PADEU | PADEU **19**(2007),17-27
ISBN 963 463 557

New solutions in the direct problem of dynamics

C. Blaga[1], M.-C. Anisiu[2],G. Bozis[3]

[1]Faculty of Mathematics and Computer Science,
"Babeş-Bolyai" University, Cluj-Napoca, Romania
[2]T. Popoviciu Institute of Numerical Analysis, Romanian Academy
P.O. Box 68, 400110 Cluj-Napoca, Romania
[3]Department of Physics, Aristotle University of Thessaloniki GR-54006, Greece
E-mail: [1]cpblaga@math.ubbcluj.ro, [2]mira@math.ubbcluj.ro, [3]gbozis@auth.gr

Abstract
Given a planar potential V, we look for families of orbits $f(x,y)=c$ (determined by their slope function $\gamma = f_y/f_x$), traced by a material point of unit mass under the action of that potential. The second-order equation which relates γ and V is nonlinear in γ; to find special solutions, we consider in addition a linear first-order partial differential equation satisfied by γ. The problem does not admit always solutions; but when solutions do exist, they can be found by algebraic manipulations. Examples are given for homogeneous families γ, and for some special cases which arise in the course of reasoning.

1 Introduction

The planar direct problem of Dynamics consists in finding families of orbits $f(x,y)=c$ traced in the xy Cartesian plane by a material point of unit mass, under the action of a given potential V.

Any family of orbits is determined by its 'slope function' $\gamma = f_y/f_x$, the subscripts denoting partial derivatives. There are two equations relating the functions V, γ (and their derivatives):

(i) the first order equation in V, given by Szebehely (1974) (equation (8) below), which is associated with the energy dependence on the family f;

(ii) the energy-free second order linear equation in V, given by Bozis (1984) and written below in the form (6)-(7).
These equations, born in the framework of the inverse problem, are rearranged here in order to face the direct problem, as suggested by Bozis (1995). The difficulty with the second order equation arises from its nonlinearity in the unknown family γ. This is why in several papers additional information on the families of orbits (sometimes on the given potentials also) was used in order to obtain solutions of the direct problem. Homogeneous families produced by homogeneous or inhomogeneous potentials were studied by Bozis and Grigoriadou (1993) and by Bozis et al (1997), as well as families of orbits with $\gamma = \gamma(x)$, corresponding to families $f(x, y) = y + h(x) = c$ (Bozis et al, 2000). Later on (Anisiu et al, 2004), the solutions of equation (6) were looked for in a class of functions verifying a linear PDE

$$r(x, y)\gamma_x + \gamma_y = 0; \tag{1}$$

this class contains the homogeneous functions f, for which γ is homogeneous of zero degree and $r = x/y$. In all these cases γ was found as the common root of certain algebraic equations in γ, with coefficients depending on V and on derivatives of V.

In what follows we consider a given potential V and study the existence and the construction of solutions γ of the direct problem of dynamics, under the hypothesis that γ satisfies an equation of the form

$$a(x, y, \gamma)\gamma_x + b(x, y, \gamma)\gamma_y = c(x, y, \gamma). \tag{2}$$

We may suppose $b \neq 0$ and denote by $r = a/b$ and $s = c/b$.

In the following we replace (2) by the equation

$$r(x, y, \gamma)\gamma_x + \gamma_y = s(x, y, \gamma) \tag{3}$$

with r and s known functions of x, y, γ. We then develop the reasoning to check whether the given potential can be compatible with families $\gamma = \gamma(x, y)$ satisfying the condition (3).

In section 2 we give the basic partial differential equations of the direct problem and add to them two (second order) differential relations derived from (3). Then, in section 3 we obtain the algebraic equations verified by γ_x. In section 4 we obtain three algebraic equations which the required family must satisfy when γ is a homogeneous function of degree m. The resultants of the two pairs of equations must vanish and this leads to two differential conditions which all adequate potentials must satisfy. In section 5 we present some special cases and examples. A synthesis is presented in section 6.

2 Partial differential equations satisfied by γ

We consider a planar potential V under the action of which a monoparametric family of orbits

$$f(x,y) = c \tag{4}$$

can be described by a material point of unit mass. This family can be represented in a unique way by its *slope function*

$$\gamma = \frac{f_y}{f_x}. \tag{5}$$

To each γ there corresponds a unique family (4).

