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SYMMETRIC PERIODIC ORBITS IN THE ANISOTROPIC SCHWARZSCHILD-TYPE PROBLEM

VASILE MIOC[1], MIRA-CRISTIANA ANISIU[2] and MICHAEL BARBOSU[3]
[1]*Astronomical Institute of the Romanian Academy, Str. Cuţitul de Argint 5, RO-040557 Bucharest, Romania, e-mail: vmioc@aira.astro.ro*
[2]*T. Popoviciu Institute of Numerical Analysis of the Romanian Academy, P.O. Box 68, 400110 Cluj-Napoca, Romania, e-mail: mira@math.ubbcluj.ro*
[3]*SUNY Brockport, Department of Mathematics, Brockport, NY 14420, U.S.A., e-mail: mbarbosu@brockport.edu*

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Abstract. Studying the two-body problem associated to an anisotropic Schwarzschild-type field, Mioc et al. (2003) did not succeed in proving the existence or non-existence of periodic orbits. Here we answer this question in the affirmative. To do this, we start from two basic facts: (1) the potential generates a strong force in Gordon's sense; (2) the vector field of the problem exhibits the symmetries S_i, $i = \overline{1,7}$, which form, along with the identity, an Abelian group of order 8 with three generators of order 2. Resorting to S_2 and S_3, in connection with variational methods (particularly the classical lower-semicontinuity method), we prove the existence of infinitely many S_2- or S_3-symmetric periodic solutions. The symmetries S_2 and S_3 constitute an indicator of the robustness of the classical isotropic Schwarzschild-type system to perturbations (as the anisotropy may be considered).

Key words: Schwarzschild-type problems, nonlinear particle dynamics, symmetries, periodic orbits, variational methods

1. Introduction

Astronomy provides a lot of concrete situations that can be tackled via anisotropic mathematical models. The anisotropy of the gravitational constant was discussed by many authors (e.g. Will, 1971, Vinti, 1972). The two-dimensional galactic models also join this class of problems. We further mention: motions around a luminous accretion disk, far orbits around binary stars, orbits around stars with unequal luminosity over surface (pulsars, stars with spots), orbits in proto-stellar systems (and in the proto-solar, too), motion of bodies (from dust to satellites) around planets, in the field of radiation re-emitted by these ones (Saslaw, 1978; Mioc and Radu, 1992). Even the celebrated model of Hénon and Heiles (1964) involves anisotropy.

Gutzwiller (1971, 1973, 1977) defined the anisotropic Kepler problem (namely the anisotropic two-body problem associated to the Newtonian potential) with an essential goal: to identify links between classical and quantum mechanics. Devaney (1978, 1981) and Casasayas and Llibre (1984) went deeper into this problem.

According to a suggestion formulated by Diacu (1996), the anisotropic Manev problem (associated to a classic potential of the form $\alpha/r + \beta/r^2$) was tackled with a more ambitious purpose: to find connections between classical, quantum, and relativistic mechanics. Important results in this problem were obtained by Craig et al. (1999), Diacu and Santoprete (2001, 2002), Santoprete (2002).

Combining Gutzwiller's anisotropy with Schwarzschild's (1916) potential (of the form $\alpha/r + \beta/r^3$), Mioc et al. (2003; hereafter Paper I) considered the anisotropic Schwarzschild-type problem. They used the powerful tools of the theory of dynamical systems to depict the main features of the global flow.

However, Paper I did not offer an answer to a crucial question for all dynamical systems: does the model admit periodic solutions? In this paper we solve this problem in the affirmative.

In Section 2 we recall the basic equations of the problem in configuration-momentum coordinates. We show that the anisotropic Schwarzschild-type potential generates a strong force in Gordon's (1975) sense. We also show that the corresponding vector field benefits of seven symmetries S_i, $i = \overline{1,7}$, which form, along with the identity, an Abelian group of order 8 with three generators of order 2.

In Section 3 we expose some basic notions related to Sobolev spaces, which are the natural frame for finding periodic solutions by variational methods. To get certain families of periodic orbits, we resort to the symmetries S_2, S_3 in connection with variational methods. The subspaces of symmetric periodic paths Σ_2 and Σ_3 will be divided into homotopy classes by their winding number (rotation index). The action integral A_T will be the functional whose critical points will be the periodic solutions of the Schwarzschild-type problem.

Section 4 contains auxiliary results, which allow us to prove the existence of critical points on some subsets with symmetry and topological constraints, which will be critical points for A_T on the whole space.

Section 5 presents the main results of our endeavours. Following the methods used by Gordon (1975), Ambrosetti and Coti Zelati (1993), and Coti Zelati (1994) for general strong force fields, and by Diacu and Santoprete (2002) for the anisotropic Manev problem, we resort to the classical lower-semicontinuity method (Tonelli, 1915) to find a minimizer of the action integral in each class, avoiding the collision-type or escape-type minimizers. We prove that, for any pre-assigned period, the action integral has a critical point in each homotopy class with nonnull winding number in Σ_2 and Σ_3. This implies the existence of infinitely many S_i-symmetric ($i = 2, 3$) periodic solutions of the anisotropic Schwarzschild-type problem.

Some remarks are to be formulated here. Even if the idea of using variational methods to find periodic orbits in the planar three-body problem for a potential force of the type $1/r^n$ with $n \geqslant 2$ goes as far back as the end of the 19th century (Poincaré, 1896), effective results appeared only relatively recently. The most celebrated result in this context was obtained by Chenciner and Montgomery (2000), who connected symmetries to variational principles to find a new periodic solution of the three-body problem with equal masses. As it was mentioned by Anisiu (1998) for the Manev problem, and it is obviously true for the Schwarzschild one, in these cases (unlike in the Newtonian case), the force field is ‘strong’ (according to Gordon’s (1975) definition), which makes variational methods easier to apply. Interesting results of this type were obtained by Bertotti (1991) for the restricted three-body problem.

