The inverse problem of dynamics for families in parametric form

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Abstract

The two-dimensional inverse problem of dynamics is considered for nonconservative force fields, both in inertial and rotating frames. The families of curves are given in parametric form $x = F(\lambda, b)$, $y = G(\lambda, b)$, b varying along the monoparametric family of planar curves and λ being the parameter describing a specific curve. The special case of the force fields generated by a potential in an inertial field, already studied by Bozis and Borghero [3], is derived as well as the corresponding one in rotating frames.

1 Introduction

The central aim of the inverse problem consists in finding the potential (or force field) which can give rise to a monoparametric family of planar curves traced by a unit mass material point P. The family of orbits was generally given (Szebehely [8], Broucke and Lass [4], Szebehely and Broucke [9], Bozis [1]) in the form

$$f(x, y) = c. \tag{1}$$

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More details are to be found in the monograph paper of Bozis [2].

Recently the problem of finding the potential when the family is given in parametric form was taken into account by Bozis and Borghero [3], because there are models inspired from the physical reality which give rise to such families.

In the following we study the problem of finding general force fields which can produce families of orbits a priori given in parametric form. For families having the form (1) the problem was considered by Pal and Anisiu [7].

2 The inverse problem in an inertial frame

The inverse problem for families of the type (1) was considered at first by Dainelli [5] and Whittaker [10]. It consists in finding the components $X(x, y)$ and $Y(x, y)$ of the force under whose action a particle P of unit mass will describe a given family of orbits, the motion being governed by the system

$$\begin{aligned} \ddot{x} &= X \\ \ddot{y} &= Y, \end{aligned} \tag{2}$$

the dots representing differentiation with respect to the time t. The family of orbits was given in the implicit form (1).

We consider the corresponding problem for the case of a family of orbits given in the form

$$x = F(\lambda, b) \qquad y = G(\lambda, b), \tag{3}$$

where the parameter b varies from member to member of the family (as c did for (1)) and λ stands for the parameter varying along each curve for a fixed b. Partial derivatives will be denoted by subscripts.

The transformation (3) is considered a C^2 one-to-one correspondence between a domain D in the xy Cartesian plane and a domain D' in the λb plane, with the Jacobian

$$J = F_\lambda G_b - F_b G_\lambda \tag{4}$$

different from zero. The fact that the Jacobian (4) is different from zero at a point assures the bijectivity and the existence of a differentiable inverse of (3) only on a neighbourhood of that point; if it is different from zero at each point of $\mathbb{R}^2$, we have in general only local bijectivity for the transformation. If an extra condition is added, as for example weak coercivity (Zeidler [11])

$$\|(F, G)\| \to \infty \ \textit{whenever} \ \|(\lambda, b)\| \to \infty, \tag{5}$$

then the transformation is globally bijective. So, imposing the condition that the Jacobian is different from zero at each point of a domain D (or $\mathbb{R}^2$) we obtain generally only local results.

Besides the condition (4) on the Jacobian, we impose that

$$\Delta = F_{\lambda\lambda} G_\lambda - G_{\lambda\lambda} F_\lambda, \tag{6}$$

$$\beta = F_\lambda^2 + G_\lambda^2 \tag{7}$$

are also different from zero.

The problem is now to find $X, Y \in C(D)$ such that the system (2) to have as solutions the curves of the family (3), which means that there exists $l : (t_0, t_1) \to (\lambda_0, \lambda_1)$ of C^2-class so that the functions

$$x(t) \equiv F(l(t), b) \qquad y(t) \equiv G(l(t), b) \tag{8}$$

verify the system (2).

