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# PROGRAMMED MOTION WITH HOMOGENEITY ASSUMPTIONS

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**Abstract.** We consider the problem : Given a planar region $T_{orb}$ described by one inequality $g(x, y) \leq c_0$, find the potentials $V = V(x, y)$ which can generate monoparametric families of orbits $f(x, y) = c$ (also to be found) lying exclusively in the region $T_{orb}$. We make assumptions on the homogeneity of both the function $g(x, y)$ describing the boundary of the region $T_{orb}$ and of the slope function $\gamma(x, y) = f_y/f_x$ of the required family. We show that, under certain conditions, the slope function $\gamma(x, y)$ can be obtained as the common solution of two algebraic equations. The theoretical results are illustrated by an example.

## 1. INTRODUCTION

Monoparametric families of orbits $f(x, y) = c$, which are produced by a given potential $V(x, y)$ and which have 'slope function' $\gamma(x, y) = f_y/f_x$, satisfy the second order nonlinear PDE (Bozis 1995)

$$\gamma^2\gamma_{xx}-2\gamma\gamma_{xy}+\gamma_{yy}=\frac{-(\gamma\gamma_x-\gamma_y)}{V_x+\gamma V_y}(\gamma_x V_x-(2\gamma\gamma_x-3\gamma_y)V_y-\gamma(V_{xx}-V_{yy})-(\gamma^2-1)V_{xy}), \quad (1)$$

where the subscripts denote partial derivatives. Families of straight lines, for which it is $\gamma\gamma_x - \gamma_y = 0$ and $V_x + \gamma V_y = 0$ (Bozis & Anisiu 2001), are excluded from our study.

The inequality (Bozis & Ichtiaroglou 1994)

$$B(x, y) \geq 0, \quad (2)$$

where

$$B(x, y) = \frac{V_x + \gamma V_y}{-(\gamma\gamma_x - \gamma_y)}, \quad (3)$$

determines the region $T_{orb}$ of the $xy$ plane where the potential $V(x, y)$ creates *real* orbits or *real parts* of the orbits belonging to the family $\gamma(x, y)$.

Conversely, we can select a specific region $T_{orb}$ of the $xy$ plane which *we want to make the exclusive allowed region* for certain unknown families created by an unknown potential.

We restrict ourselves to regions which are described by *one* inequality, say

$$b(x,y) \geq 0, \tag{4}$$

and impose the condition that the function $B(x,y)$ (corresponding to the pair $(V,\gamma)$) defines the same region (2) as the inequality (4) does. We interpret this by stating that there must exist a *nonvanishing* function $\Theta(x,y)$, in a region $T_0$ broader from the region $T_{orb}$, such that

$$B(x,y) = b(x,y)\Theta(x,y), \tag{5}$$

$$\Theta(x,y) \geq 0 \text{ for } (x,y) \in T_0 \text{ and } \tilde{\Theta}(x,y) \neq \infty, \tag{6}$$

where $\tilde{\Theta}(x,y)$ denotes the (one-variable) function $\Theta(x,y)$ evaluated at the points of the curve $b(x,y)=0$.

Bozis (1996) solved the problem of finding the force fields which produce a given family of orbits in a fixed in advance region, and Anisiu & Bozis (2000) considered the conservative case for the families $f(x,y) = y - H(x)$ and a given region.

## 2. BASIC PROGRAMMED MOTION PROBLEM

The function $B$ satisfies the second order linear equation (Bozis 1995, Anisiu 2003)

$$-B_{xx} + k^* B_{xy} + B_{yy} = \lambda^* B_x + \mu^* B_y + \nu^* B, \tag{7}$$

$$\begin{aligned} k^* &= \frac{1-\gamma^2}{\gamma}, \quad \lambda^* = \frac{\gamma_x + 2\gamma\gamma_y}{\gamma}, \\ \mu^* &= \frac{2\gamma\gamma_x - 3\gamma_y}{\gamma}, \quad \nu^* = \frac{2(\gamma_x\gamma_y - \gamma_{yy} + \gamma\gamma_{xy})}{\gamma}. \end{aligned} \tag{8}$$