Another remark concerns the existence of S_i-symmetric ($i = 2, 3$) periodic solutions. Such solutions exist in the isotropic Schwarzschild-type problem (see Stoica and Mioc, 1997; Mioc, 2002). Their persistence (even deformed) in the anisotropic case (regarded as a perturbation of the isotropic case) makes the symmetries S_2, S_3 constitute an indicator of the robustness of the system to perturbations.

We tried to expose in some detail the necessary mathematical tools, in connection with which we must not forget two issues: (a) almost all such mathematical methods were born from and intended to tackle concrete astronomical situations; (b) our present results add new, important features to the dynamics of the anisotropic Schwarzschild-type problem.

2. Basic Equations and Properties

The *2D* anisotropic Schwarzschild-type problem is described by the two-degrees-of-freedom system of ODE

$$\dot{\mathbf{q}} = \partial H(\mathbf{q}, \mathbf{p})/\partial \mathbf{p}, \quad \dot{\mathbf{p}} = -\partial H(\mathbf{q}, \mathbf{p})/\partial \mathbf{q}, \tag{1}$$

where

$$\mathbf{q} = (q_1, q_2) \in \mathbb{R}^2 \setminus \{(0,0)\}, \quad \mathbf{p}(= \dot{\mathbf{q}}) = (p_1, p_2) \in \mathbb{R}^2$$

stand, respectively, for the configuration and momentum of a two-particle system originated in one of the particles. The Hamiltonian H has the form

$$H(\mathbf{q}, \mathbf{p}) = |\mathbf{p}|^2/2 - W(q) \tag{2}$$

in which the potential function $W\colon \mathbb{R}^2\backslash\{(0,0)\} \to \mathbb{R}$ has the expression (Paper I):

$$W(q_1, q_2) = (\mu q_1^2 + q_2^2)^{-1/2} + b(\mu q_1^2 + q_2^2)^{-3/2}, \tag{3}$$

where $\mu > 0$ and $b > 0$ are parameters.

We mention that the Lagrangian $L(\mathbf{q}, \mathbf{p}) = |\mathbf{p}|^2/2 + W(\mathbf{q})$ of the anisotropic Schwarzschild-type problem (1) is

$$L(q_1, q_2, p_1, p_2) = (p_1^2 + p_2^2)/2 + (\mu q_1^2 + q_2^2)^{-1/2} + b(\mu q_1^2 + q_2^2)^{-3/2} \tag{4}$$

and has the property that $L(\mathbf{q}, \mathbf{p}) > 0$.

Equations (1) define the motion of a unit-mass particle with respect to another unit-mass particle in an anisotropic plane, namely a plane in which the attraction forces act differently in every direction. The force function $W(\mathbf{q})$ characterizes the anisotropy of the plane as a function of the parameter μ. For $\mu > 1$ the attraction is the strongest in the q_1-direction and the weakest in the q_2-direction; for $\mu < 1$ the situation is reversed. For $\mu = 1$ we retrieve the classical Schwarzschild-type two-body problem, whose global flow was fully depicted by Stoica and Mioc (1997). We shall consider, without loss of generality, that $\mu > 1$.

One sees that the Hamiltonian (2) is the sum of the kinetic $(K(\mathbf{p}(t)) = |\mathbf{p}(t)|^2/2)$ and potential $(-W(\mathbf{q}(t)))$ energies. It provides the first integral of energy

$$H(\mathbf{q}(t), \mathbf{p}(t)) = \tilde{h}, \ t \in \mathbb{R}, \tag{5}$$

where $\tilde{h}$ stands for the energy constant.

Notice that, unlike in the classical Schwarzschild model, the force derived from the potential function W is not central; the anisotropy of the plane destroys the rotational invariance. Consequently, the angular momentum $C(t) = \mathbf{p}(t) \times \mathbf{q}(t)$ is not conserved; it does not provide a first integral.

A basic property of system (1) with W given by (3) is that it satisfies the strong force condition.

DEFINITION 1 (Gordon, 1975). A potential function $W\colon \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}$ generates a *strong force* if there exist a neighbourhood N of (0, 0) and a C^2-class function $U\colon N \setminus \{(0,0)\} \to \mathbb{R}$ such that

(i) $U(q_1, q_2) \to -\infty$ as $(q_1, q_2) \to (0,0)$;

(ii) $W(q_1, q_2) \geqslant (\partial U/\partial q_1)^2 + (\partial U/\partial q_2)^2 = |\nabla U|^2$ for all (q_2, q_2) in $N \backslash \{(0,0)\}$.

Remark 2. If there exist the constants $c, R > 0$ such that the potential function $W\colon \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}$ verifies the inequality

$$W(\mathbf{q}) \geqslant \frac{c}{|\mathbf{q}|^2}, \quad 0 < |\mathbf{q}| < R, \tag{6}$$

then W generates a strong force, with $U(\mathbf{q}) = \sqrt{c}\ln(|\mathbf{q}|)$ and $N = \{\mathbf{q} : |\mathbf{q}| < R\}$. Condition (6), which is easier to be verified, appears as the definition of strong forces in many papers.

