# THE PHOTOGRAVITATIONAL MODEL OF CONSTANTIN POPOVICI IN A MANEV-TYPE FIELD

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*Abstract.* The photogravitational model of Constantin Popovici combines the Newtonian attraction force with a radiative repelling one. We consider a post-Newtonian attraction force, namely the one generated by a Manev-type potential, and the same repelling force defined by Popovici. It is proved that for this new problem the integration of the equations of motion can be performed in a similar way as in Popovici's model. The study of the equilibria reveals specific situations for the Manev-Popovici model, as the existence of saddle points (which are unstable).
Key words: photogravitational models – equilibria.

## 1. INTRODUCTION

Constantin Popovici (1923, 1940) considered a two-body problem based on a Newtonian attraction force

$$F_N = -A/r^2 \tag{1}$$

and a modified radiative force

$$F_R = R\left(1 - c^{-1}\dot{r}\right)/r^2, \tag{2}$$

$r = \sqrt{x^2 + y^2}$ denoting the distance between the two particles, $c$ the speed of light, $A$, $R$ positive constants.

Anisiu (1995) studied the relative motion governed by the corresponding system of equations written as

$$\begin{aligned} \ddot{x} &= -\left(k + l\dot{r}\right)r^{-3}x \\ \ddot{y} &= -\left(k + l\dot{r}\right)r^{-3}y, \end{aligned} \tag{3}$$

where

$$k = A - R,\ \ l = Rc^{-1} > 0. \tag{4}$$

Brought to polar coordinates, the system (3) becomes

$$\begin{cases} \ddot{r} - r\dot{\theta}^2 = -(k + l\dot{r})/r^2 \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0. \end{cases} \tag{5}$$

The dots denote differentiation with respect to the time *t*.

The system (5) admits the first integral of angular moment

$$r^2\dot{\theta} = C, \tag{6}$$

$C$ denoting the angular moment constant.

It was proved by Anisiu (1995) that the system (5), for $C \neq 0$, has as a solution the planar curve given by

$$r^{-1} = q + u_0, \tag{7}$$

with $q = kC^{-2}, \alpha = l(2C)^{-1}$ and

$$u_0 = \begin{cases} e^{-\alpha\theta}\left(C_1 e^{(\alpha^2-1)^{1/2}\theta} + C_2 e^{-(\alpha^2-1)^{1/2}\theta}\right), \text{ if } |\alpha| > 1 \\ (C_1 + C_2\theta)e^{-\alpha\theta}, \text{ if } |\alpha| = 1 \\ e^{-\alpha\theta}\left(C_1 \sin(1-\alpha^2)^{1/2}\theta + C_2 \cos(1-\alpha^2)^{1/2}\theta\right), \text{ if } 0 < |\alpha| < 1. \end{cases} \tag{8}$$

The time *t* is given by $t = \frac{1}{C}\int r^2 d\theta + C_3$.

The first equation of (5) can be written using the angular moment constant as

$$\ddot{r} + lr^{-2}\dot{r} = C^2 r^{-3} - kr^{-2}, \tag{9}$$

and this was shown to admit a unique linear stable equilibrium

$$r_0 = C^2 k^{-1} = q^{-1} \tag{10}$$

for $k > 0$, and no equilibria for $k \leq 0$.

A qualitative analysis of the equilibria was done by Mioc (2001, 2002), and by Mioc and Blaga (2001, 2002).

For $C = 0$ the body moves on a line passing through the attractive body, the motion being governed by the equation

$$\ddot{r} = -(k + l\dot{r})r^{-2}.$$

This equation admits (linear stable) equilibria if and only if $k = 0$, any value $r_0 > 0$ being an equilibrium. The change of variables $r(t) = w, \dot{r}(t) = v(w)$ gives $\ddot{r}(t) = v\frac{dv}{dw}$;

denoting $v' = \frac{dv}{dw}$, the equation reduces to the first order equation with separable variables

$$w^2 w v' + lv + k = 0.$$

We have then $t = \int \frac{1}{v(w)} dw + c.$

## 2. THE BASIC EQUATIONS OF POPOVICI'S MODEL IN A MANEV-TYPE FIELD

Instead of the Newtonian attraction force (1), Manev (1924) considered a more general one of the type

$$F_M = -A / r^2 - B / r^3 \quad , \tag{11}$$

with $A, B > 0$. Rich information on the development of the research related to Manev -type fields can be found in Mioc and Stoica (1995 a, b), Delgado et al (1996), Mioc and Stavinschi (1999), and Diacu et al (2000).

