FAMILIES OF ORBITS IN CONSERVATIVE FIELDS OF HÉNON-HEILES TYPE

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Abstract. The equations of the inverse problem of dynamics are used in order to obtain planar and spatial potentials of Hénon-Heiles type which give rise to some special families of curves. The curves of such a family can be traced by a material point of unit mass, with suitable initial conditions, moving under the action of the specific potential. We determine the region where the motion is possible, as well as the total energy of the particle.

Key words: celestial mechanics – inverse problem of dynamics – families of orbits

1. INTRODUCTION

The planar Hénon-Heiles potential (1964) $V(x,y)=(x^2+y^2)/2-y^3/3+x^2y$ represents a model of the gravitational attraction of a galaxy. The considered potential was constructed by adding to the potential of a planar oscillator two terms of third degree in the coordinates. The Hénon-Heiles potential emerges from other problems too (Boccaletti and Pucacco, 1996, p. 123): consider a system representing the motion of three particles on a circle acted upon by exponentially decreasing forces. Apply some canonical transformations and obtain a Hamiltonian system. By expanding the Hamiltonian in series of the generalized coordinates up to the third-degree terms, one obtains a non-integrable Hénon-Heiles Hamiltonian.

The form of a planar potential of Hénon-Heiles type is

$$V(x,y)=\frac{1}{2}(x^2+\omega^2y^2)+ay^3+bx^2y, \tag{1}$$

with $a,b,\omega\in R,\ \omega\neq 0$ so that at least one of a, b is different from zero.

More recently, spatial potentials of Hénon-Heiles type have been considered. They consist also of cubic perturbations of harmonic oscillators and describe the motion in the central region of a galaxy (Ferrer et al, 2002), as well as the motion of the nuclei in small molecules (Gutzwiller, 1990).

Their form is

$$U(x,y)=\frac{1}{2}(x^2+\omega_1^2y^2+\omega_2^2z^2)+az^3+(bx^2+cy^2)z, \tag{2}$$

with $a,b,c,\omega_1,\omega_2\in R,\ \omega_1,\omega_2\neq 0$ so that at least one of a, b, c is different from zero.

Using the tools of the inverse problem of dynamics, we shall find potentials of type (1), respectively (2), which generate some given families of curves described by particles of unit mass under the influence of those potentials.

2. THE PLANAR PROBLEM

We consider the following version of the inverse problem for one material point of unit mass, moving in the xy inertial Cartesian plane. Given a family of curves

$$f(x,y)=c \tag{3}$$

with f of C^3-class (continuous and with continuous derivatives up to third order on a domain of the plane) and such that $f_x^2 + f_y^2 \neq 0$, find the potentials V of C^2-class under whose action, for appropriate initial conditions, the particle will describe the curves of that family. The equations of motion are

$$\ddot{x} = -V_x \qquad \ddot{y} = -V_y,$$

where the dots denote derivatives with respect to the time t, and the subscripts partial derivatives. By making use of the energy integral, Szebehely (1974) proved that the potential V is a solution of a first order partial differential equation, written by Bozis (1983) as

$$V_x + \gamma V_y + \frac{2\Gamma(E(f) - V)}{1 + \gamma^2} = 0, \qquad (4)$$

where $E(f)$ denotes the total energy, which is constant on each curve of the family (3). The functions

$$\gamma = \frac{f_y}{f_x} \ \ and \ \ \Gamma = \gamma\gamma_x - \gamma_y$$

are related to the geometry of the family (γ representing the slope and Γ being proportional to the curvature).

If the family (3) consists of straight lines ($\Gamma = 0$), the potential must satisfy the equation (Bozis and Anisiu, 2001)

$$V_x V_y (V_{xx} - V_{yy}) = V_{xy}(V_x^2 - V_y^2). \qquad (5)$$

The straight lines are traced with arbitrary energy.

