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FAMILIES OF PLANAR ORBITS IN ONE-VARIABLE CONSERVATIVE FIELDS

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Abstract. We discuss a simple case of the planar inverse problem of Dynamics, considering a one-dimension potential $V = v(x)$. For the families which satisfy a differential condition, the specific potentials can be obtained by quadratures. The isoenergetic families of orbits which can be described under the action of a potential $V = v(x)$ are displayed too.

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1. INTRODUCTION

The inverse problem of Dynamics consists in finding the force fields, conservative or not, which generate the motion in a dynamical system, knowing in advance a family of orbits. Historically, the first results are due to Newton, who in his monumental work *Principia* [12] found the forces that make a particle to move on ellipses. At the end of the XIXth century, the problem was considered again for several families of orbits by Bertrand [3], Dainelli [10] and Jukovski [11], their results being exposed in the well-known book of Whittaker [14]. The paper which raised again the interest in this problem was that of Szebehely [13], where a way was sought to determine the Earth's potential on the basis of satellites' movement. The book [1] contains an introduction to the planar inverse problem of Dynamics.

Although linear in the unknown potential $V(x, y)$, the basic equations of the inverse problem (equations (4) and (7) below) cannot generally be solved. Certain limiting assumptions either on the orbits or on the potential may make the problem solvable. Such is, for instance, the planar motion of a unit mass under the action of an one-dimension potential $V = v(x)$. Evidently, in this case, the two second-order ODEs of motion (equations (2) below) can be solved to completion and the pertinent three-parametric family of orbits can be found by successive quadratures.

Here we treat the above simple problem in the framework of the inverse problem so that the student becomes acquainted with the relevant basic tools and eventually compare results and also get an insight into the connection between geometrical and dynamical aspects in Mechanics (i. e. curves in the Oxy plane as geometrical entities and possible motion on these curves now considered as orbits).

2. THE EQUATIONS OF THE PLANAR INVERSE PROBLEM

The planar inverse problem aims to the finding of the potentials $V = V(x, y)$ which can produce the planar motion of a particle of unit mass along a given family of curves

$$f(x, y) = c. \tag{1}$$

The equations of motion of the particle are

$$\ddot{x} = -V_x \qquad \ddot{y} = -V_y, \tag{2}$$

where the indices denote partial derivatives.

The system (2) admits the energy integral $\dot{x}^2 + \dot{y}^2 = 2(E - V)$, the total energy E being constant on each trajectory of the system. The family of curves (1) being given, in the case when it will appear as a family of trajectories of the system (2), we shall denote the energy by $E = E(f)$, emphasizing the fact that it is constant on each member of the family.

Using the functions

$$\gamma = \frac{f_y}{f_x}, \quad \Gamma = \gamma\gamma_x - \gamma_y, \tag{3}$$

Szebehely's equation [13], which expresses the connection between the potential and the given family, was written in a simpler form in [5]

$$V_x + \gamma V_y + \frac{2\Gamma\,(E(f) - V)}{1 + \gamma^2} = 0. \tag{4}$$

The function γ is related to the slope of the curves in the family, and Γ to their curvature. The reader can easily see that functions $\gamma = \gamma(x, y)$ and families $f(x, y) = c$ are in an one-to-one correspondence. On the other hand, since $\mathrm{d}\,y/\,\mathrm{d}\,x = -1/\gamma(x, y)$, we understand that the PDE (4) gives all potentials $V = V(x, y)$ which can produce as orbits all the solutions of a first order ODE (in the solved form $y' = -1/\gamma(x, y)$).

The inequality $E(f) - V \geq 0$ expresses the fact that the kinetic energy cannot be negative, therefore

$$\frac{V_x + \gamma V_y}{\Gamma} \leq 0. \tag{5}$$

The meaning of the inequality (5) was discussed in [7]. Supposing that (i) $\Gamma \neq 0$, (ii) all pertinent functions are sufficiently smooth and introducing the notation

$$\kappa = \frac{1}{\gamma} - \gamma, \quad \lambda = \frac{\Gamma_y - \gamma\Gamma_x}{\gamma\Gamma}, \quad \mu = \lambda\gamma + \frac{3\Gamma}{\gamma}, \tag{6}$$

Bozis [6] derived the free of energy equation of the second order in V

$$-V_{xx} + \kappa V_{xy} + V_{yy} = \lambda V_x + \mu V_y. \tag{7}$$

In the case of families of straight lines, for which $\Gamma = 0$, the corresponding equation is of first order and it reads

$$V_x + \gamma V_y = 0. \tag{8}$$

The equations of the inverse problem of Dynamics are presented in detail by Bozis [8] and Anisiu [2].

