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Strong forces in celestial mechanics

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Abstract

Strong forces in celestial mechanics have the property that the particle moving under their action can describe periodic orbits, whose existence follows in a natural way from variational principles. The Newtonian potential does not give rise to strong forces; we prove that potentials of the form $-1/r^\alpha$ produce strong forces if and only if $\alpha \geq 2$. Perturbations of the Newtonian potential with this property are also examined.

KEY WORDS: celestial mechanics; force function
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1 Introduction

Strong forces were considered in 1975 by Gordon [2], when he tried to obtain existence results of periodic solutions in the two-body problem by means of variational methods. As it is well-known, the planar motion of a body (e. g. the Earth) around a much bigger one (e. g. the Sun) is classically modelled by the system

$$\begin{aligned} \ddot{x}_1 &= -\frac{x_1}{r^3} \\ \ddot{x}_2 &= -\frac{x_2}{r^3} \end{aligned} \qquad (1)$$

with $r = \sqrt{x_1^2 + x_2^2}$, or, alternatively

$$\ddot{x} = -\nabla V, \tag{2}$$

with the Newtonian potential

$$V = -\frac{1}{r} \tag{3}$$

and $x = (x_1, x_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$. The potential V has a singularity at the origin of the plane. Even if one works in a class of 'noncollisional' loops ($x(t) \neq (0,0)$, $\forall t \in \mathbb{R}$), the extremal offered by a variational principle will be the limit of a sequence of such loops, hence we have no guarantee that it will avoid the origin. Gordon remarked that for other type of conservative forces, called by him *strong forces*, the extremals are not collisional trajectories.

2 Main results

We shall consider systems of the type

$$\ddot{x} = \nabla W, \tag{4}$$

where $x = (x_1, ..., x_N) \in \mathbb{R}^N$, and $W \in C^2(\mathbb{R}^N \setminus \{0\})$ is the force function ($W = -V$). We shall denote by $|\cdot|$ the Euclidean norm in $\mathbb{R}^N$. The cases physically meaningful are those with $N \in \{1, 2, 3\}$.

Definition 1 *(Gordon [2]) The system (4) satisfies the* strong force *(SF) condition if and only if there exists a neighbourhood $\mathcal{N}$ of the origin 0 of $\mathbb{R}^N$ and a function $U \in C^2(\mathbb{R}^N \setminus \{0\})$ such that*

(i) $U(x) \to -\infty$ as $x \to 0$;

(ii) $W(x) \geq |\nabla U(x)|^2$ for all x in $\mathcal{N} \setminus \{0\}$.

Remark 2 *As a matter of fact, one can choose another differentiable norm instead of the Euclidean one, hence in Definition 1 $\mathcal{N}$ may be supposed to be the unit ball $\{x \in \mathbb{R}^N : |x| < 1\}$.*

Remark 3 *If the force function W gives rise to a strong force, the function aW (with $a > 0$) has the same property; this happens also for each function $W_1 \in C^2(\mathbb{R}^N \setminus \{0\})$ with $W_1(x) \geq W(x)$, $x \in \mathcal{N} \setminus \{0\}$.*

The example given by Gordon to illustrate the definition is $W(x) = 1/|x|^2$; he remarks also that $W(x) = 1/|x|$, corresponding to the Newtonian potential, is not strong, fact which determines him to say that 'it is disappointing that the gravitational case is excluded by the SF condition'. Nevertheless he obtained existence results for periodic orbits in strong force fields, and this was the starting point for applying systematically the variational methods in celestial mechanics. It is interesting to mention that, in 1896, Poincaré [5] had the same idea of using the least action principle to find periodic orbits in the planar three-body problem, for a force of the type $1/r^n$ with $n \geq 2$ (excluding again the Newtonian potential). By that time the variational methods were not formulated in a rigourous way, and there was a strong belief that Newtonian potential governs the motion of celestial bodies, so Poincaré's result remained for years purely theoretical.

