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On L^p Norms and the Spectral Radius of Operators in Hilbert Spaces

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Abstract. We prove that $\lim_{p\to\infty} \|f\|_{p+1}^{p+1} / \|f\|_p^p = \|f\|_\infty$ for $f \neq 0$ in the Bochner space $L_E^\infty(\mu)$, where $(E, |\cdot|)$ is a Banach space and $(X, \mathcal{A}, \mu)$ a finite measure space. We discuss also the existence of $\lim_{n\to\infty} \left\|T^{n+1}\right\| / \|T^n\|$ for continuous linear operators T in Hilbert spaces.

Key Words: L^p norms, linear operators, spectral radius.

MSC 2000: 46E30, 47A75

1 A limit involving L^p and L^∞ norms

Let $(X, \mathcal{A}, \mu)$ be a measure space. If μ is finite and $f \in L^\infty(\mu)$, the L^∞ norm of the real function f can be obtained as the limit

$$\|f\|_\infty = \lim_{p\to\infty} \|f\|_p . \tag{1}$$

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This result can be found in [9], p. 34.

It is known that for a sequence of real numbers $a_p > 0$, the equality

$$\lim_{p\to\infty} (a_p)^{1/p} = \lim_{p\to\infty} \frac{a_{p+1}}{a_p}$$

holds, provided that the second limit exists (Stolz-Cesàro) [1, p. 150].

The problem we are going to solve is:

For $a_p = \|f\|_p^p$, does the limit $\lim_{p\to\infty} \dfrac{a_{p+1}}{a_p}$ exist?

Remark 1.1 *There are known several conditions on the sequences* a_n, b_n *insuring that* $\dfrac{a_n}{b_n} \to L \implies \dfrac{a_{n+1} - a_n}{b_{n+1} - b_n} \to L$. *They apply for example for Traian Lalescu's sequence [5]:* $\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!} \to 1/e$. *Similar sequences were studied by T. Popoviciu [7] and recently by many other mathematicians.*

As a special case, for $a_n := \ln a_n$ *and* $b_n := n$, *it follows that* $\sqrt[n]{a_n} \to L \implies \frac{a_{n+1}}{a_n} \to L$.

Unfortunately, these conditions do not apply for the problem to be studied. We prove directly the following result (in Bochner spaces).

Theorem 1.1 *Let* $(E, |\cdot|)$ *be a Banach space,* $(X, \mathcal{A}, \mu)$ *a finite measure space* ($\mu(X) < \infty$) *and* $f \in L_E^\infty(\mu)\backslash\{0\}$. *Then*

$$\lim_{p\to\infty} \frac{\int |f|^{p+1} \,\mathrm{d}\mu}{\int |f|^p \,\mathrm{d}\mu} = \|f\|_\infty .$$

Proof. Replacing f by $|f| / \|f\|_\infty$, one may suppose that $0 \le f \le 1$

and $\|f\|_\infty = 1$. Let us denote

$$r_p = \frac{\int |f|^{p+1} \,\mathrm{d}\mu}{\int |f|^p \,\mathrm{d}\mu}.$$

Then $0 \le r_p \le 1$, so,

$$\limsup_{p\to\infty} r_p \leq 1. \tag{2}$$

For $0 < a < 1$ we denote $A_a = \{x \in X : f(x) \geq a\}$, $B_a = X \backslash A$. We have $\mu(A_a) > 0$ because $\|f\|_\infty = 1$. We show that

$$\lim_{p\to\infty} \frac{\int_{B_a} f^p \,\mathrm{d}\mu}{\int_{A_a} f^p \,\mathrm{d}\mu} = 0.$$

Let us choose b so that $a < b < 1$. Then $A_b \subseteq A_a$ and

$$0 \leq \frac{\int_{B_a} f^p \,\mathrm{d}\mu}{\int_{A_a} f^p \,\mathrm{d}\mu} \leq \frac{a^p \mu(B_a)}{\int_{A_b} f^p \,\mathrm{d}\mu} \leq \frac{a^p \mu(B_a)}{b^p \mu(A_b)} = \left(\frac{a}{b}\right)^p \frac{\mu(B_a)}{\mu(A_b)} \to 0 \ (p \to \infty).$$