The first partial derivatives of $V$ are related to $B$ by

$$V_x = -B(\gamma\gamma_x - \gamma_y) + \frac{1}{2}\gamma(B_y - \gamma B_x), \quad V_y = -\frac{1}{2}(B_y - \gamma B_x). \tag{9}$$

*Remark* If $\gamma$ is homogeneous of degree zero, then so is $k^*$, whereas $\lambda^*, \mu^*$ are of degree $-1$ and $\nu^*$ of degree $-2$. If $B(x,y)$ is weighted homogeneous of degrees e.g. $n_1$ and $n_2$, then the entire equation (7) will lead to a weighted homogeneous expression of degrees $n_1 - 2$ and $n_2 - 2$.

We suppose that a region is given by the unique inequality (4). The *basic programmed motion problem* is: What families can be created in the given region (4) and which potentials do generate them? We introduce the function $B(x,y)$, as given by (5), into the equation (7) and we obtain the linear in $\Theta$ PDE

$$b\left(-\Theta_{xx} + K\Theta_{xy} + \Theta_{yy}\right) = L\Theta_x + M\Theta_y + N\Theta, \tag{10}$$

where

$$\begin{aligned} &K = k^*, L = \lambda^* b + 2b_x - k^* b_y, M = b\mu^* - k^* b_x - 2b_y, \\ &N = \nu^* b + \lambda^* b_x + \mu^* b_y + b_{xx} - k^* b_{xy} - b_{yy}. \end{aligned} \tag{11}$$

## 3. HOMOGENEITY ASSUMPTIONS

The remark in the preceding section shows that the problem becomes simpler if the functions are homogeneous, therefore we suppose that:

(i) The allowed region is given by (4), where

$$b = c_0 - x^m b_0(z), \quad z = \frac{y}{x}, \quad b_0 \neq 0. \tag{12}$$

(ii) The slope function $\gamma$ is homogeneous of degree zero, i.e.

$$\gamma = \gamma(z). \tag{13}$$

(iii) The function $\Theta$ is also homogeneous of degree $k$, i.e.

$$\Theta(x, y) = x^k \Theta_0(z), \quad \Theta_0 \neq 0. \tag{14}$$

Then, equation (10) becomes

$$R_1 x^k + R_2 x^{m+k} = 0. \tag{15}$$

Both $R_1$ and $R_2$ must vanish identically, resulting in a system of two ODEs of the form

$$2\Theta_0(z\gamma + 1)\ddot{\gamma} + 2\Theta_0 z\dot{\gamma}^2 + k_1\dot{\gamma} + k_0 = 0 \tag{16}$$

$$2b_0\Theta_0(z\gamma + 1)\ddot{\gamma} + 2b_0\Theta_0 z\dot{\gamma}^2 + m_1\dot{\gamma} + m_0 = 0, \tag{17}$$

where $k_1$, $m_1$ are linear in $\Theta_0$ and $\dot{\Theta}_0$, and $k_0$, $m_0$ in $\Theta_0$, $\dot{\Theta}_0$ and $\ddot{\Theta}_0$.

Our hypotheses ($b_0 \neq 0$, $\Theta_0 \neq 0$ and straight lines excluded) assure that

$$b_0\Theta_0(1 + \gamma z) \neq 0, \tag{18}$$

therefore the equations (16) and (17) are equivalent to

$$\dot{\gamma} = \frac{\Gamma_2\gamma^2 + \Gamma_1\gamma + \Gamma_0}{\Delta_1\gamma + \Delta_0}, \quad 2(1 + \gamma z)\ddot{\gamma} + 2z\dot{\gamma}^2 + K_1\dot{\gamma} + K_0 = 0, \tag{19}$$

where

$$\Gamma_2 = \Gamma_{00} + \Gamma_{01}w, \quad \Gamma_1 = \Gamma_{10} + \Gamma_{11}w, \quad \Gamma_0 = -\Gamma_2 \tag{20}$$