Remark 1 As it is easily seen from (4), a potential V which depends only on one variable cannot generate families of straight lines ($\Gamma = 0$), except in the trivial case $V = const$. In what follows we shall consider $\Gamma \neq 0$.

3. POTENTIALS DEPENDING MERELY ON ONE VARIABLE

We study only the case of potentials $V = v(x)$, because $V = v(y)$ can be reduced to this one by interchanging the roles of the variables x, y and by considering the family $\tilde{f}(y, x) = c$.

The equation (7) becomes very simple when $V = v(x)$, namely

$$-v''(x) = \lambda v'(x), \tag{9}$$

where primes denote differentiation with respect to x. In equation (9), the function λ must depend merely on the variable x, so it has to satisfy $\partial\lambda/\partial y = 0$. Using the expression of λ from (6b), we find the following necessary and sufficient condition to be satisfied by the family γ in order that λ depends merely on x:

$$\gamma\Gamma\left(\Gamma_{yy} - \gamma\Gamma_{xy}\right) + \Gamma_y\left(\gamma^2\Gamma_x - \gamma\Gamma_y - \Gamma\gamma_y\right) = 0. \tag{10}$$

Equation (10) is a differential condition (of the third order in $\gamma(x, y)$) satisfied by all families γ generated by one-dimension potentials. On the other hand, for a given $\lambda = \ell(x)$, the potential $v(x)$ corresponding to each family γ is found uniquely from (9) (up to the multiplicative and additive constants c_1 and c_2). It is

$$v(x) = c_1 \int \Lambda(x)\,\mathrm{d}\,x + c_2, \tag{11}$$

where

$$\Lambda(x) = \exp\left(-\int \ell(x)\,\mathrm{d}\,x\right). \tag{12}$$

From (5) it follows that real motion is possible in the region

$$\frac{v'(x)}{\Gamma} \leq 0, \tag{13}$$

with the energy obtained from (4)

$$E(f) = v(x) - \frac{1+\gamma^2}{2\Gamma} v'(x). \tag{14}$$

For a given $\lambda = \ell(x)$, the potential $V = v(x)$ is found from (11), and we face the direct problem of Dynamics: find the families of orbits compatible with a given potential. We remark that in the frame of the direct problem pairs of one-dimension potentials $V = v(x)$ and one-dimension families γ have been found in [9]. Condition (10) may also be written as a PDE of the second order in $\gamma(x, y)$, i. e.

$$\gamma^2\gamma_{xx} - 2\gamma\gamma_{xy} + \gamma_{yy} + (\gamma_x + \gamma\ell(x))(\gamma\gamma_x - \gamma_y) = 0. \tag{15}$$

Solutions of (15) may be found in special cases. Thus, e. g.

(i) Looking for solutions $\gamma = g(x)$, we find

$$\gamma = \pm\left(k_1 \int \Lambda(x)\,\mathrm{d}\,x + k_2\right)^{\frac{1}{2}}, \tag{16}$$

where $\Lambda(x)$ is given by (12) and k_1, k_2 are new integration constants. So, all families (16) are compatible with the potential (11).

For $k_1 = c_1$, $k_2 = c_2$ we see that the potential $v(x)$ can create the families $\gamma = \pm\sqrt{v(x)}$.

Remark 2 From Szebehely's equation (4) it follows that the energy on all members of a family (16) is given by $E = -c_1\,(k_2 + 1)\,/k_1$. This means that the family is isoenergetic, and the possibility that families with this property are traced in the presence of a one-dimension potential will be studied in section 4.

(ii) In particular for $\ell = \ell_0/x$, ($\ell_0 = const.$) and for functions γ of the form $\gamma = \gamma(y/x)$, the PDE (15) becomes

$$(1 + \gamma z)\ddot{\gamma} + z\dot{\gamma}^2 + (2 - \ell_0)\gamma\dot{\gamma} = 0, \tag{17}$$

where $z = y/x$ and dots denote differentiation in z.