THEOREM 3. *The potential function W given by* (3) *generates a strong force.*

Proof. From the expression (3) of the potential function W and from $\mu > 1$ it is obvious that $W(q_1, q_2) \geqslant b(\mu q_1^2 + q_2^2)^{-3/2} \geqslant b\mu^{-3/2}(q_1^2 + q_2^2)^{-3/2} \geqslant b\mu^{-3/2} |\mathbf{q}|^2$ for $0 < |\mathbf{q}| < 1$, hence W satisfies (6) with $c = b\mu^{-3/2}$ and $R = 1$. □

By (2) and (3), the motion Equations (1) explicitly read.

$$\begin{aligned} \dot{q}_1 &= p_1, \\ \dot{q}_2 &= p_2, \\ \dot{p}_1 &= -\mu(\mu q_1^2 + q_2^2)^{-3/2}(1 + 3b(\mu q_1^2 + q_2^2)^{-1})q_1, \\ \dot{p}_2 &= -(\mu q_1^2 + q_2^2)^{-3/2}(1 + 3b(\mu q_1^2 + q_2^2)^{-1})q_2. \end{aligned} \tag{7}$$

An important property of these equations can be stated as

THEOREM 4. *The vector field* (7) *benefits of seven symmetries* $S_i = S_i(q_1, q_2, p_1, p_2, t)$, $i = \overline{1,7}$, as follows:

$$\begin{aligned} S_1 &= (q_1, q_2, -p_1, -p_2, -t), \\ S_2 &= (q_1, -q_2, -p_1, p_2, -t), \\ S_3 &= (-q_1, q_2, p_1, -p_2, -t), \\ S_4 &= (q_1, -q_2, p_1, -p_2, t), \\ S_5 &= (-q_1, q_2, -p_1, p_2, t), \\ S_6 &= (-q_1, -q_2, -p_1, -p_2, t), \\ S_7 &= (-q_1, -q_2, p_1, p_2, -t). \end{aligned} \tag{8}$$

The set $G = \{I\} \cup \{S_i \mid i = \overline{1,7}\}$, *endowed with the usual composition law* "$\circ$", *forms a symmetric Abelian group with an idempotent structure, isomorphic to* $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, *where I denotes the identity. This group owns seven proper subgroups isomorphic to Klein's group.*

Proof. It is easy to check that Equations (7) are invariant to the transformations (8). As regards the Abelian group structure of G, it suffices to construct and examine the following composition table:

$\circ$	I	S_1	S_2	S_3	S_4	S_5	S_6	S_7
I	I	S_1	S_2	S_3	S_4	S_5	S_6	S_7
S_1	S_1	I	S_4	S_5	S_2	S_3	S_7	S_6
S_2	S_2	S_4	I	S_6	S_1	S_7	S_3	S_5
S_3	S_3	S_5	S_6	I	S_7	S_1	S_2	S_4
S_4	S_4	S_2	S_1	S_7	I	S_6	S_5	S_3
S_5	S_5	S_3	S_7	S_1	S_6	I	S_4	S_2
S_6	S_6	S_7	S_3	S_2	S_5	S_4	I	S_1
S_7	S_7	S_6	S_5	S_4	S_3	S_2	S_1	I

Since every element is its own inverse, the idempotent structure is obvious.

Observe that, among the symmetries (8), only three are independent. Consider, for instance, that these ones are S_1, S_2, S_3. The relations $S_1 \circ S_2 = S_4$, $S_1 \circ S_3 = S_5$, $S_2 \circ S_3 = S_6$, $S_1 \circ S_2 \circ S_3 = S_7$ are immediate. Every other three independent symmetries generate the remaining four ones. This means that G is an Abelian group of order eight with three generators of order two. By the Fundamental Theorem of Abelian Groups, G is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

As to the proper subgroups of G, let us denote them by $G_{ijk} = \{I, S_i, S_j, S_k \mid i \neq j \neq k \neq i, S_i \circ S_j = S_k\}$. We can immediately check that the only sets $\{i, j, k\}$ that fulfil this condition lead to the subgroups $\tilde{G}_{124}, \tilde{G}_{135}, \tilde{G}_{167}, \tilde{G}_{236}, \tilde{G}_{257}, \tilde{G}_{347}, \tilde{G}_{456}$. All these subgroups are of order four with two generators of order two, hence isomorphic to Klein's subgroup. This completes the proof. □

In Paper I we have shown that Theorem 4 also holds for the motion equations expressed in collision-blow-up or infinity-blow-up McGehee-type coordinates (McGehee, 1973, 1974). The respective groups of 4 symmetries, G_0 and G_∞, are isomorphic to G. This is not a trivial result, because the phase spaces corresponding to G_0 and to G_∞ contain the supplementary boundary manifolds of collision and infinity, respectively.

As mentioned in the introductory section, in Paper I the important question about the existence or not of periodic orbits remained open. This question will be answered in the affirmative in what follows, where the symmetries (8) and the strong force property of the potential will play a premier role.

3. The Functional Background

In order to get certain families of periodic orbits, we shall use some symmetry and topological constraints in connection with a variational principle, finding periodic orbits not as minimizers, but as extremal values of the action integral. To this end we shall resort to results concerning periodic solutions of fixed period for symmetric, singular, Lagrangian systems (see Gordon 1975; Ambrosetti and Coti Zelati, 1993; Coti Zelati, 1994), as the system (7) is.