The nonlinear second order differential equation relating potentials and orbits in the form suitable for the direct problem (Bozis, 1995) is

$$\gamma^2\gamma_{xx} - 2\gamma\gamma_{xy} + \gamma_{yy} = h, \tag{6}$$

where

$$h = \frac{\gamma\gamma_x - \gamma_y}{V_y\gamma + V_x}\left(-\gamma_x V_x + (2\gamma\gamma_x - 3\gamma_y)V_y + \gamma(V_{xx} - V_{yy}) + (\gamma^2 - 1)V_{xy}\right). \tag{7}$$

Szebehely's equation (1974) involving the total energy $E(f)$ is (Bozis, 1983)

$$V_x + \gamma V_y + \frac{2\Gamma}{1+\gamma^2}(E(f) - V) = 0, \tag{8}$$

where

$$\Gamma = \gamma\gamma_x - \gamma_y. \tag{9}$$

In order to solve (8) for $E(f)$, the condition $\Gamma \neq 0$ must be imposed, hence it follows also that $V_x + \gamma V_y \neq 0$. The case $\Gamma = 0$ was studied in detail by Bozis and Anisiu (2001) and will be considered in section 5. If for a given V we can find a solution γ of (6), equation (8) will allow us to find the energy along each member of the family, namely

$$E(f) = V - \frac{(V_x + \gamma V_y)\left(1+\gamma^2\right)}{2\Gamma}. \tag{10}$$

The real parts of the orbits of the family are lying in the region defined by the inequality (Bozis and Ichtiaroglou, 1994)

$$\frac{V_x + \gamma V_y}{\Gamma} \leq 0. \tag{11}$$

As we have mentioned in the Introduction, the special families of orbits we are going to consider are those for which equation (3) is also satisfied. We differentiate it with respect to x and obtain

$$r\gamma_{xx} + \gamma_{xy} = -r_{001}{\gamma_x}^2 - r_{100}\gamma_x + s_{001}\gamma_x + s_{100}. \tag{12}$$

Then we differentiate (3) with respect to y

$$r\gamma_{xy} + \gamma_{yy} = -r_{001}\gamma_x\gamma_y - r_{010}\gamma_x + s_{001}\gamma_y + s_{010}. \tag{13}$$

For the functions r and s, which depend on the three variables x, y, γ, we adopt the three-subscripts notation, e. g. $\partial^{i+j+k}s/\partial x^i\partial y^j\partial\gamma^k = s_{ijk}$. The system of equations (6), (12) and (13) allows us to obtain the second order derivatives of γ in terms of γ and its first order derivatives.

3 Algebraic equations satisfied by γ_x

We solve the system of equations (6), (12) and (13) with respect to γ_{xx}, γ_{xy} and γ_{yy}. These second order derivatives depend on γ, γ_x, γ_y, on r, s, and their first-order derivatives and, of course, on the first and second order derivatives of V. In fact, considering (3), we can express γ_y in terms of γ_x. We introduce the notations

$$\begin{aligned}
\Pi &= (\gamma + r)^2 \left(V_y\gamma + V_x\right)\\
K &= -2(r_{001} - 1)V_y\gamma^2 + \left[(5r - 2rr_{001})V_y - (2r_{001} + 1)\,V_x\right]\gamma +\\
&\quad + r\left[3rV_y - (1 + 2r_{001})\,V_x\right]\\
L &= V_{xy}\gamma^3 + \left[V_{xx} - V_{yy} + rV_{xy} - 2\left(r_{100} - s_{001}\right)V_y\right]\gamma^2\\
&\quad + \left[r\left(V_{xx} - V_{yy}\right) - V_{xy} + 2\left(s_{001} - r_{100}\right)V_x +\right.\\
&\quad \left. + \left(-rr_{100} + r_{010} + sr_{001} - 5s + 2rs_{001}\right)V_y\right]\gamma\\
&\quad - rV_{xy} + \left(-rr_{100} + r_{010} + sr_{001} + s + 2rs_{001}\right)V_x - 6rsV_y,\\
M &= (-sV_{xy} + 2s_{100}V_y)\gamma^2 + \left[s\left(V_{yy} - V_{xx}\right) + 2s_{100}V_x +\right.\\
&\quad \left. + \left(rs_{100} - s_{010} - ss_{001}\right)V_y\right]\gamma\\
&\quad + sV_{xy} + \left(rs_{100} - s_{010} - ss_{001}\right)V_x + 3s^2V_y.
\end{aligned} \tag{14}$$