For $\ell_0 = 2$, a first integration of (17) leads to

$$(1 + z\gamma)\dot{\gamma} - \frac{1}{2}\gamma^2 = k_1 \tag{18}$$

and, for $k_1 > 0$, a second integration leads to

$$\frac{2k_1 z - \gamma}{2k_1 + \gamma^2} - \frac{\arctan\frac{\gamma}{\sqrt{2k_1}}}{\sqrt{2k_1}} = k_2. \tag{19}$$

Sporadic solutions of (17) may be found for other values of ℓ_0. Thus e. g. for $\ell_0 = 3$, we obtain from (17) the families $\gamma = k_0 z - 1/(2z)$ ($k_0 = const.$) corresponding to the two-parametric family $f(x, y) = y/(x^2 + 2k_0 y^2) = c$, derived by the potential $-1/x^2$.

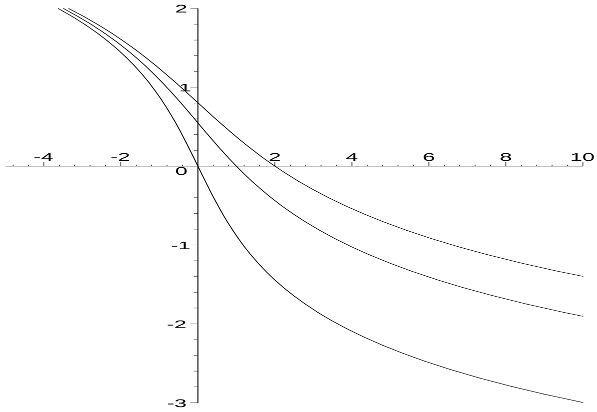

Fig. 1 Curves of the family in example 1 with $c = 1, 3, 5$

In the next examples we give planar families of curves which fulfil the differential condition (10), and we get the potentials $V = v(x)$ under whose action a material point of unit mass can describe the curves of these families.

Example 1. For the family

$$f(x, y) = \exp(2y) + 2x \exp(y) = c \tag{20}$$

we get $\lambda = -1/x$. The one-variable potential is given by $v(x) = c_1 x^2$, and the curves of the family can be traced all over the plane for $c_1 < 0$, with the energy $E(f) = -c_1\,(f + 1)$. For the family (20) we have $\gamma = x + \exp(y)$ and this γ satisfies, of course, the PDE (15).

Example 2. To the family

$$f(x, y) = x/\sqrt[3]{y^2} = c \tag{21}$$

it corresponds $\lambda = 2/x$. The potential $v(x) = c_1/x$ allows the curves of the family to be traced all over the plane, for $c_1 < 0$, with the energy $E(f) = -9c_1/(4f^3)$. To the family (21) there corresponds $\gamma = -2/(3z)$ $(z = y/x)$ and, since $\ell_0 = 2$, this γ satisfies the equation (18) leading to $k_1 = 0$.

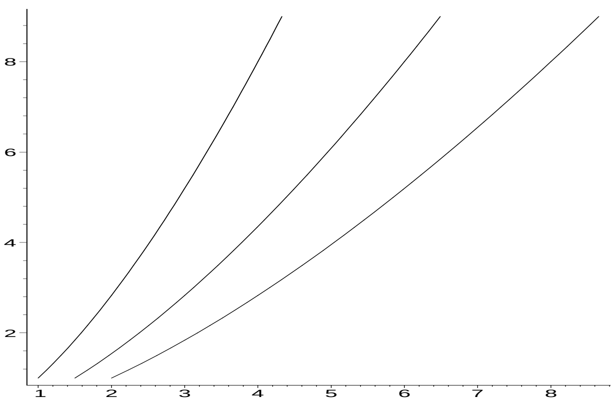

Fig. 2 Curves of the family in example 2 with $c = 1, 1.5, 2$

4. ALL ISOENERGETIC FAMILIES CREATED BY ONE-DIMENSION POTENTIALS

A family of orbits (1) is called *isoenergetic* if the constant value of the energy is the same on all members of the family, i. e. $E(f) = E_0$. Isoenergetic families of orbits are easier to handle, because in this case the unknown function $E(f)$ in (4) is just a constant. Good reasons to study isoenergetic families have been discussed by Borghero and Bozis [4], who solved to completion the inverse problem of Dynamics for isoenergetic families created by homogeneous potentials $V(x, y)$.

