An estimation of a generalized divided difference in uniformly convex spaces

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Abstract

The rest in some approximation formulae can be expressed in terms of a generalized divided difference on three knots. We provide an estimation of such a divided difference for functions defined on a uniformly convex space.

KEY WORDS: uniformly convex space; Fréchet derivative; generalized divided difference

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1 Introduction

The convexity properties of the functions were used by Tiberiu Popoviciu to give estimations of the rest in some approximation formulae. A synthesis of this type of results can be found in the book [4].

Several theorems of representation of linear functionals were proved by Raşa [5], [6]. To mention two of them, let $E$ denote a locally convex Hausdorff real space and $X$ a compact convex metrizable subset of $E$; for $f \in C(X)$, $x, y \in X$ and $a \in [0, 1]$, we denote

$$ (x, a, y; f) = (1 - a) f (x) + a f (y) - f ((1 - a) x + a y).$$

(1)

We remark that, since $X$ is a metrizable space, there exist strictly convex functions in $C' (X)$; we denote by $\varphi$ such a function.
Theorem 1 Let $L : C(X) \to \mathbb{R}$ be a linear functional such that $L(g) > 0$ for each strictly convex function $g \in C(X)$. Then for every $f \in C(X)$ there exists $x, y \in X$, $x \neq y$ and $a \in (0, 1)$ such that

$$L(f) = L(\varphi) \frac{(x, a, y; f)}{(x, a, y; \varphi)}.$$ 

We consider now $C(X)$ endowed with the uniform norm.

Theorem 2 Let $L : C(X) \to \mathbb{R}$ be a continuous and linear functional such that $L(g) \geq 0$ for each convex function $g \in C(X)$. Then for every $f \in C(X)$ there exists $x, y \in X$, $x \neq y$ and $a \in (0, 1)$ such that

$$L(f) = L(\varphi) \frac{(x, a, y; f)}{(x, a, y; \varphi)}.$$ 

For the special case $E = \mathbb{R}$, $X = [0, 1]$ and $\varphi(t) = t^2$, $t \in [0, 1]$, Ivan and Raşa [3] showed that

$$(x, a, y; \varphi) = (1 - a)x^2 + ay^2 - ((1 - a)x + ay)^2 = a(1 - a)(x - y)^2,$$

for all $x, y, a \in [0, 1]$. In this case it follows that

$$\frac{(x, a, y; f)}{(x, a, y; \varphi)} = [x, (1 - a)x + ay, y],$$

where the last expression is the classical divided difference of the real function $f$ on the knots $x$, $(1 - a)x + ay$ and $y$. In the general case,

$$[x, a, y; f, \varphi] := \frac{(x, a, y; f)}{(x, a, y; \varphi)}$$

(2)

with $(x, a, y; f)$ given by (1) was then named generalized divided difference on three knots.

2 Main results

We give an estimate of the generalized divided difference (2) in the case of a real uniformly convex space.
Let \((E, \|\cdot\|)\) be a real smooth uniformly convex space and \(X\) a compact subset of \(E\). Consider the (strictly convex) function \(\varphi_r \in C(X)\) given by

\[
\varphi_r(x) = \|x\|^r, \ x \in X,
\]

where \(1 < r \leq 2\).

We need upper and lower bounds for the expression \((x, a, y; f)\). An upper bound for \(|(x, a, y; f)|\) was found in [3], for \(f\) twice Fréchet differentiable on an open set \(Y\) and \(\|f''(y)\| \leq M\) for each \(y \in Y\), namely

\[
|(x, a, y; f)| \leq \frac{M}{2} a(1 - a) \|x - y\|^2.
\]

(3)

It was proved for Hilbert case, but it can be shown that (3) holds in our setting too.