Our first concern is to find out which functions $W_\alpha \in C^2(\mathbb{R}^N \backslash \{0\})$,

$$W_\alpha(x) = \frac{1}{|x|^\alpha}, \ \alpha > 0, \tag{5}$$

satisfy the SF condition.

Theorem 4 *For $\alpha > 0$, W_α from (5) satisfies the SF condition if and only if $\alpha \geq 2$.*

Proof. Let $\alpha \geq 2$, $\mathcal{N} = \left\{x \in \mathbb{R}^N : |x| < 1\right\}$ and $U = \ln|x|$. It follows that $|\nabla U(x)|^2 = 1/|x|^2 \leq W_\alpha(x)$ whatever x in $\mathcal{N}\backslash\{0\}$, hence W_α satisfies the SF condition.

Let us consider now $0 < \alpha < 2$. We suppose that there exists a function U as in Definition 1. We fix $x_0 \in \mathbb{R}^N \backslash \{0\}$; for $\mu, \lambda > 0$, $\mu < \lambda < 1/|x_0|$ we evaluate

$$|U(\lambda x_0) - U(\mu x_0)| = \left| \int_\mu^\lambda dU(tx_0, x_0)dt \right| \leq |x_0| \int_\mu^\lambda |\nabla U(tx_0)| \, dt.$$

Using (ii) we obtain

$$|x_0| \int_\mu^\lambda |\nabla U(tx_0)| \, dt \leq |x_0| \int_\mu^\lambda \frac{1}{|tx_0|^{\alpha/2}} dt =$$

$$|x_0|^{1-\alpha/2} \int_\mu^\lambda \frac{1}{t^{\alpha/2}} dt = |x_0|^{1-\alpha/2} \frac{1}{1-\alpha/2} \left(\lambda^{1-\alpha/2} - \mu^{1-\alpha/2}\right),$$

hence $|U(\lambda x_0) - U(\mu x_0)| \leq |x_0|^{1-\alpha/2} (1-\alpha/2)^{-1} \left(\lambda^{1-\alpha/2} - \mu^{1-\alpha/2}\right)$. In this last relation we make $\mu \to 0_+$ and we obtain the contradiction $|x_0|^{1-\alpha/2} \cdot (1-\alpha/2)^{-1} \lambda^{1-\alpha/2} \geq \infty$. It follows that, for any $0 < \alpha < 2$, W_α does not satisfy the SF condition. ■

In view of Remark 3, we have

Corollary 5 *Each function $W \in C^2(\mathbb{R}^N \backslash \{0\})$ with $W(x) \geq a/|x|^\alpha$ ($a > 0$, $\alpha \geq 2$) satisfies the SF condition.*

Example 6 *Corollary 5 includes among the functions which satisfy the SF condition those related to various perturbations of the Newtonian force. One of them, with great physical significance, corresponds to the Manev potential [4] and is given by*

$$W_M = m\left(\frac{1}{r} + \frac{3m}{2c^2}\frac{1}{r^2}\right), \tag{6}$$

m being the gravitational parameter of the two-body system and c the speed of light. This potential is a good substitute for relativity theory at the solar system's level. It was mentioned as a strong force by Anisiu [1]. The advances in the qualitative understanding of the motion in a Manev-type field are exposed in [3]. A potential of Manev-type was studied by Newton himself, and he showed that the force generated by such a potential produces a precessionally elliptic orbit.

Schwarzschild [6] solved the relativistic analog of the classical Kepler problem and derived the force function

$$W_S = GM\left(\frac{1}{r} + \frac{b}{r^3}\right), \tag{7}$$

where G is the gravitational constant, M is the mass of the field-generating body and b a positive constant. The motion in a Schwarzschild field, with implications in astrophysics, is studied by Stoica and Mioc [7].

We can establish precisely what perturbations of the Newtonian potential are strong or not.

