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ON FIXED POINT THEOREMS FOR MAPPINGS DEFINED ON SPHERES IN METRIC SPACES

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In the last years, the interest in metric fixed point theorems appeared again. New proofs were given [1, 2] for theorems of Hardy-Rogers, Ćirić - Reich - Rus.

In this note we study the case of mappings defined only on spheres, not on the entire metric space, following the well-known theorem:

THEOREM 1. *Let* (X,d) *be a complete metric space,* $f:\overline{B}(z,r) \to X$ *a* k-*contraction,* $k \in [0,1)$, i.e.

(1) $$d(fx,fy) \le kd(x,y) \text{ for all } x,\ y \text{ in } \overline{B}(z,r) = \{x \in X : d(x,z) \le r\}.$$

If

(2) $$d(z,fz) \le r(1-k),$$

then f *has a unique fixed point* $u \in \overline{B}(z,r)$.

Proof. The condition on fz implies the fact that $f(\overline{B}(z,r)) \subseteq \overline{B}(z,r)$. Indeed, let $x \in \overline{B}(z,r)$. We estimate $d(z,fx)$:

$$d(z,fx) \le d(z,fz) + d(fz,fx) \le r(1-k) + kd(z,x) \le r.$$

It follows that $f(\overline{B}(z,r)) \subseteq \overline{B}(z,r)$ and Banach theorem applies in this complete metric space. ∎

This theorem extends easily to Hardy - Rogers contractions.

THEOREM 2. *Let* (X,d) *be a complete metric space,* $f:\overline{B}(z,r) \to X$ *a Hardy - Rogers contraction, i.e. there exists* $a_i \geq 0$, $i = \overline{1,5}$, $A = \sum_{i=1}^{5} a_i < 1$ *such that*

(3) $d(fx,fy) \leq a_1 d(x,y) + a_2 d(x,fx) + a_3 d(y,fy) + a_4 d(x,fy) + a_5 d(y,fx)$ *for all* x,y *in* $\overline{B}(z,r)$.

If

(4) $$d(z,fz) \leq \frac{2(1-A)}{2+A-a_1} r ,$$

then f *has a unique fixed point* $u \in \overline{B}(z,r)$.

Proof. To prove that $f(\overline{B}(z,r)) \subset \overline{B}(z,r)$, we use a symmetric form of (3) which is obtained evaluating also $d(fy,fx)$ and adding,

(5) $$d(fx,fy) \leq a_1 d(x,y) + \frac{a_2+a_3}{2}[d(x,fx)+d(y,fy)] + \frac{a_4+a_5}{2}[d(x,fy)+d(y,fx)]$$ for each x,y in $\overline{B}(z,r)$.

Let x be in $\overline{B}(z,r)$, hence $d(x,z) \leq r$. We estimate

$$d(z,fx) \leq d(z,fz) + d(fz,fx) \leq$$

$$\leq d(z,fz) + a_1 d(z,x) + \frac{a_2+a_3}{2}[d(z,fz)+d(x,z) + d(z,fx)] + \frac{a_4+a_5}{2}[d(z,fx)+d(x,z)+d(z,fz)] =$$

$$= (1 + \frac{a_2+a_3+a_4+a_5}{2}) d(z,fz) + (a_1 + \frac{a_2+a_3+a_4+a_5}{2}) \cdot d(x,z) + \frac{a_2+a_3+a_4+a_5}{2} d(z,fx) =$$

$$= (1 + \frac{A-a_1}{2}) d(z,fz) + \frac{A+a_1}{2} d(x,z) + \frac{A-a_1}{2} d(z,fx).$$

It follows

$$(1 - \frac{A-a_1}{2}) d(z,fx) \leq (1 + \frac{A-a_1}{2}) d(z,fz) + \frac{A+a_1}{2} d(x,z) \leq$$

$$\leq (1-A)r + \frac{A+a_1}{2} r = (1 - \frac{A-a_1}{2}) r.$$

Dividing by $1 - \frac{A-a_1}{2} > 0$ we obtain $d(z,fx) \leq r$ and Hardy - Rogers theorem applies and it assures the existence and uniqueness of the fixed point u. ∎

Remark 1. For the Ćirić - Reich - Rus contractions ($a_1 = a$, $a_2 = a_3 = b$, $a_4 = a_5 = 0$, $a + 2b < 1$), condition (4) becomes:

$$d(z,fz) \leq \frac{1-a-2b}{1+b} r .$$

For the Kannan contractions ($a_1 = a_4 = a_5 = 0$, $a_2 = a_3 = b < \frac{1}{2}$), condition (4) writes

$$d(z,fz) \leq \frac{1-2b}{1+b} r.$$

For the Banach contractions ($a_1 = k$, $a_i = 0$, $i = \overline{2,5}$), condition (4) becomes (2).

Similar results hold for the more general contractions defined by Rus [3].

Let $\varphi : \mathbb{R}_+^5 \to \mathbb{R}_+$ be a continuous function such that

(a) if $r_i \leq s_i$, $i = \overline{1,5}$, then $\varphi(r) \leq \varphi(s)$;

(b) $\phi(r) < r$ for each $r > 0$, where $\phi(r) = \varphi(r,r,r,r,r)$;

(c) $r - \phi(r) \to \infty$ for $r \to \infty$.

For (X,d) a metric space, a function $f: \overline{B}(z,r) \to X$ is a φ - contraction if

(6) $d(fx,fy) \leq \varphi(d(x,y)), d(x,fx), d(y,fy), d(x,fy), d(y,fx))$ for all $x,y \in \overline{B}(z,r)$.