Then, the second-order derivatives of γ can be expressed as

$$\gamma_{xx} = \frac{1}{\Pi}\left(K\gamma_x^2 + L\gamma_x + M\right)$$

$$\gamma_{xy} = -\frac{1}{\Pi}\left\{(rK + r_{001}\Pi)\,\gamma_x^2 + [rL + (r_{100} - s_{001})\,\Pi]\,\gamma_x + rM - s_{100}\Pi\right\} \quad (15)$$

$$\gamma_{yy} = \frac{1}{\Pi}\left\{\left(r^2K + 2rr_{001}\Pi\right)\gamma_x^2 + \left[r^2L + (rr_{100} - r_{010} - sr_{001} - 2rs_{001})\,\Pi\right]\gamma_x + r^2M - (rs_{100} + ss_{001} + s_{010})\,\Pi\right\}.$$

Remark 1 As we have already mentioned, we have $V_y\gamma + V_x \neq 0$. The case $\gamma+r = 0$ will be studied later. For the moment we suppose that the denominator Π in (15) is different from zero.

Working with (15) we find that the two compatibility conditions $(\gamma_{xx})_y = (\gamma_{xy})_x$ and $(\gamma_{xy})_y = (\gamma_{yy})_x$ produce *one single relation* which, after substituting γ_{xx}, γ_{xy} and γ_{yy} given by (15) and γ_y from (3), reduces to a third-degree algebraic equation in γ_x

$$\Gamma_3\gamma_x^3 + \Gamma_2\gamma_x^2 + \Gamma_1\gamma_x + \Gamma_0 = 0. \quad (16)$$

The coefficient Γ_3 of γ_x^3 in (16) is given by

$$\Gamma_3 = (\gamma + r)^2\,(V_y\gamma + V_x)\left[(V_y\gamma + V_x)\,(\gamma + r)\,r_{002} - 2\,(V_y\gamma + V_x)\,r_{001}^2 + (2V_y\gamma + 3rV_y - V_x)\,r_{001}\right]. \quad (17)$$

In the last factor of Γ_3, all the terms contain the derivatives of r with respect to its third variable γ. It follows that, if r depends merely on x and y, equation (16) is in fact at most of second degree in γ_x. There are significant situations when this condition is fulfilled, as in the case of functions γ homogeneous of order $m \neq 0$, which verify

$$x\gamma_x + y\gamma_y = m\gamma. \quad (18)$$

The coefficient Γ_0 in (16) reads

$$\Gamma_0 = a_1s^3 + a_2s^2 + a_3s - (V_y\gamma + V_x)\,(a_4s_{200} + a_5s_{110} + a_6s_{020} + a_7s_{100} + a_8s_{010}). \quad (19)$$

It follows that the coefficient $\Gamma_0 = 0$ if $s(x, y, \gamma) = 0$. After a factorization by γ_x, equation (16) is again of second degree.

Remark 2 When γ satisfies the condition (1) (case studied by Anisiu et al, 2004), in equation (16) $\Gamma_3 = \Gamma_0 = 0$. Therefore γ_x is the solution of an equation of first degree. This happens, for example, for γ homogeneous of order 0.

In what follows, to ease the algebra, we shall assume that the functions r and/or s are of a form that makes equation (16) of second degree, i. e.

$$G_2\gamma_x^2 + G_1\gamma_x + G_0 = 0. \tag{20}$$

We differentiate (20) with respect to x and substitute the second-order derivatives of γ from (15) and γ_y from (3); the result will be an equation of third order in γ_x

$$H_3\gamma_x^3 + H_2\gamma_x^2 + H_1\gamma_x + H_0 = 0. \tag{21}$$

Our calculations have shown that equation (21) is of second degree if $s = 0$; but it will be of third degree for homogeneous functions of order m. In order that (20) and (21) have a common solution, the necessary and sufficient condition is that their resultant is null. This is a first condition that γ has to fulfil.