If $\Gamma \neq 0$, by eliminating the energy from (4) (using the fact that $E_y / E_x = f_y / f_x$) Bozis (1984) obtained the energy-free equation of second order

$$-V_{xx} + \kappa V_{xy} + V_{yy} = \lambda V_x + \mu V_y, \qquad (6)$$

where

$$\kappa = \frac{1}{\gamma} - \gamma, \ \ \lambda = \frac{\Gamma_y - \gamma\Gamma_x}{\gamma\Gamma}, \ \ \mu = \lambda\gamma + \frac{3\Gamma}{\gamma}. \qquad (7)$$

Bozis and Ichtiaroglou (1994) have shown that only the curves of the family (3) or parts of them which are situated in the plane region

$$\frac{V_x + \gamma V_y}{\Gamma} \leq 0 \qquad (8)$$

can be described by the unit mass particle. The basic equations (4) and (6) of the planar inverse problem of dynamics present the connection between geometry and dynamics. Their derivation and other related results are exposed by Bozis (1995) and by Anisiu (2003, 2004a).

Szebehely obtained the first order equation (4) intending to determine the potential of the earth by means of satellite observations, while Bozis used equation (6) to check if a given family of orbits may be generated in the plane of symmetry outside a material concentration.

From the class of Hénon-Heiles type potentials (1), we look for those compatible with a family of polytropic curves

$$f(x, y) = x^{-p} y, \ \ p \in Z \setminus \{0\}. \qquad (9)$$

The problem was considered by Anisiu (1998, p. 128) and Anisiu and Pal (1999). For the family (9) we obtain

$$\gamma = -\frac{x}{py} \ \ and \ \ \Gamma = \frac{(1-p)x}{p^2 y}.$$

For p=1, the family (9) represents a family of straight lines. In this case the potential (1) must satisfy equation (5). Replacing (1) in equation (5), we find that no potential of this type can give rise to a family of straight lines.

For families (9) with p different from 0 and from 1 we use equation (6) to obtain the potential, then inequality (8) to obtain the admissible region, and equation (4) for the energy. We find that the potential

$$V_1(x,y) = \frac{1}{2}(x^2 + 16y^2) + b(x^2 + \frac{16}{3}y^2)y$$

generates the family $f_1(x,y) = x^{-4}y$ in the region described by $1 + (x^2 + 8y^2)b/(12y) \le 0$, with the energy $E_1(f_1) = -b/(24 f_1)$. Another potential, namely

$$V_2(x,y) = \frac{1}{2}(x^2 + 4y^2) + ay^3,$$

gives rise to the family $f_2(x,y) = x^2 y$ in the region $ay + 2 \le 0$, with the energy $E_2(f_2) = -af_2/4$. No other values of p make the family (9) compatible with a potential of the type (1).

3. THE SPATIAL PROBLEM

We consider now the three-dimensional family of curves

$$\varphi(x,y,z) = c_1, \ \ \psi(x,y,z) = c_2, \tag{10}$$

with φ, ψ of C^3-class and with

$$\begin{vmatrix} \varphi_y & \varphi_z \\ \psi_y & \psi_z \end{vmatrix} \ne 0. \tag{11}$$

We can suppose that any other determinant (containing derivatives with respect to x and y, or to x and z) is different from zero, and proceed accordingly. We deal with the following version of the inverse problem: find the potentials U of C^2-class under whose action, for appropriate initial conditions, a material point of unit mass, whose motion is described by

$$\ddot{x} = -U_x \quad \ddot{y} = -U_y \quad \ddot{z} = -U_z,$$

will trace the curves of the family (10).

We shall use the notation (Bozis and Kotoulas, 2004a, Anisiu, 2004b)

$$\begin{gathered} \alpha = \frac{\varphi_z\psi_x - \varphi_x\psi_z}{\varphi_y\psi_z - \varphi_z\psi_y}, \ \ \beta = \frac{\varphi_x\psi_y - \varphi_y\psi_x}{\varphi_y\psi_z - \varphi_z\psi_y}, \\ A = \alpha_x + \alpha\alpha_y + \beta\alpha_z, \ \ B = \beta_x + \alpha\beta_y + \beta\beta_z. \end{gathered} \tag{12}$$

We have to deal with the special case when A=B=0. In this case, analized in detail by Bozis ans Kotoulas (2004b), the family (10) consists of straight lines. The potential satisfies the partial differential equations

$$\begin{gathered} U_{xy}(U_x^2 - U_y^2) - U_xU_y(U_{xx} - U_{yy}) + U_z(U_xU_{yz} - U_yU_{xz}) = 0 \\ U_{xz}(U_x^2 - U_z^2) - U_xU_z(U_{xx} - U_{zz}) + U_y(U_xU_{yz} - U_zU_{xy}) = 0. \end{gathered} \tag{13}$$