We first need some notations. Given a number $T > 0$, let us denote by $C^\infty([0, T], \mathbb{R}^2)$ the space of T-periodic C^∞ cycles (loops) f: $[0, T] \to \mathbb{R}^2$. We define the inner products

$$\langle f, g\rangle_{L^2} = \int_0^T f(t)g(t)\mathrm{d}t, \tag{9}$$

$$\langle f, g\rangle_{H^1} = \int_0^T (\dot{f}\dot{g} + fg)\mathrm{d}t = \langle \dot{f}, \dot{g}\rangle_{L^2} + \langle f, g\rangle_{L^2}, \tag{10}$$

and let $\|f\|_{L^2} = (\int_0^T |f|^2\mathrm{d}t)^{1/2}$ and $\|f\|_{H^1} = (\int_0^T (|\dot{f}|^2 + |f|^2)\mathrm{d}t)^{1/2}$ be the corresponding norms. Then the completion of $C^\infty([0, T], \mathbb{R}^2)$ with respect to $\|\cdot\|_{L^2}$ is denoted by L^2 and is the space of square integrable functions. The completion with respect to $\|\cdot\|_{H^1}$ is denoted by H^1 and is the Sobolev space of all absolutely continuous T-periodic functions that have L^2 derivatives defined almost everywhere. The space H^1 is compactly embedded in the space of continuous functions on $[0, T], C^0([0, T], \mathbb{R}^2)$ with $\|f\| = \max\{|f(t)| : t \in [0, T]\}$ (Gordon, 1975).

The Schwarzschild potential (3) has a singularity at the origin of $\mathbb{R}^2$, hence we shall denote by

$$\Lambda = \{f \in H^1 \,|\, f(t) \neq (0, 0) \quad \textit{for all } t \in [0, T]\} \tag{11}$$

the open subset of the cycles in H^1 which do not pass through the origin (noncollisional cycles). For the noncollisional cycles in Λ it makes sense to define the *angle function* $\vartheta_f \in C^0([0, T], \mathbb{R})$ by

$$\cos\vartheta_f(t) = f_1(t)\Big/\sqrt{f_1^2(t) + f_2^2(t)}, \quad \sin\vartheta_f(t) = f_2(t)\Big/\sqrt{f_1^2(t) + f_2^2(t)},$$

which measures the angle between the positive q_1-axis and the vector $f(t)$ in the mathematically positive direction. The *rotation index* (or *winding number*) $w(f)$ will represent the growth of the angle function during a period, measured in units of full rotations of f, that is

$$w(f) = \frac{\vartheta_f(0) - \vartheta_f(T)}{T}$$

The rotation index is always an integer which shows how many times the continuous cycle $f: [0, T] \to \mathbb{R}^2 \setminus \{(0,0)\}$ 'winds around' the origin, having positive values for counterclockwise rotations and negative for clockwise ones. Identifying S^1 with $\mathbb{R}/[0, T]$ one has $w(f) = \deg(f)$, where $\deg(f)$ is the degree of the circle map $F: S^1 \to S^1, F(t) = f(t)/|f(t)|$ (Amann, 1990). It follows

$$\Lambda = \bigcup_{k \in \mathbb{Z}} \Lambda_k,$$

where $\Lambda_k = \{f \in \Lambda | w(f) = k\}$. We mention that Λ_0 contains the loops which are homothetic in $\mathbb{R}^2 \setminus \{(0,0)\}$ to a point.

We shall use some of the 'natural' symmetries $S_i, i = \overline{1,7}$ of the system (7).

Let us denote by $\Sigma_i, i = \overline{1,7}$ the subsets of H^1 formed by S_i-symmetric cycles, namely those which satisfy $S_i(f(t)) = f(t)$. It is clear that each Σ_i is a subspace of H^1. In the sequel we shall provide orthogonal decompositions of H^1 in terms of its subspaces Σ_i with $i \in \{2,3\}$ and $i \in \{1,7\}$, respectively.

LEMMA 5. *The subspaces Σ_i with $i \in \{1,2,3,7\}$ are closed, weakly closed, and complete with respect to $\| \cdot \|_{H^1}$, hence they are Sobolev spaces. Moreover,*

$$H^1 = \Sigma_2 \oplus \Sigma_3 = \Sigma_1 \oplus \Sigma_7. \tag{12}$$

Proof. Consider a function $f = (f_1, f_2)$; it is well known that f_1 and f_2 can be written as sums of an even absolutely continuous function and an odd one, namely $f_j = f_{j,e} + f_{j,o}$, where $f_{j,e}(t) = (f_j(t) + f_j(-t))/2$ and $f_{j,o}(t) = (f_j(t) - f_j(-t))/2$, $j = 1, 2$. By virtue of this fact, we can write f as the sum of an S_2-symmetric function, $f_{S_2} = (f_{1,e}, f_{2,o})$ and an S_3-symmetric one, $f_{S_3} = (f_{1,o}, f_{2,e})$.

Now, let us consider an element $f \in \Sigma_2$. Then $\langle f, g \rangle_{H^1} = 0$ for every $g \in \Sigma_3$. This is due to the fact that, by (9) and (10),

$$\langle f, g \rangle_{H^1} = \int_0^T (\dot{f}_1 \dot{g}_1 + \dot{f}_2 \dot{g}_2) \mathrm{d}t + \int_0^T (f_1 g_1 + f_2 g_2) \mathrm{d}t, \tag{13}$$

where the second integrand is an odd function, whereas the first one is an odd function almost everywhere. Thus the above inner product is zero for every $g \in \Sigma_3$.