We obtain for $\liminf_{p\to\infty} r_p$ the following estimation

$$\liminf_{p\to\infty} r_p = \liminf_{p\to\infty} \frac{\int_{A_a} f^{p+1} \,\mathrm{d}\mu + \int_{B_a} f^{p+1} \,\mathrm{d}\mu}{\int_{A_a} f^p \,\mathrm{d}\mu + \int_{B_a} f^p \,\mathrm{d}\mu} = \liminf_{p\to\infty} \frac{\int_{A_a} f^{p+1} \,\mathrm{d}\mu}{\int_{A_a} f^p \,\mathrm{d}\mu} \cdot \frac{1 + \frac{\int_{B_a} f^{p+1} \,\mathrm{d}\mu}{\int_{A_a} f^{p+1} \mathrm{d}\mu}}{1 + \frac{\int_{B_a} f^p \,\mathrm{d}\mu}{\int_{A_a} f^p \,\mathrm{d}\mu}} = \liminf_{p\to\infty} \frac{\int_{A_a} f^{p+1} \,\mathrm{d}\mu}{\int_{A_a} f^p \,\mathrm{d}\mu} \cdot 1 \geq \liminf_{p\to\infty} \frac{\int_{A_a} f^p a \,\mathrm{d}\mu}{\int_{A_a} f^p \,\mathrm{d}\mu} = a.$$

But $a \in (0, 1)$ is arbitrary, so

$$\liminf_{p\to\infty} r_p \geq 1. \tag{3}$$

From (2) and (3) it follows $\lim_{p\to\infty} r_p = 1$. ■

Equality (1) can be obtained as a consequence of Theorem 1.1, using the Stolz-Cesàro result.

2 On a limit concerning operators with spectral radius $r(T) \neq 0$

Let E be a Hilbert space and T a linear continuous operator. Then the spectral radius $r(T)$ of the operator T is given by

$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}.$$

The result in section 1 suggest the following problem: If $r(T) \neq 0$, is it true that $\lim_{n\to\infty} \left\|T^{n+1}\right\| / \|T^n\|$ does exist?

We mention the following interesting related result due to Kellogg [3], [8, p. 240], which provides an algorithm for finding an eigenvalue for a compact self-adjoint operator.

Theorem 2.1 *Let E be a Hilbert space, T a compact self-adjoint operator, $x_0 \in E$ such that $Tx_0 \neq 0$. Then, for $x_n = T^n x_0$, one has that $x_n \neq 0$, the sequence $\|x_{n+1}\| / \|x_n\|$ is increasing and convergent to $r > 0$ such that either r or $-r$ is an eigenvalue for T.*

In [2, p. 222], the definition of operators of class $\mathcal{K}$ was given.

Definition 2.1 *The linear continuous operator T is of class $\mathcal{K}$ if for each $x \in E$, $m \in \mathbb{N}$, $m \geq 2$ and $k \in \{1, 2, ..., m-1\}$*

$$\left\|T^k x\right\| \leq C_{m,k} \|x\|^{1-\frac{k}{m}} \|T^m x\|^{\frac{k}{m}}, \tag{4}$$

where $C_{m,k}$ are constants.

Remark 2.1 *1. If T is invertible, the minimal constants $C_{m,k}$ in (4) must satisfy (see [2, p. 223])*

$$C_{m,k} \leq \left\|T^k\right\|^{1-\frac{k}{m}} \left\|T^{-(m-k)}\right\|^{\frac{k}{m}}; \tag{5}$$

if T is normal, $C_{m,k} = 1$ for each m and k.