Let r_z be equal to $\sup\{ r \in R_+ : r - \phi(r) \le d(z,fz)\}$ which is finite because $r = 0$ is in the considered set (which is hence nonvoid) and φ is continuous. We obtain a generalization of the fixed point theorem of Rus [3] which is given for φ - contractions defined on the entire space X.

THEOREM 3. *Let* (X,d) *be a complete metric space and* f: $\bar{B}(z,r) \to X$ *be a* φ *- contraction. If*

(7) $$d(z,fz) \le r - \phi(r_z),$$

then f *has a unique fixed point* $u \in \bar{B}(z,r)$, $u = \lim_{n\to\infty} f^n z$ *and* $d(f^n z,x) \le \phi^n(r_z)$.

Proof. If $fz=z$, the conclusion is obvious.

Let $fz \ne z$; one has then $r_z > 0$. Let $z_0 = z$; $z_1 = fz$ satisfies

$$d(z_0,z_1) \le r - \phi(r_z) \le r,$$

hence $z_1 \in \bar{B}(z,r)$. Suppose $z_n = f^n z \in \bar{B}(z,r)$. Then

$$d(z,z_{n+1}) \le d(z,fz) + d(fz,f^{n+1}z) \le$$

$$\le d(z,fz)+\varphi(d(z,f^n z),d(z,fz),d(f^n z,f^{n+1}z),d(z,f^{n+1}z),d(f^n z,fz)) \le$$

$$\le d(z,fz) + \phi(\mathrm{diam}\ O_f(z,n+1)),\quad O_f(z,n+1) = \{z,fz,\dots,f^{n+1}z\}.$$

Because of (6), diam $O_f(z,n+1)$ is larger than $d(f^i z,f^j z)$, $i,j \ge 1$, so there exists $p \le n + 1$ such that diam $O_f(z,n+1) = d(z,f^p z)$.

Then

$$\mathrm{diam}\ O_f(z,n+1) = d(z,f^p z) \le d(z,fz) + d(fz,f^p z) \le$$

$$\le d(z,fz) + \phi(\mathrm{diam}\ O_f(z,n+1)) \quad \text{and}$$

$$\mathrm{diam}\ O_f(z,n+1) - \phi(\mathrm{diam}\ O_f(z,n+1)) \le d(z,fz),$$

hence diam $O_f(z,n+1) \le r_z$.

It follows $d(z,z_{n+1}) \le d(z,fz) + \phi(r_z) \le r$.

The sequence $\{z_n\}_{n\in N}$ is well-defined for f: $\bar{B}(z,r) \to X$.

We shall prove now that this sequence converges. Let $n \ge 1$.

We estimate:

$$d(z_n,z_{n+p}) \le \varphi(d(z_{n-1},z_{n+p-1}),d(z_{n-1},z_n),d(z_{n+p-1},z_{n+p}),$$
$$d(z_{n-1},z_{n+p}),d(z_{n+p-1},z_n)) \le \phi(\mathrm{diam}\ O_f(z_{n-1},p+1)) \le$$
$$\le \phi^2(\mathrm{diam}\ O_f(z_{n-2},p+2)) \le \dots \le \phi^n(\mathrm{diam}\ O_f(z,p+n)) \le \phi^n(r_z).$$

But $r_z > 0$, so $\phi^n(r_z) \to 0$ $(n \to \infty)$; indeed, the sequence is descreasing and bounded and if its limit were $a \ne 0$, from the continuity of ϕ it would follow $a = \phi(a)$, contradicting the property (b).

It follows that $\{z_n\} \subset \bar{B}(z,r)$ is a Cauchy sequence, hence it converges to a limit u in $\bar{B}(z,r)$, which is a fixed point for f. Indeed, suppose $u \ne fu$. Then, for $n \in N$

$$d(u,fu) \le d(u,f^n z) + d(f^n z,fu) \le$$
$$\le d(u,f^n z) + \varphi(d(f^{n-1}z,u),d(f^{n-1}z,f^n z),d(u,fu),d(f^{n-1}z,fu),$$
$$d(u,f^n z))$$

and for $n \to \infty$

$$d(u,fu) \le \varphi(0,0,d(u,fu),d(u,fu),0) \le \phi(d(u,fu)) < d(u,fu),$$

which is a contradiction.

The uniqueness of the fixed point can be easily established in a similar way. ∎

Remark 2. Condition (3) in Theorem 2 corresponds to a φ - contraction with

$$\varphi(r_1,\dots,r_5) = \sum_{i=1}^{5} a_i r_i,\quad a_i \ge 0,\ i=\overline{1,5},\quad A = \sum_{i=1}^{5} a_i < 1.$$

In this case $r_z = \sup\{ r \ge 0;\ r - Ar \le d(z,fz)\} = \dfrac{1}{1 - A} d(z,fz)$ and condition (7) in Theorem 3 becomes

$$d(z,fz) \le r - \frac{A}{1 - A} d(z,fz),\ \text{i.e.}$$

$$d(z,fz) \le (1 - A)r.$$

This condition is less restrictive than (4), but when condition (4) is applicable the sequence of succesive approximations starting from each x in $\overline{B}(z,r)$ converges to the fixed point u.

R E F E R E N C E S

1. J.Lopez - Gomez, A fixed point theorem for discontinuous operators, Glasnik Mat. **23** (1988), 115 - 118.
2. L.Nova, G., Fixed point theorems for some discontinuous operators, Pacific J. Math. **123** (1986), 189 - 196.
3. I.A.Rus, Metrical fixed point theorems, University of Cluj - Napoca, Department of Mathematics, 1979 (mimeographed).

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This paper is in final form and no version of it is or will be submitted for publication elsewhere.