Denote by $\Sigma_2^\perp = \{g \in \Sigma_2 | \langle f, g \rangle_{H_1} = 0, \forall_g \in \Sigma_2\}$ the space orthogonal to Σ_2. It is easy to see that $\Sigma_2^\perp$ is closed and that $\Sigma_3 \subset \Sigma_2^\perp$. To prove that $\Sigma_3 = \Sigma_2^\perp$, suppose that there exists $\varphi \in \Sigma_2^\perp$ such that $\varphi \neq 0$ and $\varphi \notin \Sigma_3$. Then write $\varphi = \varphi s_2 + \varphi s_3$ and compute $\langle \varphi s_2, \varphi \rangle_{H_1} = \langle \varphi s_2, \varphi s_2 + \varphi s_3 \rangle_{H^1} = \langle \varphi s_2, \varphi s_2 \rangle_{H^1} = \| \varphi s_2 \|_{H^1} = \delta > 0$. But this contradicts the hypothesis that $\varphi \in \Sigma_2^\perp$, therefore $\Sigma_3 = \Sigma_2^\perp$. So Σ_3 and, consequently, Σ_2 are closed and such that

$H^1 = \Sigma_2 \oplus \Sigma_3$. In addition, since H^1 is complete, Σ_2 and Σ_3 are complete. Lastly, Σ_2 and Σ_3 are weakly closed because they are norm-closed subspaces.

The statements for Σ_1 and Σ_7 can be proved similarly, by writing $f = f_{S_1} + f_{S_7}$, with $f_{S_1} = (f_{1,e}, f_{2,e})$ and $f_{S_7} = (f_{1,o}, f_{2,o})$. This completes the proof. □

Consider the sets $\tilde{\Sigma}_i = \Sigma_i \cap \Lambda$ of symmetric noncollisional cycles, which are open submanifolds of the spaces Σ_i. We shall say that a cycle in $\tilde{\Sigma}_i$ is of class $\Lambda_k, k \in \mathbb{Z}$, if its winding number about the origin of the coordinate system is k, namely if it performs k loops around the origin. We remind that k is positive for counterclockwise rotations and negative else. The family $(\Lambda_k)_{k \in \mathbb{Z}}$ provides a partition of $\tilde{\Sigma}_i$ into homotopy classes (components).

We shall describe the geometric properties of the cycles in $\Sigma_i, i = \overline{1,7}$. A cycle f is in Σ_1 if and only if $f_j(-t) = f_j(t),\ j = 1, 2$, hence the cycles in $\tilde{\Sigma}_1$ will have null winding number ($\tilde{\Sigma} \subset \Lambda_0$). The cycles in Σ_2 and Σ_3 have mirror symmetry with respect to the q_1, respectively to the q_2-axis. Those in Σ_4 and Σ_5 are lying on the q_1, respectively q_2-axis, hence the cycles in $\tilde{\Sigma}_4$ and $\tilde{\Sigma}_5$ will have again null winding number. The cycles in Σ_6 reduce themselves to the single point (0, 0). The cycles in Σ_7 are symmetric with respect to the origin (0, 0) and all of them pass through (0, 0), hence they are all collisional.

We are interested in noncollisional families of cycles with nonnull winding numbers; Σ_2 and Σ_3 are the only subspaces of H^1 among the seven subspaces corresponding to the natural symmetries $S_i, i = \overline{1,7}$, which satisfy those requirements.

Remark 6. To be more clear for a nonmathematician reader, we summarize what we have done so far. We fixed a period T and we defined the Sobolev space H^1 of all absolutely continuous T-periodic functions. We defined the subsets Σ_i of H^1 formed by S_i-symmetric cycles (characterized by the symmetries $S_i, i = \overline{1,7}$, of the motion Equation (7)). Moreover, we have shown that the couples (Σ_2, Σ_3), (Σ_1, Σ_7) cover – via direct sum – the whole space H^1. Being interested only in cycles that are noncollisional or nonescape type, or do not represent quasiperiodic orbits, we showed that only Σ_2 and Σ_3 fulfil these conditions. To continue our mathematical endeavours, we divided every subset $\tilde{\Sigma}_i$ in homotopy classes via the winding number k; this will be useful further down.

We shall define now the action integral, whose extremal values will provide symmetric periodic orbits. The *action integral* $A_T\colon \Lambda \to \mathbb{R}$ between the instants 0 and T, along a cycle f whose Euclidean coordinate representation is $\mathbf{q} = (q_1, q_2)$, has the expression

$$A_T(f) = \int_0^T L(\mathbf{q}(t), \mathbf{p}(t))\mathrm{d}t, \tag{14}$$

with the positive Lagrangian function given by (4). The appearance of the quadratic terms $p_1^2 = \dot{q}_1^2$ and $p_2^2 = \dot{q}_2^2$ in the expression of the Lagrangian function makes the Sobolev space H^1 adequate for this problem. The Schwarzschild potential belongs to a class of potentials (with $W(q) \geqslant 0$ for $\mathbf{q} \neq (0,0)$ and $W(\mathbf{q}) \to 0$ for $|\mathbf{q}| \to \infty$) for which it can be shown (Coti Zelati, 1994) that $\inf_{f \in \Lambda} A_T(f) = 0$, and all minimizing sequences are unbounded.

In order to obtain periodic solutions of (7) we are forced to minimize the functional A_T on subsets Λ_k of Λ, chosen by using symmetry and topological constraints. After selecting an adequate subset, we shall use a direct method of the calculus of variation, i.e. the *lower-semicontinuity method* (e.g., Struwe 1996) and get a minimizer in that subset, which will be proved to be an extremal value of the functional A_T. Finally we show that the extremal values, which belong to the Sobolev space H^1, are regular enough to constitute classical periodic solutions of (7).

