Spatial Families of Orbits in 2D Conservative Fields

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Abstract. In the framework of the 3D inverse problem of dynamics, we establish the conditions which must be fulfilled by a spatial family of curves to possibly be described by a unit mass particle under the action of a 2D potential $V = v(y,z)$, and give a method to find the potential.

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INTRODUCTION

In the 3D inverse problem of Dynamics, assuming that a two-parametric set of orbits

$$f(x,y,z) = c_1, \quad g(x,y,z) = c_2 \tag{1}$$

in the $Oxyz$ space can be traced by a unit mass particle in the presence of an unknown potential $V = V(x,y,z)$, one aims to finding the potential (see [1] for a review of the results up to 1995).

Bozis and Kotoulas [2] and also Anisiu [3] produced a system of two linear in $V(x,y,z)$ PDEs, one of the first and one of the second order, which will be used in what follows.

We shall treat the problem for the special case of potentials $V = v(y,z)$.

THE EQUATIONS OF THE PROBLEM

We deal with two-parametric families of orbits written in the form (1), which are in an one-to-one correspondence with a pair (α,β) of 'slope functions' defined by

$$\alpha = \frac{f_z g_x - f_x g_z}{f_y g_z - f_z g_y}, \quad \beta = \frac{f_x g_y - f_y g_x}{f_y g_z - f_z g_y}. \tag{2}$$

The indices denote partial derivatives.

Let us assume that $\alpha_0 \neq 0$ and adopt the notation

$$\bar{\varepsilon} = (1,\alpha,\beta), \alpha_0 = \bar{\varepsilon}\,\mathrm{grad}\,\alpha, \beta_0 = \bar{\varepsilon}\,\mathrm{grad}\,\beta, \Theta = 1+\alpha^2+\beta^2, n = \frac{\Theta}{\alpha_0}, n_0 = \bar{\varepsilon}\,\mathrm{grad}\,n. \tag{3}$$

We shall consider exclusively potentials of the form $V = v(y,z)$ and families (1) with $\alpha_0 \neq 0$. In this case the system of the two PDEs mentioned in the introduction becomes

$$v_z = Gv_y,\ \Theta(\alpha+\beta G)v_{yy} + \Psi v_y = 0, \tag{4}$$

where $G = \beta_0/\alpha_0,\ \Psi = \beta\Theta G_y + \alpha_0\left(n_0 - 2(\alpha+\beta G)\right)$.

For any compatible pair of potential $v(y,z)$ and orbit (α,β) real motion is allowed in the region $-v_y/\alpha_0 \geq 0$ ([2], [3]).

COMPATIBLE POTENTIALS $V = v(y,z)$ AND FAMILIES (1)

It is seen from (4a) that the function G must be independent of x, i. e.

$$G_x = 0. \tag{5}$$

We assume that $\alpha+\beta G \neq 0$ and we put $H = \Psi/\Theta(\alpha+\beta G)$. For the PDE (4b), now written as $v_{yy} + Hv_y = 0$, to have a solution of the form $v(y,z)$ it must be

$$H_x = 0. \tag{6}$$

From (4b) we get

$$v_y = D(z)\exp(-\int^y H(u,z)\,\mathrm{d}u) \tag{7}$$

and from (4a) we obtain v_z. From the compatibility condition $(v_{yz} = v_{zy})$ for v_y and v_z, as these are given by (7) and (4a) respectively, there follows the homogeneous linear first order ODE $D'(z) = JD(z)$, where J depends merely on z if and only if

$$G_{yy} - G_y H - GH_y + H_z = 0. \tag{8}$$

Proposition For $\alpha_0 \neq 0$, $\alpha+\beta G \neq 0$ and for any family (α,β) satisfying the conditions (5), (6) and (8), there exists a two-dimension compatible potential $v = v(y,z)$. The potential is given by the (compatible) equations (7) and (4a). Real motion is allowed in the region defined by the inequality $D(z)/\alpha_0 \leq 0$.

If $\alpha_0 \neq 0,\ \alpha+\beta G = 0$ and $\Psi \neq 0$ it follows that $v = const$. If $\alpha_0 \neq 0,\ \alpha+\beta G = 0$ and $\Psi = 0$, equation (4b) is satisfied identically and the potential is found from (4a), provided that the condition (5) holds, and it will not be uniquely determined.

Example For the family $f(x,y,z) = x^4yz^3,\ g(x,y,z) = x^2yz$, we get $\alpha_0 \neq 0$, $\alpha+\beta G \neq 0$ and obtain the compatible potential $V(x,y,z) = -\left(y^2+z^2\right)^2$.

The case $\alpha_0 = 0$ and that of potentials depending on (x,y) or (x,z) will be treated elsewhere.

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REFERENCES

1. G. Bozis, *Inverse Problems* **11**, 687-708 (1995).
2. G. Bozis, and T. A. Kotoulas, *Inverse Problems* **21**, 343-356 (2005).
3. M.-C. Anisiu, *Inverse Probl. Sci. Eng.* **13**, 545-558 (2005).