Let us seek compatible pairs of potentials of the form $V = v(x)$ and isoenergetic families of orbits $f(x, y) = c$ traced with total energy $E = E_0 = const$. As $V_x = v'(x)$, the PDE (4) reads

$$\frac{2\Gamma}{1+\gamma^2} = -\frac{v'(x)}{E_0 - v(x)}. \tag{22}$$

For the ODE (22) to admit of appropriate solutions, we must have

$$\frac{\Gamma}{1+\gamma^2} = m(x), \tag{23}$$

i.e.

$$\Gamma_y(1+\gamma^2) - 2\gamma\gamma_y\Gamma = 0. \tag{24}$$

This last condition (of the second order in $\gamma(x, y)$) is the analogue of the condition (10) and must be fulfilled by all families traced isoenergetically in the presence of the one-dimension potential given by

$$v(x) = E_0 + c_1 M^2, \tag{25}$$

where

$$M(x) = \exp\left(\int m(x)\,\mathrm{d}\,x\right). \tag{26}$$

If m is considered to be given, then M is known from (26), $v(x)$ is known from (25) and we face the direct problem. All families $\gamma(x, y)$ produced by

$v(x)$ are to be found from the first order PDE (23), whose subsidiary system is

$$\frac{\mathrm{d}\,x}{\gamma} = \frac{\mathrm{d}\,y}{-1} = \frac{\mathrm{d}\,\gamma}{m(x)(1+\gamma^2)}. \tag{27}$$

If the first term in (27) is put equal to the third, it gives

$$k_1 = \frac{1+\gamma^2}{M^2} \tag{28}$$

and, if put equal to the second, it gives

$$k_2 = y + T(x, k_1), \tag{29}$$

where

$$T(x, k_1) = \int \frac{\mathrm{d}\,x}{\sqrt{k_1 M^2 - 1}}. \tag{30}$$

So, the general solution of (23) is given by

$$y + S(x,\gamma) = A\left(\frac{1+\gamma^2}{M^2}\right), \tag{31}$$

where

$$S(x,\gamma) = T\left(x, k_1 = \frac{1+\gamma^2}{M^2}\right) \tag{32}$$

and A is an arbitrary function of its argument.

In conclusion: All families $\gamma(x,y)$ given implicitly by (31) are isoenergetically traced by the potential (25).

Applying for $m = -1/x$, we obtain successively: $M(x) = 1/x$, $k_1 = x^2(1+\gamma^2)$, $T(x,k_1) = -\sqrt{k_1 - x^2}$, $S(x,\gamma) = -x\gamma$. Selecting in (31) $A(u) = u$, we find that the family $\gamma = \left(-x + \sqrt{x^2 + 4x^2 y - 4x^4}\right)/2x^2$ is compatible with the potential $v(x) = -1/x^2$, traced with total energy $E = 0$.

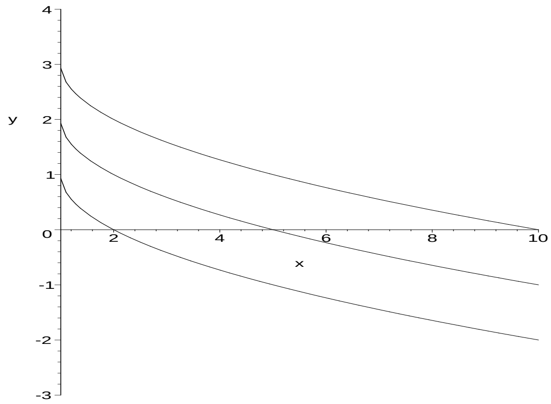

Fig. 3 Curves of the family in example 3 with $c = 1, 2, 3$

EXAMPLE 3. The family

$$f(x, y) = y + \sqrt{x - 1} \tag{33}$$

is traced isoenergetically with $E = E_0$ by the potential $v(x) = E_0 + 3c_1/4 - c_1 x$ in the region $x > 1$ for $c_1 > 0$.

5. CONCLUSIONS

For a known family of curves (1) for which $\lambda = \ell(x)$, with λ from (6b), the potential $V = v(x)$ which can generate this family, with suitable chosen initial conditions, is given by (11) with Λ from (12).

All the families of curves for which λ depends merely on x satisfy the differential condition (10), of the third order in $\gamma(x, y)$.

The totality of the isoenergetic families of curves which are described under the action of a potential $V = v(x)$ is given by (31).

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