On the closedness of sets with the fixed point property for contractions

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Abstract

It is proved by examples that there are (connected) non-closed sets with the fixed point property for contractions in complete metric spaces. In a Banach space, a convex set with nonvoid interior having the fixed point property for contractions is necessarily closed.

1. Introduction

In his communication [4], I. A. Rus mentioned the following result of Hu from 1967, which gives a characterization of metric completeness:

Theorem 1.1. [2] *A metric space (X, d) is complete if and only if for each closed subset Y of X any contraction $f : Y \longrightarrow Y$ has a fixed point.*

Hu has made the remark that "closed subset" can be replaced by "infinite denumerable closed set", and it is sufficient to consider contractions with a given constant r.

As stated in [4], for the fixed point structure theory it would be desirable to have a result related to that of Hu, namely: given a complete metric space (X, d) and a nonvoid subset Y such that any contraction $f : Y \longrightarrow Y$ has a fixed point, the subset Y is necessarily closed. Unfortunately, some examples given by Connell [1] in 1959 for cross products show, as Subrahmanyam has emphasized in [3], that there are non-closed subsets of a complete metric space for which each contraction has a fixed point. The paper of Subrahmanyam includes an abstract generalization of such an example.

In the second section we describe an example of Connell and the way it provides a non-closed set on which each contraction (in fact any continuous function) has a fixed point. The set is connected but not path connected. A second example, also appearing in Connell's paper, provides a path connected, non-closed set in $\mathbb{R}^2$ on which each continuous function has a fixed point. In fact, Connell was not concerned about the connectedness properties of these sets.

In the third section we prove a theorem which, in the setting of Banach spaces, gives a class of sets which are necessarily closed if they have the fixed point property for contractions. So in this case Rus' problem has an affirmative answer.

The final section contains some remarks on another class of sets and on the way of providing contractions without fixed points in the case of normed spaces.

2. Examples in complete metric spaces of non-closed sets having the fixed point property for contractions

In the paper [1], Connell was interested in giving examples of bounded, but non-closed sets Y, each continuous function $f : Y \longrightarrow Y$ having a fixed point. Actually these examples are also good for our purposes. As these sets have in addition some connectedness properties, we mention the following definitions.

A topological space X is *connected* if it cannot be written as a disjoint union of two open nonvoid sets; it is *path connected* if for every pair x_1, x_2 of points in X there exists a continuous function $\phi : [0, 1] \longrightarrow X$ such that $\phi(0) = x_1$ and $\phi(1) = x_2$.

Example 1. *A set $Y \subseteq \mathbb{R}^2$ which is non-closed but has the fixed point property for each contraction $f : Y \longrightarrow Y$. The set Y is connected but not path connected.*

Let there be given the function $\varphi : [0,1] \longrightarrow [0,1]$,

$$\varphi(t) = \begin{cases} \sin \dfrac{\pi}{1-t}, t \neq 1 \\ 1, t = 1. \end{cases}$$

In $\mathbb{R}^2$ one considers the connected, but not path connected set

$$Y = \{(t, \varphi(t)) \in \mathbb{R}^2 : 0 \leq t \leq 1\},$$

for which $\overline{Y} = Y \cup (\{1\} \times [-1,1])$, so Y is non-closed. The set Y is connected because $Y_0 \subseteq Y \subseteq \overline{Y_0}$, where $Y_0 = \{(t, \varphi(t) \in \mathbb{R}^2 : 0 \leq t < 1\}$ is connected as a continuous image of the interval [0,1). The assertion that Y is not path connected follows from the fact that no path can join the point (0,0) to (1,1), because otherwise Y would be locally connected, which obviously is not the case.