Remark 7. To sketch in physical terms the minimization of the action, recall that the Lagrangian of our problem represents the sum of two positive terms: the kinetic energy K and the force function W (the negative of the potential energy). Also recall that K and W are not independent each other, they being related by the energy integral (in which the constant of energy must be negative, provided the positiveness of W). Since $K, W > 0$, any minimization of their sum involves the minimization of both K and W; both push the trajectory away from the field-generating centre. But the limit imposed by the fixed energy-level and by the fixed value of T stops the orbit expansion to a finite value, which can lead to a periodic orbit.

4. Auxiliary Results

The first statement establishes the connection between the solutions of the Schwarzschild-type problem (7) and the extremals (critical points) of the functional A_T given by (14).

PROPOSITION 8. *The set of noncollisional cycles* Λ *given by* (11) *is an open subset of* H^1 *and the functional* A_T *is in the class* $C^1(\Lambda, \mathbb{R})$ *with*

$$\mathrm{d}A_T(f)[h] = \int_0^T \Big(\langle \dot{f}, \dot{h} \rangle + \langle \nabla W(f), h \rangle \Big) \mathrm{d}t; \tag{15}$$

if a cycle f with coordinate expression $\mathbf{q} = (q_1, q_2) \in \Lambda$ *is a critical point of* A_T *on* Λ, *then f is a classical periodic solution of* (7).

Proof. The first affirmation is standard (see Struwe, 1990). Let us consider $f \in \Lambda$ which is a critical point of A_T on Λ. Then f is continuous, as well as $\nabla W(f(t))$. We take the scalar product of

$$\nabla W(f(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \nabla W(f(s))\mathrm{d}s$$

with $h \in H^1, h(0) \neq 0$, then integrate on [0, T] and obtain

$$\int_0^T \langle \nabla W(f(t)), h\rangle \mathrm{d}t = -\int_0^T \left\langle \int_0^t \nabla W(f(s))\mathrm{d}s, \dot{h} \right\rangle \mathrm{d}t.$$

Because f is a critical point, from (15) we get

$$\int_0^T \left\langle \dot{f} - \int_0^t \nabla W(f(s))\mathrm{d}s, \dot{h} \right\rangle \mathrm{d}t = 0 \tag{16}$$

for each $h \in H^1, h(0) \neq 0$. It follows that

$$\dot{f} - \int_0^t \nabla W(f(s))\mathrm{d}s = \text{const a.e.} \tag{17}$$

Since $f \in \Lambda$ we obtain via Sobolev embedding that $\dot{f} \in C^0$, and from (17) that $\dot{f} \in C^1$. We integrate (16) by parts and get

$$\langle \dot{f}(T), h(T)\rangle - \langle \dot{f}(0), (0)\rangle = \int_0^T \langle \ddot{f}(t) - \nabla W(f(t)), h(t)\rangle \mathrm{d}t.$$

The right-hand side is null, and since $h(0) \neq 0$ it follows. $\dot{f}(T) = \dot{f}(0)$. □

To an element $f \in \Lambda$ we associate a curve f_U, with U given in Remark 2:

$$f_U(t) = (f(t), U(f(t))) \in \mathbb{R}^3, 0 \leqslant t \leqslant T. \tag{18}$$

The next two lemmas contain results from Gordon's (1975) paper and rely on the fact that the force is strong.

LEMMA 9. *For each* $f \in \Lambda$, *the following inequality holds*:

$$\text{arc length}(f_U) \leqslant (2A_T(f))^{1/2}\left(T^{1/2} + (A_T(f))^{1/2}\right).$$

Proof. We start from the definition of *arc length* (f_U) and make use of Cauchy's inequality to obtain

$$\begin{aligned}\text{arc length}(f_U) &= \int_0^T \left|\frac{\mathrm{d}}{\mathrm{d}t} f_U(t)\right| \mathrm{d}t \leqslant \int_0^T \left(|\dot{f}(t)| + |\nabla U(f(t))| \cdot |\dot{f}(t)|\right) \mathrm{d}t \leqslant \\ &\leqslant T^{1/2} \left(\int_0^T |\dot{f}(t)|^2 \mathrm{d}t\right)^{1/2} + \left(\int_0^T |\nabla U(f(t))|^2 \mathrm{d}t\right)^{1/2} \cdot \left(\int_0^T |\dot{f}|^2 \mathrm{d}t\right)^2 \leqslant \\ &\leqslant T^{1/2} \left(\int_0^T |\dot{f}(t)|^2 \mathrm{d}t\right)^{1/2} + \left(\int_0^T W(f(t)) \mathrm{d}t\right)^{1/2} \cdot \left(\int_0^T |\dot{f}|^2 \mathrm{d}t\right)^{1/2} \leqslant \\ &\leqslant (2A_T(f))^{1/2} (T^{1/2} + (A_T(f))^{1/2}). \qquad \square\end{aligned}$$

Lemma 9 will allow us to prove the corresponding of Gordon's geometrical lemma for the anisotropic Schwarzschild potential.