Let us consider now $A \neq 0$ and $B \neq 0$. The corresponding of Szebehely's equation (4), derived by Váradi and Érdi (1983), is

$$\alpha U_x - U_y - \frac{2A(E(\varphi,\psi) - U)}{1 + \alpha^2 + \beta^2} = 0. \tag{14}$$

The potential U satisfies also a first order equation

$$\frac{\alpha U_x - U_y}{A} = \frac{\beta U_x - U_z}{B}, \tag{15}$$

where α, β, A and B from (12) depend on the derivatives of φ and ψ up to the second order, and a second order one (Bozis and Kotoulas, 2004a, Anisiu, 2004b)

$$-U_{xx} + kU_{xy} + U_{yy} + pU_{yz} + qU_{xz} = lU_x + mU_y, \tag{16}$$

where

$$k = \frac{1}{\alpha} - \alpha,\ p = \frac{\beta}{\alpha},\ q = -\beta,\ l = \frac{3A}{\alpha} - \alpha m,\ m = \frac{A_x + \alpha A_y + \beta A_z}{\alpha A}. \tag{17}$$

The motion is possible only in the region determined by Shorokov (1988)

$$\frac{\alpha U_x - U_y}{A} \geq 0. \tag{18}$$

If only one of A and B is different from zero, say $B \neq 0$, equation (15) is replaced by

$$\alpha U_x - U_y = 0. \tag{19}$$

In this case, the second order equation is (Anisiu, 2005)

$$-U_{xx} + k^* U_{xy} + U_{yy} + p^* U_{yz} + q^* U_{xz} = l^* U_x + m^* U_y, \tag{20}$$

where

$$k^* = \frac{1}{\beta} - \beta,\ p^* = \frac{\alpha}{\beta},\ q^* = -\alpha,\ l^* = \frac{3B}{\beta} - \beta m^*,\ m^* = \frac{B_x + \alpha B_y + \beta B_z}{\beta B}. \tag{21}$$

The inequality which describes the region where real motion takes place is

$$\frac{\beta U_x - U_z}{B} \geq 0. \tag{22}$$

Using the tools of the spatial inverse problem summarized above we seek potentials (2) which give rise to a family of curves

$$\varphi(x,y,z) = x^{-r} z,\ \ \psi(x,y,z) = y^{-s} z,\quad r,s \in Z \setminus \{0\}. \tag{23}$$

For this family we have

$$\alpha = \frac{ry}{sx},\ \beta = \frac{rz}{x},\ A = \frac{r(r-s)y}{s^2 x^2},\ B = \frac{r(r-1)z}{x^2}.$$

For r=s=1 we are in the presence of a family of straight lines. Substituting U from (2) in equations (13) we find out that no such a potential can produce the mentioned family of straight lines.

For $r = s \neq 1$ we have $A = 0,\ B \neq 0$. The equations (19) and (20) are satisfied by two cubic potentials presented by Anisiu (2005). Here we find also the region where the particle can describe the curves of the corresponding family, and the total energy of the particle. We obtain the potential

$$U_1(x,y,z) = \frac{1}{2}(x^2 + y^2 + 16z^2) + \frac{16b}{3} z^3 + b(x^2 + y^2) z,$$

which gives rise to the family $\varphi_1(x,y,z)=x^{-4}z,\ \psi_1(x,y,z)=y^{-4}z$ in the region $1+(x^2+y^2+8z^2)b/(12z)\le 0$, with the energy $E_1(\varphi_1,\psi_1)=(\sqrt{-b/\varphi_1}+\sqrt{-b/\psi_1})^2/24.$ For $r=s=-2,$ the potential

$$U_2(x,y,z)=\frac{1}{2}(x^2+y^2+4z^2)+az^3$$

generates the family $\varphi_2(x,y,z)=x^2z,\ \psi_2(x,y,z)=y^2z$ in the region $az+2\le 0$, with the energy $E_2(\varphi_2,\psi_2)=-a(\varphi_2+\psi_2)/4.$ For all the other values of r and s in (23), the corresponding family of curves cannot be generated by cubic potentials (2).

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