If \(f\) is a convex function, \((x, a, y; f)\) is \(\geq 0\) and is related with the modulus of uniform strict convexity. We recall some definitions from [7], [2]. The modulus of uniform strict convexity at \(x\) (named gage of uniform convexity in [7]) is

\[
\mu_f(x, t) = \inf_{\substack{y \in \text{dom}(f) \\|x-y\|=t \\lambda \in (0,1)}} (x, \lambda, y; f) / \lambda(1 - \lambda), \ t \geq 0.
\]

A related function is

\[
\overline{\mu}_f(x, t) = \inf_{\substack{y \in \text{dom}(f) \\|x-y\|=t}} \left( f(x) + f(y) - 2f \left( \frac{x + y}{2} \right) \right).
\]

One has [2]:

\[
\frac{1}{2} \mu \leq \overline{\mu} \leq \mu.
\]

(4)

The function \(f\) is said to be uniformly convex at \(x\) if \(\mu_f(x, t) > 0\) for each \(t > 0\). The modulus of total convexity of \(f\) at \(x\) is defined by

\[
\nu_f(x, t) = \inf_{\substack{y \in \text{dom}(f) \\|x-y\|=t}} \left( f(y) - f(x) - d^+ f(x, y - x) \right)
\]

(5)

where \(d^+ f(x, h)\) denotes the directional derivative (it exists for \(f\) convex).

One has

\[
\nu_f \geq \mu_f,
\]

(6)
but a reversed inequality holds only when $f$ is Fréchet differentiable, namely in this case there exists a positive constant $\alpha$ such that
\[ \overline{\mu}_f(x,t) \geq \alpha \nu_f(x,\frac{t}{2}). \quad (7) \]

We are interested now in the case $f(x) = \varphi_r(x) = \|x\|^r$ for $r > 1$.

**Lemma 3** If $E$ is a uniformly convex space for which the norm is smooth, then for each $R > 0$ there exists a positive constant $K$ such that
\[ \mu_{\varphi_r}(z,t) \geq K t^r \]
for each $z \in E$, $\|z\| \leq R$.

**Proof.** Denote by $\delta_E$ the modulus of uniform convexity of the space. Using theorem 1 in [1], there exists a positive constant $K_1 > 0$ such that
\[ \nu_{\varphi_r}(z,t) \geq r K_1 t^r \int_0^1 \tau^{r-1} \delta_E \left( \frac{\tau t}{2 (\|z\| + \tau t)} \right) d\tau. \quad (8) \]
Using the well known fact that $t \mapsto \delta_E(t)/t$ is increasing, one obtains
\[ \delta_E \left( \frac{\tau t}{2 (\|z\| + \tau t)} \right) \geq \delta_E \left( \frac{\tau t}{2 (R + \tau t)} \right) > 0, \]
so the integral in (8) is $\geq K_2$. It follows that $\nu_{\varphi_r}(z,t) \geq t^r K_1 K_2$. Because $\varphi_r$ is Fréchet differentiable, one can use (7) and get
\[ \overline{\mu}_{\varphi_r}(z,t) \geq \alpha \left( \frac{t}{2} \right)^r r K_1 K_2. \]
Using (4) we have
\[ \mu_{\varphi_r}(z,t) \geq 2 \alpha \left( \frac{t}{2} \right)^r r K_1 K_2 = K t^r. \]

**Theorem 4** Let $Y$ be an open set, $X \subset Y \subset E$ and $f : Y \to \mathbb{R}$ a function with continuous second order Fréchet derivative, such that $\|f''(y)\| \leq M$ for all $y \in Y$. Then there exists a constant $K$ depending only on $X$ such that
\[ \|[x,a,y; f, \varphi_r]\| \leq KM \quad (9) \]
for all $x, y \in X$, $x \neq y$ and $a \in (0, 1)$.
Proof. As we have mentioned above,

\[ |(x, a, y; f)| \leq \frac{M}{2} a(1 - a) \|x - y\|^2.\]

Using lemma 3 with \( R \) such that \( X \subseteq B(0, R) \), one has

\[ (x, a, y; \varphi_r) \geq a(1 - a) \mu_{\varphi_r} (x, \|x - y\|) \geq a(1 - a) K_1 \|x - y\|^r,\]

and then

\[ |[x, a, y; f, \varphi_r]| \leq \frac{M}{2K_1} \|x - y\|^{2-r} \leq KM.\]

In the special case when \( E \) is a Hilbert space we have

\[ (x, a, y; \varphi_2) = a(1 - a) \|x - y\|^2,\]

and the following result obtained in [3] holds.

**Corollary 5** Let \( E \) be a Hilbert space and \( r = 2 \). In the conditions of theorem 4, the inequality (9) is satisfied with \( K = 1/2 \).

**References**


[5] I. Raşa, Sur les fonctionelles de la forme simple au sens de T. Popoviciu, 


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