Using the notation (4), the system for the two-body problem with Manev attraction force and Popovici's modified radiative force becomes

$$\begin{cases} \ddot{x} = -(k + Br^{-1} + l\dot{r})r^{-3} x \\ \ddot{y} = -(k + Br^{-1} + l\dot{r})r^{-3} y, \end{cases} \tag{12}$$

or, in polar coordinates,

$$\begin{cases} \ddot{r} - r\dot{\theta}^2 = -(k + Br^{-1} + l\dot{r})r^{-2} \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0. \end{cases} \tag{13}$$

There exists again a first integral of angular moment (6), $C$ denoting the angular moment constant. The system (13) is more complicated than the one considered by Popovici, where the term containing $B$ was missing. The interesting fact is that it can be solved in a similar way as (5) was solved by Anisiu (1995).

## 3. THE GENERAL SOLUTION OF THE BASIC EQUATIONS

The system (13) can be integrated and we have

THEOREM 1. If $C \neq 0$, the solution of (13) is

$$r^{-1} = k/(C^2 - B) + u_0 \, , \tag{14}$$

with $\alpha = l(2C)^{-1}$, $\beta = 1 - BC^{-2}$ and

$$u_0 = \begin{cases} e^{-\alpha\theta}(C_1 e^{(\alpha^2-\beta)^{1/2}\theta} + C_2 e^{-(\alpha^2-\beta)^{1/2}\theta}), \text{ if } \alpha^2 > \beta \\ (C_1 + C_2\theta)\, e^{-\alpha\theta}, \text{ if } \alpha^2 = \beta \\ e^{-\alpha\theta}\left(C_1 \sin\left(\beta-\alpha^2\right)^{1/2}\theta + C_2 \cos\left(\beta-\alpha^2\right)^{1/2}\theta\right), \text{ if } \alpha^2 < \beta, \end{cases} \tag{15}$$

for $\beta \neq 0;$ in the case when $\beta = 0$ ,

$$r^{-1} = \frac{k}{Cl}\theta + C_1 e^{-2\alpha\theta} + C_2\,; \tag{16}$$

in both cases the time is given by $t = \dfrac{1}{C}\int r^2 d\theta + C_3$ .

If $\mathrm{C} = 0$, the body is moving on a straight line passing through the attractor body, the motion being governed by the equation

$$\ddot{r} + lr^{-2}\dot{r} = -\left(k + Br^{-1}\right)r^{-2}. \tag{17}$$

Equation (17) can be reduced to the second type (class B) Abel equation (see Kamke (1943), p. 27)

$$w^3 wv' + lwv + kw + B = 0, \tag{18}$$

and $t = \displaystyle\int \frac{1}{v(w)} dw + c.$

*Proof*. Let us consider at first $C \neq 0.$ Inserting $\dot{\theta}$ from (6) in the first equation in (13), we obtain

$$\ddot{r} = -\left(k + Br^{-1} + l\dot{r}\right)r^{-2} + C^2 r^{-3}. \tag{19}$$

We regard *r* as a function of *θ* and we denote by $r', r''$ the derivatives of *r* with respect to *θ*. Inserting

$$\dot{r} = Cr'r^{-2},\; \ddot{r} = C^2\left(r''r - 2r'^2\right)r^{-5}$$

in (19) we get

$$C^2 r''r - 2C^2 r'^2 - \left(C^2 - B\right)r^2 = -\left(kr^3 + lcr'r\right).$$

With the substitution $r = u^{-1}$ we obtain a linear nonhomogeneous second order differential equation

$$u'' + 2\alpha u' + \beta u = q\,, \tag{20}$$

where $q = kC^{-2}, \alpha = l\left(2C\right)^{-1}$ and $\beta = 1 - BC^{-2}$ .

The characteristic equation of (20) is

$$z^2 + 2\alpha z + \beta = 0$$

and has the roots $z_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \beta}$. It follows that for $\beta \neq 0$, $u = k/(C^2 - B) + u_0$, with $u_0$ given by (15). For $\beta = 0$, $u = \frac{k}{Cl}\theta + C_1 e^{-2\alpha\theta} + C_2$.

From (6) it follows that $t = \frac{1}{C}\int r^2 d\theta + C_3$.

If $C = 0$, we have $\dot{\theta} = 0$ and the first equation in (13) becomes (17). Performing the change of variables $r(t) = w, \dot{r}(t) = v(w)$, we obtain the first order equation (18) and $t = \int \frac{1}{v(w)} dw + c.$ □

*Remark 1*. From the solution (14)-(15) of the Manev-Popovici system (13) we can formally obtain the solution (7)-(8) of the Newtonian case (5), considering $B = 0$ (hence $\beta = 1$). The solution (16) is specific for the Manev-Popovici system.

## 4. EXISTENCE OF EQUILIBRIA

For $C \neq 0$, we have nonradial motion and the equilibria for equation (19) will be given by

$$-\left(k + Br_0^{-1}\right)r_0 + C^2 = 0,$$

or

$$kr_0 = C^2 - B.$$

There will be a unique equilibrium $r_0 = \left(C^2 - B\right)/k$ for $k \neq 0$ and $\left(C^2 - B\right)/k > 0$; for $k \neq 0$ and $\left(C^2 - B\right)/k \leq 0$, or $k = 0$ and $C^2 \neq B$, there are no equilibria; for $k = 0$ and $C^2 = B$ every $r_0 > 0$ is an equilibrium.