Let us suppose that the resultant of (20) and (21) is null. We express γ_x^2 from (20) and substitute it in the first two terms of (21), then again in the result. It follows that γ_x is given by

$$\left(H_3G_1^2 - H_3G_2G_0 - H_2G_2G_1 + H_1G_2^2\right)\gamma_x + H_3G_1G_0 - H_2G_2G_0 + H_0G_2^2 = 0. \tag{22}$$

If the coefficient of γ_x is different from zero, we can express γ_x as a function of γ

$$\gamma_x = -\frac{H_3G_1G_0 - H_2G_2G_0 + H_0G_2^2}{H_3G_1^2 - H_3G_2G_0 - H_2G_2G_1 + H_1G_2^2}, \tag{23}$$

and γ_y from (3) as

$$\gamma_y = s - r\gamma_x. \tag{24}$$

We write the compatibility condition $(\gamma_x)_y = (\gamma_y)_x$, in which we replace γ_x by (23) and γ_y by (24); we obtain a second condition on γ.

From (23) and (24) we can express, after differentiation, $\gamma_{xx}, \gamma_{xy}, \gamma_{yy}$ in terms of γ and derivatives of V up to the fifth order. We insert these values in the basic equation (6), and then the values of γ_x and γ_y from (23) and (24). We obtain a third condition on γ. In order to obtain solutions of the problem under consideration, these three necessary conditions must be satisfied.

If the coefficient of γ_x in (22) is zero and the other term is not zero, we have no solution for our problem. If both coefficients in (22) are null, we are left with equation (20).

As an application to the reasoning developed in this section, we shall study first the case of functions γ which are homogeneous of order m.

4 Functions γ homogeneous of order m

Let us suppose that γ satisfies (18), hence we have $r = x/y$ and $s = m\gamma/y$. As stated above, the first equation in γ_x (20) is of second degree; its coefficients are in this case polynomials in γ. This will happen for the coefficients of the third-degree equation (21) too.

The three conditions on γ are in this case polynomials in γ. For a common solution to exist, a necessary condition is that the resultants of the two pairs of polynomials vanish. The resultants are equal to their Sylvester determinants (Mishina and Proskuryakov, 1965, p. 164). Thus we obtain two necessary conditions to be satisfied by the potential V and the function γ.

When we start working with a given potential V and a fixed degree of homogeneity for γ, we do not expect the problem to have always a solution. It is advisable to try to factor the first polynomial in γ (the resultant of (20) and (21)) and to check directly if the homogeneous functions γ are compatible with the potential. Proceeding this way we avoid lengthy calculations.

Example 1 *Let us consider $V(x, y) = -x^4 - y^2$ and look for functions γ homogeneous of first order. The polynomials (20) and (21) are of second, respectively third, degree and their resultant is*

$$R_1 = \gamma^5 \left(y\gamma - x^2\right) \left(y\gamma + x^2\right) \left(y\gamma + x\right)^3 \left(y\gamma + 2x^3\right)^4 P_{10}. \tag{25}$$

The index of P denotes in these examples the degree of the respective polynomial in γ. The second condition, which follows from the compatibility $(\gamma_x)_y = (\gamma_y)_x$, reads

$$R_2 = \left(y\gamma - x^2\right) \left(y\gamma + x^2\right) \left(y\gamma + x\right) \left(y\gamma + 2x^3\right) P_{12}. \tag{26}$$

Finally, the condition obtained from the basic equation (6) is

$$R_3 = \left(y\gamma - x^2\right) \left(y\gamma + x^2\right) \left(y\gamma + 2x^3\right) P_{22}. \tag{27}$$

The three polynomials in γ have in common two homogeneous solutions of first order, namely $\gamma_1 = x^2/y$ and $\gamma_2 = -x^2/y$, which correspond to the families $f_1 = ye^{-1/x}$, and $f_2 = ye^{1/x}$ and are compatible with the given potential.

5 Special cases and other examples

The case $r = -\gamma$, $s = 0$ $(\Gamma = 0)$

From the equation (8) it follows that

$$\gamma = -\frac{V_x}{V_y} \tag{28}$$

and only potentials $V(x,y)$ satisfying the differential condition

$$V_x V_y (V_{xx} - V_{yy}) = (V_x^2 - V_y^2) V_{xy} \tag{29}$$

are generating families having $\Gamma = 0$ (Bozis and Anisiu, 2001). So then, for our problem, we see immediately if the given potential satisfies or not the condition (29) and, if the potential is admissible, we readily check whether or not the pertinent γ, given by (28), satisfies the pre-assigned condition (3).