LEMMA 10. *The functional A_T has the property that, for any $a > 0$, there exists $\delta = \delta(a) > 0$ such that if $q \in H^1$ and $A_T(q) \leqslant a$, then $|q(t)| \geqslant \delta$ for all $t \in \mathbb{R}$, i.e. $\{q \in H^1 : A_T(q) \leqslant a\}$ is bounded away from zero.*

Proof. Let us suppose that there is no such δ; it means that for each $n \in \mathbb{N}$ there exist $t_n^* \in [0, T)$ and $f_n \in H^1$ for which $A_T(f_n) \leqslant a$ and $\inf\{|f_n(t)| : t \in [0, T]\} = |f_n(t_n^*)| < 1/n$. We have $f_n \in C^0$. If $\|f_n\| \to 0$ it follows that $\int_0^T W(f_n(t))dt \to \infty$ as $n \to \infty$, hence $A_T(f_n)$ is unbounded, contradiction. If $\|f_n\| \not\to 0$, there exist $\varepsilon > 0$ and a subsequence of f_n, denoted also f_n, and $t_n \in [0, T]$ such that $|f_n(t_n)| = \varepsilon, \forall_n \in \mathbb{N}$. The *arc length* of f_{nU} between t_n and t_n^* is greater than the length of the corresponding straight line, hence

$$\text{arc length } (f_{nU}) \geqslant |U(t_n, f_n(t)) - U(t_n^*, f_n(t_n^*))| \to \infty \text{ as } n \to \infty.$$

In view of Lemma 9, this contradicts the boundedness of $A_T\ (f_n)$. $\square$

Remark 11. As stated by Gordon(1975), the loops in $\Lambda \backslash \Lambda_0$ cannot be continuously moved off to infinity without either passing through (0,0) or having its *arc length* become infinite (for every c_1, there exists a compact subset K_{c_1} of $\mathbb{R}^2$ which contains every smooth cycle which is homotopic to f in $\mathbb{R}^2 \backslash \{(0,0)\}$ and has arc length $\leqslant c_1$).

The next result is a special case of Palais' (1979) principle of symmetric criticality, as it was presented by Chenciner (2002).

Let us consider an orthogonal (by isometries) representation ρ of a finite group G in the real Hilbert space H^1 such that, for any γ in G,

$$A_T(\rho(\gamma) \cdot f) = A_T(f). \tag{19}$$

We denote by H_ρ^1 the linear subspace of H^1 formed by the elements which are invariant under the representation ρ, and by $A_{T,\rho}$ the restriction of the action A_T to H_ρ^1. The principle of symmetric criticality asserts that a critical point for the restriction of the action on the subspace H^1_ρ is a critical point of A_T on the whole loop space H^1.

PROPOSITION 12. *Any critical point of $A_{T,\rho}$ is a critical point of A_T.*

Proof. Using the ρ-invariance of A_T given in (19), we have that, for any $\gamma \in G$

$$\mathrm{d}A_T(\rho(\gamma)\cdot f)[h] = \mathrm{d}A_T(f)[\rho(\gamma)^{-1}\cdot h].$$

Each $f \in H^1_\rho$ satisfies $\mathrm{d}A_T(f)[\rho(\gamma)^{-1}\cdot h] = \mathrm{d}A_T(f)[h]$. The H^1-gradient $\nabla A_T(f)$ will satisfy $\langle \rho(\gamma)\cdot \nabla A_T(f), h\rangle = \langle \nabla A_T(f), \rho(\gamma)^{-1}\cdot h\rangle = \langle \nabla A_T(f), h\rangle$. This means that $\nabla A_T(f)$ belongs to H^1_ρ, hence each critical point of $A_{T,\rho}$ is a critical point of A_T. □

Remark 13. The symmetry S_2 is given by the representation s_2 of the group $\mathbb{Z}_2$ acting as

$$(s_2\cdot f)(t) = (f_1(-t), -f_2(-t)),$$

while S_3 acts through the representation s_3

$$(s_3\cdot f)(t) = (-f_1(-t), f_2(-t)),$$

In these cases, a critical point of the action restricted to the invariant subspace with respect to the symmetry will be a critical point of A_T on H^1.

The fact that the elements for which the action is bounded are bounded away from zero prevents the critical points from being collisional solutions of the system (7). Another property of the action, namely its coercivity, avoids the critical points at infinity. A real functional A defined on a Hilbert space with the norm $\|\cdot\|$ is *coercive* if $A(x_n) \to +\infty$ for all sequences x_n such that $\|x\| \to +\infty$.

To end, we still need to recall some definitions and issues. Let X be a topological space, and consider $\Psi : X \to \mathbb{R}$. Then Ψ is *lower-semicontinuous* if and only if $\Psi^{-1}(-\infty, a]$ is closed for every $a \in \mathbb{R}$, in which case Ψ is bounded from below and reaches its infimum on every compact subset of X. If X is a Hausdorff space (thus compact subsets are necessarily closed), the following result (known as Weierstrass' theorem) holds:

PROPOSITION 14. *Consider a real-valued function $\Psi : X \to \mathbb{R}$, where X is a Hausdorff space, such that $\Psi^{-1}(-\infty, a]$ is compact for every $a \in \mathbb{R}$. Then Ψ is lower-semicontinuous bounded from below and reaches its infimum on X.*

Remark 15. In physical terms, we have shown that the symmetric solutions that lead to critical points of the action do not encounter either collision or escape, thus being periodic orbits of the problem.

Now we have at our disposal all the necessary mathematical instruments and results to prove the existence of periodic orbits in the anisotropic Schwarzschild-type problem.

5. Main Results

We state now the central result of our paper.

THEOREM 16. *For any* $T > 0$ *and any* $k \in Z\backslash\{0\}$, *there exists at least one* S_i*-symmetric* $(i = 2, 3)$ *periodic solution of the anisotropic Schwarzschild-type problem, with period T and winding number k.*

Proof. Let $i \in \{2, 3\}$ be fixed and X a component of $\tilde{\Sigma}_i$ that consists of non-simple cycles. Endow X with the weak topology it inherits from Σ_i. We intend to apply Proposition 14 with $\Psi = A_T$, thus we shall show that $Y_a = X \cap A_T^{-1}(-\infty, a]$ is bounded and weakly closed in Σ_i, hence weakly compact.