By differentiating in (8) we obtain the velocity components of the particle P

$$\dot{x} = F_\lambda \dot{l} \qquad \dot{y} = G_\lambda \dot{l} \tag{9}$$

and the acceleration components

$$\ddot{x} = F_{\lambda\lambda} \dot{l}^2 + F_\lambda \ddot{l} \qquad \ddot{y} = G_{\lambda\lambda} \dot{l}^2 + G_\lambda \ddot{l}. \tag{10}$$

Replacing (8) and (10) in (2) and denoting

$$\overline{X}(\lambda, b) = X(F(\lambda, b), G(\lambda, b)) \qquad \overline{Y}(\lambda, b) = Y(F(\lambda, b), G(\lambda, b)) \tag{11}$$

we obtain

$$\begin{aligned} F_{\lambda\lambda} \dot{l}^2 + F_\lambda \ddot{l} = \overline{X}(l, b) \\ G_{\lambda\lambda} \dot{l}^2 + G_\lambda \ddot{l} = \overline{Y}(l, b), \end{aligned} \tag{12}$$

where l has the argument t, while F, G and their derivatives have the argument $(l(t), b)$. It follows that

$$\dot{l}^2 = \frac{\begin{vmatrix} \overline{X}(l, b) & F_\lambda \\ \overline{Y}(l, b) & G_\lambda \end{vmatrix}}{\Delta},$$

where Δ is given by (6), so the value of $\dot{l}$ at a given t depends only on $l(t)$ and b. For an arbitrary function $k \in C^1(D', \mathbb{R}_+)$, we consider

$$\dot{l}(t) = \pm\sqrt{k(l(t), b)}.$$

Then it follows

$$\ddot{l}\,(t) = \frac{1}{2} k_\lambda\,(l\,(t)\,, b)$$

and using the relations (12) we obtain

$$\overline{X}(\lambda, b) = kF_{\lambda\lambda} + \tfrac{1}{2} k_\lambda F_\lambda \quad \overline{Y}(\lambda, b) = kG_{\lambda\lambda} + \tfrac{1}{2} k_\lambda G_\lambda, \tag{13}$$

where F, G, k and their derivatives have the argument (λ, b). So (13) provides a general form of the force field admitting the family of orbits (3) as solutions of the system (2).

In the case when the force field is generated by a potential, i.e. there exists $v \in C^1\,(D)$ so that

$$X = -v_x \; Y = -v_y \tag{14}$$

we have also

$$\frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) = E - v$$

hence $k = \frac{2(E-v)}{\beta}$, with β given by (7). Inserting in (13) the relations (14) and eliminating the terms in k_λ, one has

$$k = \frac{F_\lambda v_y - G_\lambda v_x}{\Delta}.$$

Let the potential in the coordinates λ, b be

$$V\,(\lambda, b) = v\,(x\,(\lambda, b)\,, y\,(\lambda, b))\,.$$

We have

$$v_x = J^{-1}\,(V_\lambda G_b - V_b G_\lambda) \quad v_y = J^{-1}\,(-V_\lambda F_b + V_b F_\lambda)\,,$$

with J given by (4). Equating the two forms obtained for k and denoting

$$\alpha = -\,(F_\lambda F_b + G_\lambda G_b)\,,$$

it follows

$$E = V + \frac{\beta}{2J\Delta}\,(\alpha V_\lambda + \beta V_b)\,,$$

which is exactly the equation obtained by Bozis and Borghero [3] directly for the system having the right-side terms given by (14), and for the family of curves (3).

3 The inverse problem in a rotating frame

A problem similar to that of Dainelli for the system (2) and the family of orbits (1) was studied for a system in a rotating frame by Pal and Anisiu [7]. Such a system has the form

$$\begin{aligned} \ddot{x} - 2\dot{y} &= X \\ \ddot{y} + 2\dot{x} &= Y. \end{aligned} \tag{15}$$