As another viewpoint, let us discuss briefly the following two alternatives, possibly leading to an affirmative answer of our problem:
(i) Let us fix the condition (3) but allow the potential $V(x,y)$ to be free. In this case we must inquire whether there exist common solutions for the PDE (29) and the PDE

$$r^* V_y V_{xx} + (V_y - r^* V_x) V_{xy} - V_x V_{yy} + s^* V_y^2 = 0, \tag{30}$$

where

$$r^*(x,y) = r\left(x, y, \gamma = -\frac{V_x}{V_y}\right) \text{ and } s^*(x,y) = s\left(x, y, \gamma = -\frac{V_x}{V_y}\right). \tag{31}$$

The compatibility of these two equations may be checked in a straightforward way.
(ii) Let us consider a potential $V(x,y)$ satisfying the condition (29), i.e. a potential which produces the family (28) of straight lines and let the functions r and s in (3) be at our disposal. In this case we are led to infinitely many choices for r and s for which the condition (30) is satisfied. Indeed, we can take

$$r(x,y,\gamma) = -\gamma + \left(\gamma + \frac{V_x}{V_y}\right) A(x,y,\gamma) \text{ and } s(x,y,\gamma) = \left(\gamma + \frac{V_x}{V_y}\right) B(x,y,\gamma), \tag{32}$$

where A and B are arbitrary functions with the unique provision that the pertinent functions $A^*(x,y)$ and $B^*(x,y)$ (defined as indicated in (31)) do not become infinite. By choosing the functions r and s as in (32), we have $r^*(x,y) = V_x/V_y$ and $s^*(x,y) = 0$, hence condition (30) is identical to (29).

The case $r = -\gamma$, $s \neq 0$

In this case $\Pi = 0$ in the first of the equations (14) and the formulae (15) are meaningless. Let us suppose that $r(x, y, \gamma) = \gamma$ identically. The condition (3) becomes

$$\gamma\gamma_x - \gamma_y = -s, \tag{33}$$

where s may depend on all three variables x, y and γ. We suppose here that s is not identically null, to avoid that (33) coincides with $\Gamma = 0$ (treated above).

From the derivatives of (33) with respect to x and y, we find

$$\gamma^2\gamma_{xx} - 2\gamma\gamma_{xy} + \gamma_{yy} = s_{010} - \gamma s_{100} + s\gamma_x + ss_{001}. \tag{34}$$

So, in view of (7) and (34), equation (6) may be written as

$$\begin{aligned} &s(V_x + \gamma V_y)s_{001} = (V_x + \gamma V_y)(\gamma s_{100} - s_{010}) \\ &+s\left[\gamma(V_{yy} - V_{xx}) + (1 - \gamma^2)V_{xy} + 3sV_y\right]. \end{aligned} \tag{35}$$

The above equation (35) replaces the PDE (6) and its meaning is the following: In order that the given potential $V(x, y)$ supports a family γ, the "given" function $s(x, y, \gamma)$ in (33) must satisfy the PDE (35). In other words, for our problem to admit of an affirmative answer, the required function $\gamma(x, y)$ and the "given" function $s(x, y, \gamma)$ must satisfy both equations (33) and (35). To check if these equations have common solutions γ we proceed as follows: From (35) we can express (by differentiation) γ_x and γ_y in terms of γ and insert them into (33), which then will become an equation of the form

$$F(x, y, \gamma) = 0. \tag{36}$$

Finally we check whether equations (36) and (35) have or do not have common solutions $\gamma(x, y)$.

Example 2 *Let us find solutions of (33) with $s(x, y, \gamma) = -6x/y^2$ which represent families compatible with the potential*

$$V(x, y) = 4x^2 + y^2 + 8x^4 - 2x^2y^2 - y^4 + x^3. \tag{37}$$

Condition (35) is in this case a second-degree polynomial equation in γ, which has the solutions

$$\gamma = \frac{2x}{y} \text{ and } \gamma = \frac{2x\left(17x^2 + 7y^2\right)}{y\left(2x^2 - 2y^2 + 1\right)}. \tag{38}$$

The first one is a solution of our problem.

It may happen that $\Pi = 0$ for some particular functions γ. In such a case, we have to check if this particular γ satisfies equation (3). In the affirmative case, we put the values of V and γ in (6)-(7) and, if we obtain an identity, we have a solution of our problem.