We show that each continuous function $f : Y \longrightarrow Y$ (hence each contraction) has a fixed point.Let us suppose that there is such a function f without fixed points. Denoting with indices 1 and 2 the first, respectively the second component of a point in $\mathbb{R}^2$, we define

$$A = \{x = (x_1, x_2) \in Y : f(x)_1 < x_1\},$$
$$B = \{x = (x_1, x_2) \in Y : f(x)_1 > x_1\}.$$

Both A and B are open in Y, since f is continuous. They are nonvoid, because $(1,1) \in A$ and $(0,0) \in B$. More than that, $X = A \cup B$. Indeed, we have $f(x)_1 \neq x_1$ (if we suppose $f(x)_1 = x_1$, it will follow $f(x)_2 = x_2$, since φ is a function, and $x = (x_1, x_2)$ would be a fixed point for f, contradiction with our assumption).

To summarize, A and B are open disjoint nonvoid sets such that $Y = A \cup B$, which contradicts the fact that Y is connected.

Example 2. *A set $Z \subseteq \mathbb{R}^2$ which is non-closed but has the fixed point property for each contraction $g : Z \longrightarrow Z$. Obviously, the set Z is in this case path connected and a fortiori connected.*

Let $I_n \subseteq \mathbb{R}^2, I_0 = [0,1] \times \{0\}$ and $I_k = \left\{\dfrac{1}{k}\right\} \times [0,1], k \in \mathbb{N}^*$. The set $Z = \cup_{n=0}^{\infty} I_n \subseteq \mathbb{R}^2$ is path connected, hence connected. We have $\overline{Z} = Z \cup (\{0\} \times [0,1])$, so Z is non-closed. Each continuous function $g : Z \longrightarrow Z$ (hence each contraction) has a fixed point.

Let us suppose that there is a continuous function $g : Z \longrightarrow Z$ without fixed points. The continuous function $\widetilde{g} : I_0 \longrightarrow I_0$ given by $\widetilde{g}(x) = (g(x)_1, 0)$ has obviously a fixed point $(p, 0) \in I_0$. Since g has no fixed point, there exists $k_0 \in \mathbb{N}^*$ such that $p = \frac{1}{k_o}$, hence $g(\frac{1}{k_0}, 0) = (\frac{1}{k_0}, y), 0 < y \leq 1$.

Let us denote $y_1 = \sup\{y \in [0,1] : \exists z \in [0,1], y < z, g(\frac{1}{k_0}, y) = (\frac{1}{k_0}, z)\}$. By the continuity of g and the definition of y_1, there exists $z_1 \in [0,1], y_1 \leq z_1$ such that $g(\frac{1}{k_0}, y_1) = (\frac{1}{k_0}, z_1)$. If we suppose $y_1 < z_1$, using again the continuity of g, one can find $y_2, z_2 \in [0,1], y_1 < y_2 < z_2$ such that $g(\frac{1}{k_0}, y_2) = (\frac{1}{k_0}, z_2)$, contradiction with the definition of y_1. It follows that $y_1 = z_1$ and $(\frac{1}{k_0}, y_1)$ is a fixed point for g, contradiction with our assumption that g has no fixed points.

3. Convex subsets with nonvoid interior in Banach spaces

In this section, in the setting of Banach spaces, we give a class of sets for which one can prove that if any of their contractions has a fixed point, they are necessarily closed.

We mention the following definitions. A set A in a linear space is *convex* if from $x, y \in A$ it follows that $(1-\lambda)x + \lambda y \in A$ for each $\lambda \in (0,1)$; the *relative interior* of the convex set A is $\operatorname{ri} A = \{a \in A : \forall x \in A \backslash \{a\}, \exists\, y \in A$ such that $a = (1-\lambda)x + \lambda y$, for some $\lambda \in (0,1)\}$. In a normed space, from $\operatorname{int} A \neq \varnothing$, it follows obviously that $\operatorname{ri} A \neq \emptyset$, but the converse is not true.

We can prove now

Theorem 3.1. *Let E be a Banach space, $A \subseteq E$ a convex set with* $\operatorname{int} A \neq \varnothing$. *If each contraction $h : A \to A$ has a fixed point, then A is closed.*

Proof. We have to prove that $b \in \overline{A}$ implies $b \in A$. Let us suppose by contradiction that there is an element $b \in \overline{A} \backslash A$. Making (if necessary) a translation, we can take $b = 0$. Since $\operatorname{int} A \neq \varnothing$, there is $a \in \operatorname{int} A$. Let $\alpha, \beta > 0$ be given such $\alpha + \beta < 1$, $r = \alpha + \beta \, \|a\| < 1$.