We now look for the functions $X, Y \in C(D)$ in order that the system (15) admits as solutions the curves from the family (3). In this case, after replacing the velocity and acceleration components of the particle P in (15) one obtains

$$\begin{aligned} F_{\lambda\lambda}\dot{l}^2 + F_\lambda \ddot{l} - 2G_\lambda \dot{l} &= \overline{X}(l,b) \\ G_{\lambda\lambda}\dot{l}^2 + G_\lambda \ddot{l} + 2F_\lambda \dot{l} &= \overline{Y}(l,b), \end{aligned} \tag{16}$$

hence

$$\dot{l}^2 = \frac{\begin{vmatrix} \overline{X}(l,b) - 2G_\lambda \dot{l} & F_\lambda \\ \overline{Y}(l,b) - 2F_\lambda \dot{l} & G_\lambda \end{vmatrix}}{\Delta},$$

with Δ given by (6) and $\overline{X}$, $\overline{Y}$ by (11). So, we can consider again $\dot{l}$ as an arbitrary function of $(l(t), b)$. Let $K \in C^1(D', \mathbb{R}_+)$ be an arbitrary function and

$$\dot{l}(t) = \pm\sqrt{K(l(t), b)}.$$

Then

$$\ddot{l}(t) = \frac{1}{2}K_\lambda(l(t), b)$$

and considering (16) we can take

$$\begin{aligned} \overline{X}(\lambda, b) &= \mp 2\sqrt{K}G_\lambda + KF_{\lambda\lambda} + \tfrac{1}{2}K_\lambda F_\lambda \\ \overline{Y}(\lambda, b) &= \pm 2\sqrt{K}F_\lambda + KG_{\lambda\lambda} + \tfrac{1}{2}K_\lambda G_\lambda, \end{aligned} \tag{17}$$

where F, G, K and their derivatives have the argument (λ, b).

Now, if the force field in (15) is produced by a potential, i.e. the relations (14) take place, we obtain from (17) eliminating the terms in K_λ

$$F_\lambda v_y - G_\lambda v_x = \mp 2\beta\sqrt{K} + \Delta K. \tag{18}$$

We have an energy integral

$$\frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) = E - v$$

hence $K = 2\,(E - v)\;/\,\beta$. Replacing this value of K in (18) we obtain

$$(\alpha V_\lambda + \beta V_b)\,/J = \mp\sqrt{2\,(E - V)}\sqrt{\beta} + 2\,(E - V)\,\Delta/\beta,$$

which is the correspondent of the equation derived by Drâmbă [6], Szebehely and Broucke [9] for the system (15) with X, Y given by (14), the family of curves being (1).

Example Let the family of orbits (3) be given, with

$$\begin{aligned} F(\lambda, b) &= \exp(\lambda) - \exp(b) \\ G(\lambda, b) &= \lambda + b. \end{aligned} \tag{19}$$

This is a diffeomorphism from $\mathbb{R}^2$ in $\mathbb{R}^2$ for which $\beta = \exp(2\lambda) + 1$, $J = \exp(\lambda) + \exp(b)$, $\Delta = \exp(\lambda)$ are all different from 0. The coercivity condition (5) is also verified. In the case of the inertial frame, the form of the force field obtained in (13) is

$$\overline{X}(\lambda, b) = \big(k(\lambda, b) + \tfrac{1}{2}k_\lambda(\lambda, b)\big)\exp(\lambda) \qquad \overline{Y}(\lambda, b) = \tfrac{1}{2}k_\lambda(\lambda, b), \tag{20}$$

while in the rotating frame (17) becomes

$$\begin{aligned} \overline{X}(\lambda, b) &= \mp 2\sqrt{K(\lambda, b)} + \big(K(\lambda, b) + \tfrac{1}{2}K_\lambda(\lambda, b)\big)\exp(\lambda) \\ \overline{Y}(\lambda, b) &= \pm 2\sqrt{K(\lambda, b)}\exp(\lambda) + \tfrac{1}{2}K_\lambda(\lambda, b). \end{aligned} \tag{21}$$

The inverse of the transformation (19) can be computed explicitely, so for obtaining X and Y in the variables (x, y) it suffices to insert in formulae (20) and (21)

$$\lambda = y - \tfrac{1}{2}\ln(\sqrt{x^2 + 4\exp(y)} - x) \qquad b = \tfrac{1}{2}\ln(\sqrt{x^2 + 4\exp(y)} - x).$$

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