For $C = 0$, the possible equilibria of (17) are given by $kr_0 = -B;$ there will be a unique equilibrium $r_0 = -B/k$ for $k < 0$ and no equilibria for $k \geq 0$.

It follows

THEOREM 2. In the case of a nonradial motion $(C \neq 0)$ the equation (19) has a unique equilibrium $r_0 = \left(C^2 - B\right)/k$ if $k \neq 0$ and $\left(C^2 - B\right)/k > 0$; every $r_0 > 0$ is an equilibrium if $k = 0$ and $C^2 = B$; otherwise, there are no equilibria.

In the case of the motion on a straight line through the origin $(C = 0)$, equation (17) has a unique equilibrium $r_0 = -B/k$ for $k < 0$ and no equilibria for $k \geq 0$. □

*Remark 2*. The equilibria which are specific for the Manev-type photogravitational problem (19) with $C \neq 0$ are those obtained for $k = 0$ and $C^2 = B$.

For $C = 0$, the situation is completely different from that in the Newtonian case, when equilibria exist if and only of $k = 0$, every $r_0 > 0$ being an equilibrium.

## 5. STABILITY OF EQUILIBRIA

Let us consider the case of nonradial motion $(C \neq 0)$. For $k \neq 0$ and $(C^2 - B)/k > 0$, $r_0 = (C^2 - B)/k$ is the unique equilibrium of (19).

The eigenvalues for the linearized equation obtained from (19) are the roots of the quadratic equation

$$\lambda^2 + \frac{lk^2}{(C^2 - B)^2}\lambda + \frac{k^4}{(C^2 - B)^3} = 0,$$

which has the discriminant

$$\Delta = \frac{k^4}{(C^2 - B)^4}\left(l^2 - 4(C^2 - B)\right).$$

If $l^2 < 4(C^2 - B)$, we have two conjugate complex roots with $\operatorname{Re}\lambda_{1,2} = -\frac{lk^2}{2(C^2 - B)^2} < 0$, and $r_0$ is a stable spiral point.

If $l^2 = 4(C^2 - B)$, the equal real roots are $\lambda_1 = \lambda_2 = -\frac{lk^2}{2(C^2 - B)^2} < 0$ and $r_0$ is a stable node.

If $l^2 > 4(C^2 - B)$ and $(C^2 - B) > 0$ (hence $k > 0$ too), $\lambda_1$ and $\lambda_2$ are both real and negative, hence $r_0$ is a stable node.

If $l^2 > 4(C^2 - B)$ and $C^2 - B < 0$ (hence $k < 0$ too), then $\lambda_1\lambda_2 = \frac{k^4}{(C^2 - B)^3} < 0$ and $r_0$ is a saddle point.

The second case which provides equilibria for the nonradial motion $(C \neq 0)$ is $k = 0$ and $C^2 = B$. Each $r_0 > 0$ is an equilibrium and equation (18) becomes in this case linear, namely $\ddot{r} = -l\dot{r}$. It has the solution $r = -\frac{C_1}{l}e^{-lt} + C_2$ and each $r_0 > 0$ will be a stable equilibrium.

In the case of the radial motion $(C = 0)$, equation (17) has a unique equilibrium $r_0 = -B/k$ for $k < 0$. The eigenvalues of the linearized equation obtained from (17) are given by

$$\lambda^2 + \frac{lk^2}{B^2}\lambda - \frac{k^4}{B^3} = 0,$$

with the discriminant

$$\Delta = \frac{k^4}{B^4}\left(l^2 + 4B\right) > 0.$$

The roots $\lambda_1$ and $\lambda_2$ are in this case real and $\lambda_1\lambda_2 = -\frac{k^4}{B^3} < 0,$ hence the unique equilibrium is a saddle point.

The above analysis of the linear stability of equilibria can be summarized in THEOREM 3. In the case of nonradial motion $(C \neq 0)$, if $\left(C^2 - B\right)/k > 0,$ the unique equilibrium $r_0 = \left(C^2 - B\right)/k$ will be:

- an unstable saddle point if $C^2 - B < 0$ and $k < 0$;
- a stable node if $C^2 - B > 0, k > 0$ and $l^2 \geq 4\left(C^2 - B\right)$;
- a stable spiral point if $C^2 - B > 0, k > 0$ and $l^2 < 4\left(C^2 - B\right)$.

If $C^2 - B = 0$ and $k = 0$, each $r_0 > 0$ is a stable equilibrium.

In the case when $C = 0$, for $k < 0$ the unique equilibrium $r_0 = -B/k$ will be an unstable saddle point. □

In conclusion, the Manev-type field brings into the scene new solutions and equilibria, specified in *Remark 1* and *Remark 2*; from the point of view of stability we mention the apparition of unstable equlibria (saddle points).

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