Spatial Families of Orbits in 2D Conservative Fields

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Abstract. In the framework of the 3D inverse problem of dynamics, we establish the conditions which must be fulfilled by a spatial family of curves to possibly be described by a unit mass particle under the action of a 2D potential \( V = v(y, z) \), and give a method to find the potential.

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INTRODUCTION

In the 3D inverse problem of Dynamics, assuming that a two-parametric set of orbits

\[ f(x, y, z) = c_1, \quad g(x, y, z) = c_2 \]  

in the \( Oxyz \) space can be traced by a unit mass particle in the presence of an unknown potential \( V = V(x, y, z) \), one aims to finding the potential (see [1] for a review of the results up to 1995).

Bozis and Kotoulas [2] and also Anisiu [3] produced a system of two linear in \( V(x, y, z) \) PDEs, one of the first and one of the second order, which will be used in what follows.

We shall treat the problem for the special case of potentials \( V = v(y, z) \).

THE EQUATIONS OF THE PROBLEM

We deal with two-parametric families of orbits written in the form (1), which are in an one-to-one correspondence with a pair \((\alpha, \beta)\) of ‘slope functions’ defined by

\[ \alpha = \frac{f_z g_x - f_x g_z}{f_y g_z - f_z g_y}, \quad \beta = \frac{f_x g_y - f_y g_x}{f_y g_z - f_z g_y}. \]  

(2)

The indices denote partial derivatives.

Let us assume that \( \alpha_0 \neq 0 \) and adopt the notation

\[ \tilde{\epsilon} = (1, \alpha, \beta), \quad \alpha_0 = \tilde{\epsilon} \text{ grad } \alpha, \quad \beta_0 = \tilde{\epsilon} \text{ grad } \beta, \quad \Theta = 1 + \alpha^2 + \beta^2, \quad n = \frac{\Theta}{\alpha_0}, \quad n_0 = \tilde{\epsilon} \text{ grad } n. \]  

(3)
We shall consider exclusively potentials of the form $V = v(y,z)$ and families (1) with $\alpha_0 \neq 0$. In this case the system of the two PDEs mentioned in the introduction becomes

$$v_z = Gv_y, \quad \Theta(\alpha + \beta G)v_{yy} + \Psi v_y = 0,$$

(4)

where $G = \beta_0/\alpha_0$, $\Psi = \beta \Theta G_v + \alpha_0 (n_0 - 2(\alpha + \beta G))$.

For any compatible pair of potential $v(y,z)$ and orbit $(\alpha, \beta)$ real motion is allowed in the region $-v_y/\alpha_0 \geq 0$ ([2], [3]).

**COMPATIBLE POTENTIALS $V = v(y,z)$ AND FAMILIES (1)**

It is seen from (4a) that the function $G$ must be independent of $x$, i.e.

$$G_x = 0.$$

(5)

We assume that $\alpha + \beta G \neq 0$ and we put $H = \Psi/\Theta(\alpha + \beta G)$. For the PDE (4b), now written as $v_{yy} + H v_y = 0$, to have a solution of the form $v(y,z)$ it must be

$$H_x = 0.$$

(6)

From (4b) we get

$$v_y = D(z) \exp(-\int^y H(u,z) \, du)$$

(7)

and from (4a) we obtain $v_z$. From the compatibility condition ($v_{yz} = v_{zy}$) for $v_y$ and $v_z$, as these are given by (7) and (4a) respectively, there follows the homogeneous linear first order ODE $D'(z) = JD(z)$, where $J$ depends merely on $z$ if and only if

$$G_{yy} - G_y H - G H_y + H_z = 0.$$

(8)

**Proposition** For $\alpha_0 \neq 0$, $\alpha + \beta G \neq 0$ and for any family $(\alpha, \beta)$ satisfying the conditions (5), (6) and (8), there exists a two-dimension compatible potential $v = v(y,z)$. The potential is given by the (compatible) equations (7) and (4a). Real motion is allowed in the region defined by the inequality $D(z)/\alpha_0 \leq 0$.

If $\alpha_0 \neq 0$, $\alpha + \beta G = 0$ and $\Psi \neq 0$ it follows that $v = \text{const}$. If $\alpha_0 \neq 0$, $\alpha + \beta G = 0$ and $\Psi = 0$, equation (4b) is satisfied identically and the potential is found from (4a), provided that the condition (5) holds, and it will not be uniquely determined.

**Example** For the family $f(x,y,z) = x^4 y z^3$, $g(x,y,z) = x^2 y z$, we get $\alpha_0 \neq 0$, $\alpha + \beta G \neq 0$ and obtain the compatible potential $V(x,y,z) = -(y^2 + z^2)^2$.

The case $\alpha_0 = 0$ and that of potentials depending on $(x,y)$ or $(x,z)$ will be treated elsewhere.

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**REFERENCES**