2. For T_i, $i = 1, ..., 4$ linear continuous operators on E, the following two inequalities regarding the spectral radius

$$r\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right) \le r\left(\begin{bmatrix} \|T_1\| & \|T_2\| \\ \|T_3\| & \|T_4\| \end{bmatrix}\right),$$

$$\begin{aligned} &r(T_1T_2 + T_3T_4) \le \tfrac{1}{2}\left(\|T_2T_1\| + \|T_4T_3\|\right) \\ &+\sqrt{\left(\|T_2T_1\| - \|T_4T_3\|\right)^2 + 4\,\|T_2T_3\| \cdot \|T_4T_1\|} \end{aligned}$$

have been proved in [4].

We state the following

Conjecture 2.1 *If T is of class $\mathcal{K}$ and $r(T) \neq 0$, then* $\lim_{n\to\infty} \left\|T^{n+1}\right\| / \|T^n\|$ *do exist.*

We prove the next result mentioned in [2, p. 216].

Proposition 2.1 $r(T) = \|T\| \Leftrightarrow \|T^n\| = \|T\|^n$, *for all $n \in \mathbb{N}$.*

Proof. The spectral mapping theorem implies that $r(T^n) = r(T)^n$, so if $r(T) = \|T\|$ then $\|T\|^n = r(T)^n = r(T^n) \le \|T^n\| \le \|T\|^n$, hence $\|T^n\| = \|T\|^n$.

Conversely, if $\|T^n\| = \|T\|^n$, for all $n \in \mathbb{N}$ then $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n} = \lim_{n\to\infty} \|T\|^{n\cdot 1/n} = \|T\|$. ■

If $r(T) = \|T\|$, obviously $\left\|T^{n+1}\right\| / \|T^n\| = \|T\|$ and conjecture 2.1 holds. Note also that if T is normal, then $r(T) = \|T\|$; but T may not be normal and yet $\lim_{n\to\infty} \left\|T^{n+1}\right\| / \|T^n\|$ exists $(= r(T))$; see ex 2.3.

Remark 2.2 *Let T be the Volterra operator in $L^2([0,1])$,*

$$(Tx)(t) = \int_0^t x\mathrm{d}\lambda.$$

Then $r(T) = 0$, $\|T\| = 2/\pi = 0.6366197722...$, $\left\|T^2\right\| = 1/\alpha^2 = .2844128717...$ *where α is the smallest positive root of the equation* $(e^a + e^{-a})\cos(a) = -2$, *see [6, p. 259]. The norms $\|T^n\|$ are more difficult to find for $n > 2$.*

The next example shows that conjecture 2.1 does not hold for all linear continuous operators.

Example 2.1 *An operator with $r(T) = 1$ for which $\lim_{n\to\infty} \left\|T^{n+1}\right\| / \left\|T^n\right\|$ does not exist.*

Let $T = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$.

Then $r(T) = 1, \left\|T^n\right\| = \begin{cases} 1, & \text{for } n \text{ even} \\ \left(\sqrt{5}+1\right)/2, & \text{for } n \text{ odd} \end{cases}$. *In this case,* $\left\|T^n\right\|^{1/n} \to r(T)$ *but* $\left\|T^{n+1}\right\| / \left\|T^n\right\|$ *diverges.*

Actually, this behaviour is almost generic. We give below the Maple code computing $r(T)$ and the sequence $\left\|T^{n+1}\right\| / \left\|T^n\right\|$ for a linear operator in $\mathbb{R}^d$ $(d = 2)$ with randomly selected entries from $\{-5, -4, ..., 4, 5\}$. Note that for $d > 4$ this can be done only approximately.

We display the values of the sequences $\left\|T^{n+1}\right\| / \left\|T^n\right\|$ and $\left\|T^n\right\|^{1/n}$.