We define a function $H : E \longrightarrow E$,

$$H(x) = \alpha x + \beta \frac{\|x\|}{\|x\| + 1} a.$$

For $x \in A$, we have $u = \dfrac{\|x\|}{\|x\| + 1} a \in \operatorname{int} A$ (since $0 \in \overline{A}$, $a \in \operatorname{int} A$). Then the convex combination

$$v = \frac{\alpha}{\alpha + \beta} x + \frac{\beta}{\alpha + \beta} u$$

is contained in $\operatorname{int} A$, so $\alpha x + \beta u = (\alpha + \beta) v + (1 - \alpha - \beta) 0 \in \operatorname{int} A$ $(\alpha + \beta < 1)$. It follows that $H(A) \subseteq \operatorname{int} A \subseteq A$, hence $H(\overline{A}) \subseteq \overline{H(A)} \subseteq \overline{A}$.

We prove now that H is a contraction on E with the constant r :

$$\|H(x) - H(y)\| \leq \alpha \|x - y\| + \beta \|a\| \frac{\|x - y\|}{\|x - y\| + 1}$$

$$\leq (\alpha + \beta \|a\|) \|x - y\| .$$

Applying Banach's theorem for $H \mid_{\overline{A}}$, it follows that it has a unique fixed point, which is equal to 0 (because $H(0) = 0$).

By the hypothesis, the fixed point set of $h = H \mid_A$ is nonvoid; it is included in that of $H \mid_{\overline{A}}$, which contains exactly the point 0, so it follows that $0 \in A$, contradiction. It remains that the set A has to be closed. ■

The class of convex sets with nonvoid relative interior is larger than that considered in Theorem 3.1 The problem if the condition $\operatorname{int} A \neq \emptyset$ could be replaced by $\operatorname{ri} A \neq \emptyset$ remains open.

4. Remarks

In the case of normed spaces, there are some further comments to be done.

Remark 1. *Suppose that the problem at the end of the previous section can be answered in the affirmative, i.e. Theorem 3.1 is true with* $\operatorname{ri} A \neq \emptyset$*. Then an immediate consequence would be that any normed space with the property that each contraction has a fixed point is in fact a Banach space. Indeed, a normed space A can be considered as a convex set with* $\operatorname{ri} A \neq \emptyset$ *in its completion* $\widetilde{A} = E$*; applying the theorem, it follows that the normed space A is closed, hence a Banach space.*

Remark 2. *In a normed space which is not Banach, contractions without fixed points may exist. But to provide such contractions is not an easy task. For example, let us consider the space l^0 of all real sequences with a finite number of nonzero terms, endowed with the* sup *norm, and the shift operator* $s : l^0 \longrightarrow l^0, s(x_1, x_2, ...) = (0, x_1, x_2, ...)$. *For* $\lambda \in (0,1)$ *and* $e_1 = (1, 0, 0, ...)$, *we define* $f : l^0 \longrightarrow l^0, f(x) = \lambda s(x) + e_1$, *which is a* $\lambda-$*contraction. But it has no fixed points, because* $f(u) = u$ *implies* $u = (1, \lambda, \lambda^2, \lambda^3, ...) \notin l^0$.

References

[1] E. H. Connell, Properties of fixed point spaces, *Proc. Amer. Math. Soc.* **10**(1959), 974-979

[2] T. K. Hu, On a fixed point theorem for metric spaces, *Amer. Math. Monthly* **74**(1967), 436-437

[3] P. V. Subrahmanyam, Completeness and fixed points, *Monatsch. für Math.* **80**(1975), 325-330

[4] I. A. Rus, Maximal fixed point structures, "*Zilele academice clujene*", 18-23 nov. 1996.