Example 3 *Let us look for families γ which are compatible with the Hénon-Heiles potential*

$$V(x,y) = \frac{1}{2}\left(x^2 + 16y^2\right) + x^2 y + \frac{16}{3}y^3 \tag{39}$$

and which satisfy the equation (3) with $r(x,y,\gamma) = x/y + 3\gamma$ and $s(x,y,\gamma) = -3\gamma/(4y)$. The equality $\gamma + r = 0$ holds if and only if $\gamma = -x/(4y)$. This function verifies the equation (3) for the specified values of r and s, and, together with the potential (39), equation (6)-(7), hence it is a solution of our problem. The same family has been found by Bozis et al (1997) as a homogeneous family generated by the inhomogeneous potential (39).

Remark 3 If $V(x,y)$ and $s(x,y,\gamma)$ are left free (to be adequately determined) the possibly existing common solutions $\gamma(x,y)$ of (35) and (36) will be expressed in terms of partial derivatives of the second order in $s(x,y,\gamma)$ and of the third order in $V(x,y)$.

6 General comments

In the framework of the inverse problem of Dynamics, a monoparametric family of orbits is uniquely represented by its slope function γ defined in (5). For a given potential $V(x,y)$, the finding of some or all families generated by V amounts to the solution of the nonlinear in γ second order PDE (6). This is a task more or less impossible.

In this paper, in order to ease and make possible the solution of the problem (even by finding a subset of solutions), we add the restriction on γ expressed by the differential condition (3). In so doing, we come to have to deal with two PDEs, one of the first and one of the second order in the unknown function $\gamma(x,y)$. Therefore the very existence of a solution is not guaranteed. Yet, we showed that, if such a solution does exist, its finding may be accomplished by algebraic manipulation.

We deal basically with the direct problem, i.e. the potential is given and the orbits are to be found. The functions $r(x,y,\gamma)$ and $s(x,y,\gamma)$ are also generally given. One then might suggest to face the problem by solving for γ the first order PDE (3) and then proceed to find, among its solutions, those which are compatible with the given potential. However, this last task (possible in some of the examples presented in this paper) does not seem to be easier or performable by a straightforward way. Besides that, the finding of the general solution of (3) is not *always* possible.

The above strictly direct problem does not generally have a solution. For this reason, we may profitably deal with the two equations (3) and (6) in various ways. We can e.g. allow tentatively the potential $V(x, y)$ to be free and find compatibility conditions on it so that a solution $\gamma(x, y)$ can be found. Or, keeping $V(x, y)$ fixed, we may allow the functions r and s in (3) to be free and then adjust them properly so that we obtain a solution.

References

[1] Anisiu, M.-C., Blaga, C. and Bozis, G.: 2004, Special families of orbits in the direct problem of dynamics, *Celest. Mech. Dyn. Astron.* **88**, 245-257.

[2] Bozis, G.: 1983, Inverse problem with two-parametric families of planar orbits, *Celest. Mech.* **31**, 129-143.

[3] Bozis, G.: 1984, Szebehely's inverse problem for finite symmetrical material concentrations, *Astronom. Astrophys.* **134**(2), 360-364.

[4] Bozis, G. and Ichtiaroglou, S.: 1994, Boundary curves for families of planar orbits, *Celest. Mech. Dyn. Astron.* **58**, 371-385.

[5] Bozis, G.: 1995, The inverse problem of dynamics. Basic facts, *Inverse Problems* **11**, 687-708.

[6] Bozis, G., Anisiu, M.-C. and Blaga, C.: 1997, Inhomogeneous potentials producing homogeneous orbits, *Astron. Nachr.* **318**, 313-318.

[7] Bozis, G. and Grigoriadou, S.: 1993, Families of planar orbits generated by homogeneous potentials, *Celest. Mech. Dyn. Astron.* **57**, 461-472.

[8] Bozis, G., Anisiu, M.-C. and Blaga, C.: 2000, A solvable version of the direct problem of dynamics, *Rom. Astronom. J.* **10**(1), 59-70.

[9] Bozis, G. and Anisiu, M.-C.: 2001, Families of straight lines in planar potentials, *Rom. Astronom. J.* **11**(1), 27-43.

[10] Hénon, M. and Heiles, C.: 1964, The applicability of the third integral of motion: some numerical experiments, *Astron. J.* **69**, 73-79.

[11] Mishina, A. P. and Proskuryakov, I. V.: 1965, *Higher Algebra*, Pergamon Press, Oxford-London.

[12] Szebehely, V.: 1974, On the determination of the potential by satellite observation, in: E. Proverbio (ed.) *Proceedings of the International Meeting on Earth's Rotations by Satellite Observations*, Cagliari, Bologna, 31-35.