Example 2.2 *The operator* $T = \begin{bmatrix} 4 & 5 \\ -4 & 4 \end{bmatrix}$ *has $r(T) = 2$ and* $\left\|T^{n+1}\right\| / \left\|T^n\right\|$ *diverges.*

```
> T:=randmatrix(2,2,entries=rand(-5..5));
```

$$T := \begin{bmatrix} 4 & 5 \\ -4 & 4 \end{bmatrix}$$

```
> Digits:=15:
> interface(displayprecision=3):
> m:=30:
> max(op(map(abs,[eigenvalues(T)]))); #r(T)
```

$$2$$

```
> nT:=norm(T,2);
```

$$nT := 9/2 + 1/2 * 65^{1/2}$$

```
> S:=evalf(evalm(T/nT));
  p:=evalf(seq( norm(S&^n,2),n=1..m)):
```

$$S := \begin{bmatrix} 0.4689 & 0.5861 \\ -0.4689 & -0.4689 \end{bmatrix}$$

```
> evalf([seq(nT*p[n+1]/p[n],n=1..m-1)]);
```

[.4689, 8.531, .4689, 8.531, .4689, 8.531, .4689, 8.531,
.4689, 8.531, .4689, 8.531, .4689, 8.531, .4689, 8.531,
.4689, 8.531, .4689, 8.531, .4689, 8.531, .4689, 8.531,
.4689, 8.531, .4689, 8.531, .4689]

```
> evalf([seq(nT*p[n]^(1/n),n=1..m)]);
```

[8.531, 2.000, 3.244, 2.000, 2.673, 2.000, 2.461, 2.000,
2.350, 2.000, 2.282, 2.000, 2.236, 2.000, 2.203, 2.000,
2.178, 2.000, 2.159, 2.000, 2.143, 2.000, 2.130, 2.000,
2.119, 2.000, 2.110, 2.000, 2.103, 2.000]

Example 2.3 *However, for* $T := \begin{bmatrix} -2 & 2 \\ 5 & -3 \end{bmatrix}$ *one obtains:* $r(T) = \frac{5+\sqrt{41}}{2} \simeq 5.702$ *and the sequence* $\left\|T^{n+1}\right\| / \|T^n\|$ *converges (to* $r(A)$*). Note that the numerical results show that this sequence coverges faster than* $\|T^n\|^{1/n}$.

```
> evalf([seq(nT*p[n+1]/p[n],n=1..m-1)]);
```

[5.550, 5.719, 5.699, 5.702, 5.702, 5.702, 5.702, 5.702,
5.702, 5.702, 5.702, 5.702, 5.702, 5.702, 5.702, 5.702,
5.702, 5.702, 5.702, 5.702, 5.702, 5.702, 5.702, 5.702]

```
> evalf([seq(nT*p[n]^(1/n),n=1..m)]);
```

[6.451, 5.983, 5.894, 5.845, 5.816, 5.797, 5.783, 5.773,
5.765, 5.758, 5.753, 5.749, 5.745, 5.742, 5.739, 5.737,
5.735, 5.733, 5.731, 5.730, 5.729, 5.727, 5.726, 5.725,
5.724]

References

[1] G. M. Fichtenholz. *Differential and Integral Calculus*, v. I (Romanian). Ed. Tehnică, Bucureşti, 1964.

[2] I. M. Glazman and Iu. I. Liubici. *Linear analysis in finite dimensional spaces* (Romanian). Ed. Ştiinţifică şi Enciclopedică, Bucureşti, 1980.

[3] O.D. Kellogg. On the existence of closure of sets of characteristic functions, *Math. Ann.,* **86** (1922), 14-17.

[4] F. Kittaneh. Spectral radius inequalities for Hilbert space operators, *Proc. AMS,* **34** (2) (2005), 385-390.

[5] T. Lalescu. Problem 579 (Romanian), *Gazeta Matematică* (Bucharest), **6** (1900), 148.

[6] I. Muntean. *Special topics in Functional Analysis* (Romanian), Babeş-Bolyai University, Cluj, 1990.

[7] T. Popoviciu. On the computation of some limits (Romanian), *Gazeta Matematică* (Bucharest) Ser. A, **76** (1) (1971), 8-11.

[8] F. Riesz and B. Sz-Nagy. *Functional Analysis*, Dover, New York, 1990.

[9] K. Yoshida. *Functional Analysis.* Springer Verlag, Berlin-Göttingen-Heidelberg, 1965.