Let $f \in Y_a$, hence $f \in X$ and $0 \leqslant A_T(f) \leqslant a$. From Lemma 9 it follows that the elements of Y_a are bounded *in arc length* by the same constant, and from Lemma 10 that they are bounded away from (0, 0). The elements of X being tied to (0, 0), from Remark 11 it follows that X is bounded in C^0 norm, i.e. there exists $c > 0$ such that $\|f\| \leqslant c$ for all $f \in X$. Let $f \in Y_a$; we have

$$\|f\|_{H^1}^2 = \|f\|_{L^2}^2 + \|\dot{f}\|_{L^2}^2 \leqslant \|f\|^2 + 2A_T(f) \leqslant c^2 + 2a,$$

hence Y_a is bounded with respect to $\|\cdot\|_{H^1}$.

To show that Y_a is weakly closed, we consider a sequence $f_n \in X \cap A_T^{-1}(-\infty, a]$ which converges weakly to a cycle $f \in H^1$. The subspaces Σ_i being weakly closed (as it was proved in Lemma 5), we have that $f \in \Sigma_i$. As stated above, the cycles f_n are bounded in *arc length* and bounded away from (0, 0). The weak convergence in H^1 implies that $\lim_{n\to\infty} \|f_n - f\| = 0$ with $\|\cdot\|$ the C^0 norm. Because $f_n, n \in \mathbb{N}$, are bounded away from (0, 0), there exists $\delta > 0$ such that $|f_n(t)| \geqslant \delta$, for each $t \in [0, T]$ and $n \in \mathbb{N}$. Then

$$\delta \leqslant |f_n(t)| \leqslant |f(t)| + |f_n(t) - f(t)| \leqslant |f(t)| + \|f_n - f\|,$$

and making $n \to \infty$ it follows that $|f(t)| \geqslant \delta, t \in [0, T]$. Therefore $f \in \tilde{\Sigma}_i$ and it is in the same component of $\tilde{\Sigma}_i$ as $f_n, n \in \mathbb{N}$, hence $f \in X$.

Because of the fact that $|f_n(t)| \geqslant \delta, |f(t)| \geqslant \delta$ for $t \in [0, T]$, it follows that we may apply Fatou's lemma to obtain

$$\int_0^T W(f(t))\mathrm{d}t = \int_0^T \liminf W(f_n(t))\mathrm{d}t \leqslant \liminf \int_0^T W(f_n(t))\mathrm{d}t.$$

The H^1 norm is weakly sequentially lower-semicontinuous (see Struwe, 1996), thus

$$\|\dot{f}\|_{L^2}^2 = \|f\|_{H^1}^2 - \|f\|_{L^2}^2 \leqslant \liminf \|f_n\|_{H^1}^2 - \|f\|_{L^2}^2 = \liminf \|\dot{f}_n\|_{L^2}^2,$$

where the last equality holds because (f_n) converges strongly in L^2. We evaluate now $A_T(f) = \|\dot{f}\|_{L^2}^2/2 + \int_0^T W(f(t))\mathrm{d}t$, with W given by (3):

$$A_T(f) \leqslant \liminf \|\dot{f}_n\|_{L^2}^2 + \liminf \int_0^T W(f_n(t))\mathrm{d}t \leqslant \liminf A_T(f_n) \leqslant a,$$

hence $f \in Y_a$.

Proposition 14 implies that A_T attains its infimum on X; as a consequence of Palais' principle of symmetric criticality, and using Proposition 6, any $\bar{f}$ for which $A_T(\bar{f}) = \min\{A_T(g) : g \in X\}$ is a classical periodic solution of the system (7). □

Remark 17. Even if it does not appear explicitly in the proof of Theorem 16, the winding number k is present in X.

Remark 18. It is known that for each periodic solution of an autonomous system (as the anisotropic Schwarzschild-problem is) there exists a minimal period (see for example Amann, 1990). Let us consider a periodic solution f of period T and winding number k, and let $\tau = T/m$ be its minimal period (which leads to another value of the winding number). Applying one of the existence theorems for the period $\tau/2$ we obtain the existence of a $\tau/2$ periodic solution f_1, which is of course also periodic of period T, and surely different from f (which has the minimal period τ). Continuing this process, we obtain an infinite set of distinct T-periodic solutions (each one featured by its own winding number).

6. Conclusions

To summarize, here are some of the properties of the anisotropic Schwarzschild-type problem revealed by applying abstract mathematical results:

6.1. The vector field associated to this anisotropic problem, expressed in configuration-momentum coordinates, exhibits seven symmetries $S_i, i = \overline{1, 7}$, which, along with the identity, form a groups isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

6.2. Using variational methods, we point out the existence of S_i-symmetric $(i = 2, 3)$ periodic orbits that may have any assigned period T and winding number $k \in \mathbb{Z}\backslash\{0\}$.

6.3. Consider the anisotropic Schwarzschild-type problem to be a perturbation of the isotropic case (Stoica and Mioc, 1997) via the anisotropy parameter $\mu > 1$. Observe that anisotropy, no matter how large its size, deforms the S_i-symmetric $(i = 2, 3)$ periodic orbits of the isotropic problem (whose symmetries were pointed out by (Mioc, 2002)). but does not destroy them. This makes the symmetries $S_i(i = 2, 3)$ constitute an indicator of the robustness of the system to perturbations.

These results add new, important features to the dynamics of the anisotropic Schwarzschild-type problem.

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