Theorem 7 *For $\alpha > 0$, the perturbation of the Newtonian force $\widetilde{W}_\alpha \in C^2(\mathbb{R}^N \backslash \{0\})$ given by*

$$\widetilde{W}_\alpha(x) = \frac{1}{|x|} + \frac{b}{|x|^\alpha},\ b > 0,$$

satisfies the SF condition if and only if $\alpha \geq 2$.

Proof. The fact that $\widetilde{W}_\alpha$ satisfies the SF condition for $\alpha \geq 2$ follows directly from Corollary 5.

For $0 < \alpha \leq 1$ and $x \in \mathcal{N} \backslash \{0\}$ (as mentioned in Remark 2, we take $\mathcal{N}$ the unit ball), we have $\widetilde{W}_\alpha(x) \leq (1+b) / |x|$, so $\widetilde{W}_\alpha$ cannot satisfy the SF condition, because from Theorem 4 it follows that $W_1(x) = 1/|x|$ does not satisfy it. For $1 < \alpha < 2$, we suppose that there exists a function U as in Definition 1. We fix $x_0 \in \mathbb{R}^N \backslash \{0\}$; for $\mu, \lambda > 0$, $\mu < \lambda < 1/|x_0|$ we evaluate as in the proof of Theorem 4

$$|U(\lambda x_0) - U(\mu x_0)| \leq |x_0| \int_\mu^\lambda |\nabla U(t x_0)|\, dt \leq |x_0| \int_\mu^\lambda \sqrt{\frac{1}{|t x_0|} + \frac{b}{|t x_0|^\alpha}} dt$$

$$\leq |x_0|^{1-\alpha/2} \sqrt{1+b} \int_\mu^\lambda \frac{1}{t^{\alpha/2}} dt = |x_0|^{1-\alpha/2} \frac{\sqrt{1+b}}{1-\alpha/2} \left(\lambda^{1-\alpha/2} - \mu^{1-\alpha/2}\right).$$

It follows that

$$|U(\lambda x_0) - U(\mu x_0)| \leq |x_0|^{1-\alpha/2} \frac{\sqrt{1+b}}{1-\alpha/2} \left(\lambda^{1-\alpha/2} - \mu^{1-\alpha/2}\right)$$

and, making $\mu \to 0_+$, we obtain a contradiction. It follows that, for any $0 < \alpha < 2$, $\widetilde{W}_\alpha$ does not satisfy the SF condition. ∎

By Corollary 5 we have that each force function with $W(x) \geq a/|x|^2$, $a > 0$, satisfies the SF condition; due to the simplicity of this description, it is sometimes considered as SF definition. The next example shows that there are SF potentials which do not satisfy the mentioned inequality.

Example 8 *Define $\varphi : (0, 1/2] \to \mathbb{R}$, $\varphi(t) = \ln(-\ln t)$. The function φ can be extended to a C^3 function defined on $(0, \infty)$ by taking $\varphi(t)$ a polynomial of third degree for $t > 1/2$. Then $W(x) := \varphi'(|x|)^2$ satisfies the SF condition. Indeed, we can choose $U(x) = -\varphi(|x|)$ and we have $\lim_{x \to 0} U(x) = -\lim_{t \to 0_+} \varphi(t) = -\infty$ and*

$$|\nabla U(x)|^2 = \left| \varphi'(|x|) \frac{x}{|x|} \right|^2 = \varphi'(|x|)^2 = W(x),\ x \in \mathbb{R}^N \backslash \{0\}.$$

On the other side, let us suppose that for $|x| \leq 1/2$, $W(x) \geq a/|x|^2$, where $a > 0$. This would imply $\varphi'(t)^2 \geq a/t^2$, that is $a \leq 1/\ln^2 t$ for each $t \in (0, 1/2]$, hence $a \leq 0$, contradiction.

Note that, in this example, condition (ii) from Definition 1 is in fact an equality over $\mathbb{R}^N \backslash \{0\}$.

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