
3. \( f \) is weakly inward (i.e. \( f(x) \in \text{cl } I_A(x) \) for each \( x \in A \)),
4. For each \( x \in A \) with \( x \in f(A) \), \( u = p \circ f(u) \) is a fixed point of \( f \).
5. For each \( x \in A \), \( |f(x) - y| < \| x - y \| \) for some \( y \in A \).

Then \( f \) has a fixed point in \( A \).

**Proof.** Corollary 2 applies, so \( \text{Fix } p \neq \emptyset \). Each of the five conditions implies \( (4) \) /6/, which is in fact exactly \( \text{Fix } r \neq \emptyset \).
This means that \( f : A \to X \) is retraction with respect to \( p \) and applying Theorem 3 one has \( \text{Fix } f \neq \emptyset \).

We mention that the condition of the retraction, \( \text{Fix } f \neq \emptyset \), is necessary in Hilbert spaces the following equivalent forms:

(i) For each \( u \) in the boundary of \( A \) which is not a fixed point for \( f \), there exists \( y \) in \( A \) such that

\[
\text{Re } (f(u) - u, u - y) < 0.
\]

(ii) For each \( u \) in the boundary of \( A \) which is not a fixed point for \( f \), there exists \( y \) in \( A \) such that

\[
\| y - f(u) \| < \| u - f(u) \|.
\]

(iii) For each \( u \) in the boundary of \( A \) which is not a fixed point for \( f \),

\[
\lim \inf_{t \to 0} \frac{1}{t} d((1-t)u + tf(u), A) < \| f(u) - u \|.
\]

The equivalence follows easily from the next lemma, which relies on the fact that in a Hilbert space for a closed convex set \( A \) and \( x \in X \) one has \( p(x) = \inf_{y \in A} \text{Re } (x - a, y - a) \leq 0 \) for each \( y \) in \( A \) (\( p = P_A \) being the metric projection).

**Lemma 2.** Let \( X \) be a Hilbert space, \( A \subseteq X \) a nonvoid closed convex set, \( a \in A \), \( x \in X \). The following assertions are equivalent:

1. \( p(x) = 0 \), i.e., \( \| x - a \| \leq \| x - y \| \) for each \( y \) in \( A \).
2. \( \lim_{t \to 0} \frac{1}{t} d((1-t)a + tx, A) = 0 \),

\[
= \lim_{t \to 0} \frac{1}{t} d((1-t)a + tx, A) = 0.
\]

3. \( \lim_{t \to 0} \frac{1}{t} d((1-t)a + tx, A) \geq \| x - a \| \).

**Proof.** 1. \( \Rightarrow \) 2. The first equalities in 2. are true because

\[
\inf_{t>0} \frac{1}{t} d((1-t)a + tx, A) = \inf_{t>0} \frac{1}{t} d((1-t)a + tx, A) \leq \| x - a \|.\]

But \( \text{Re } (x-a, y) = \text{Re } (x-a, y+a) \leq 0 \), because \( y+a \in A \) and \( p(x) = 0 \).

It follows that the map \( \psi : X \setminus \{ 0 \} \to X, \psi(t) = \frac{1}{t} d(a + t(x-a), A) \) is increasing and

\[
\inf_{t>0} \psi(t) = \inf_{t>0} \| x - a \| \text{.}
\]

Now, \( \inf_{t>0} \| x - a \| = \inf_{t>0} \frac{1}{t} d(x-a, A-a) = \inf_{t>0} \frac{1}{t} d(x-a, \text{Re } (x-a, y) + \| x-a \|) \approx \| x-a \| \).

But \( \text{Re } (x-a, y) = \text{Re } (x-a, a) \leq 0 \), because \( a \in A \).

It follows that the map \( \psi \) to be minimal is increasing on \( \text{Re } (x-a, y), +\infty \), hence also on \( (1, +\infty) \). The infimum is attained on \( t = 1 \) and

\[
\inf_{t>1} \psi(t) = \inf_{t>0} \| x-a \| = \inf_{y \in A} \| x-y \| = \| x-a \| \text{,}
\]

and the implication is proved.

2. \( \Rightarrow \) 1. is obvious,

3. \( \Rightarrow \) 2. We have \( \| x-a \| \leq \inf_{t>0} \frac{1}{t} d((1-t)a + tx, A) \leq \psi(1) = d(x, A) \leq \| x-y \| \) for each \( y \) in \( A \).

**Remark.** The equivalence in Lemma 2 is also true if \( X \) is a prehilbertian space and \( A \) a complete convex set.

The assertions (i) - (iii) are all equivalent to the condition (4) in Corollary 3.
For (i) one uses the fact that $p(f(u)) \neq u$ is equivalent to the existence of $y$ in $A$ such that $\text{Re}(u - f(u), y - u) > 0$, using the characterization of the metric projection mentioned before Lemma 2 was given.

For (ii) one applies just the definition of $p(f(u))$ and obtains the inequality in (ii).

For (iii) we use the equivalence $1^\circ \Leftrightarrow 3^\circ$ which was proved in Lemma 2.

So the assertions (i) - (iii) are just reformulations of the fact that the map $f : A \to X$ is retractable on $A$ with respect to the retraction $p$.

The equivalence of (i) - (iii) was in fact proved in [9], where (i) is called the Leray-Schauder condition, (ii) the Browder-Petryshyn condition and (iii) the Gower-Ray condition. Here we emphasised the role of the properties of the metric projection in this equivalence.

We finish the paper giving a method of approximation of fixed points for maps with $p(f)$ nonexpansive in Hilbert spaces by a procedure similar to that in the proof of Theorem 1.

**THEOREM 4.** Let $X$ be a Hilbert space, $A$ a convex closed subset of $X$, $f : A \to X$ a map retractable on $A$ with respect to the metric projection such that $F = p(f) : A \to A$ is nonexpansive and $F(A)$ bounded.

Consider $F_k : A \to A$, $F_k(x) = kf(x) + (1-k)x_0$, $0 < k < 1$, $k \to 1$, $x_0 \in A$ and $x_0$ the fixed point of the contraction $F_k$.

Then $x_k \to x_0$, where $x_0$ is the fixed point of $F$ which closest to $x_0$.

Proof.

The fixed point set $Fix F$ is nonvoid; let $y_0$ be the fixed point

of $y$ which is closest to $x_0$. For $F$ one applies the approximation result in [9]; for $x_n \to x$, $x_n \in (a, 1)$, $n \in N$, one obtains firstly that $\{x_n\}_{n \in N}$ is a bounded sequence. Then a subsequence of $\{x_n\}_{n \in N}$ will converge weakly to a point $x$, using the denseness of $I - F$ one obtains $x = y_0$ and then the strong convergence of $\{x_n\}_{n \in N}$ to $y_0$.

But $Fix F = Fix f$ and the theorem is proved.

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