$$\begin{aligned} &\Gamma_{00} = (1 - k - m)r + z(\dot{r} + r^2), \quad \Gamma_{01} = 2zr - m \\ &\Gamma_{10} = m(1-2k-m) - 2(1-k-m)zr + (1-z^2)(\dot{r} + r^2) \\ &\Gamma_{11} = 2(r + mz - rz^2), \end{aligned} \tag{21}$$

$$\Delta_1 = 2(m - 2rz), \quad \Delta_0 = rz^2 - mz - 3r; \tag{22}$$

$$K_1 = K_{11}\gamma + K_{10}, \quad K_0 = K_{02}\gamma^2 + K_{01}\gamma + K_{00}, \tag{23}$$

$$K_{11} = 4zw + 2(1 - k), \quad K_{10} = -(z^2 - 3)w + kz \tag{24}$$

$$\begin{aligned} &K_{02} = (1 - k)w + z(\dot{w} + w^2) \\ &K_{01} = k(1 - k) - 2z\,(1 - k)\,w + (1 - z^2)(\dot{w} + w^2) \\ &K_{00} = -(1 - k)w - z(\dot{w} + w^2), \end{aligned} \tag{25}$$

with

$$\dot{\Theta}_0 = w\Theta_0, \quad \dot{b}_0 = rb_0. \tag{26}$$

We consider $m$, $r = \dot{b}_0/b_0$, $c_0$ (i.e. the function $b$ given by (12)) as known and we try to make compatible the two equations (19). In so doing, we prepare $\ddot{\gamma}$ from the first of equations (19), insert into the second one and obtain the quintic in $\gamma$ algebraic equation

$$\alpha_5\gamma^5 + \alpha_4\gamma^4 + \alpha_3\gamma^3 + \alpha_2\gamma^2 + \alpha_1\gamma + \alpha_0 = 0, \tag{27}$$

where the coefficients $\alpha_5, \alpha_4, ..., \alpha_0$ are functions of $z$, and of $w$ and its derivative of the first order.

We now differentiate (27) in $z$ and we obtain $\dot{\gamma}$ which we equate to $\dot{\gamma}$ given by the first of equations (19), and get

$$\beta_6\gamma^6 + \beta_5\gamma^5 + \beta_4\gamma^4 + \beta_3\gamma^3 + \beta_2\gamma^2 + \beta_1\gamma + \beta_0 = 0, \tag{28}$$

where the coefficients $\beta_6, \beta_5, ..., \beta_0$ are functions of $z$, and of $w, \dot{w}, \ddot{w}$. We are interested in the common roots of the equations (27) and (28) and this leads us to the eleventh order Sylvester determinant which is an ODE in $w$ of the second order.

We have to analyze also the case when

$$\Delta_1\gamma + \Delta_0 = 0. \tag{29}$$

If $\Gamma_2\gamma^2 + \Gamma_1\gamma + \Gamma_0 \neq 0$, the first of equations (19), hence the considered problem, has no solution. If $\Gamma_2\gamma^2 + \Gamma_1\gamma + \Gamma_0 = 0$, we express $\gamma$ from (29) and substitute it in the second equation in (19). We obtain a solution for our problem if we can find a function $w$ which gives a suitable $\Theta$.

## 4. EXAMPLE

Let us try to find families of orbits and the corresponding potentials creating them in the region

$$y \leq 1. \tag{30}$$

We can write

$$b(x, y) = 1 - y, \tag{31}$$

hence

$$m = 1,\ b_0(z) = z \text{ and } c_0 = 1. \tag{32}$$

We can now verify that, with $k = 2$, the Sylvester determinant of (27) and (28) (which for the case at hand are of degree four and five) admits of a solution $\Theta_0(z) = z^2/2$, which gives

$$\Theta(x, y) = y^2/2. \tag{33}$$

According to (5), (31) and (33), we find

$$B(x, y) = 8y^2(1 - y). \tag{34}$$

Equations (27) and (28) have the common solution $\gamma = 1/(2z)$, and from (9) we get the Hénon-Heiles type potential

$$V(x, y) = -x^2 - 4y^2 + 4y^3. \tag{35}$$

The potential (35) generates the family of curves $f(x, y) = x^2y$, traced in the region (30).

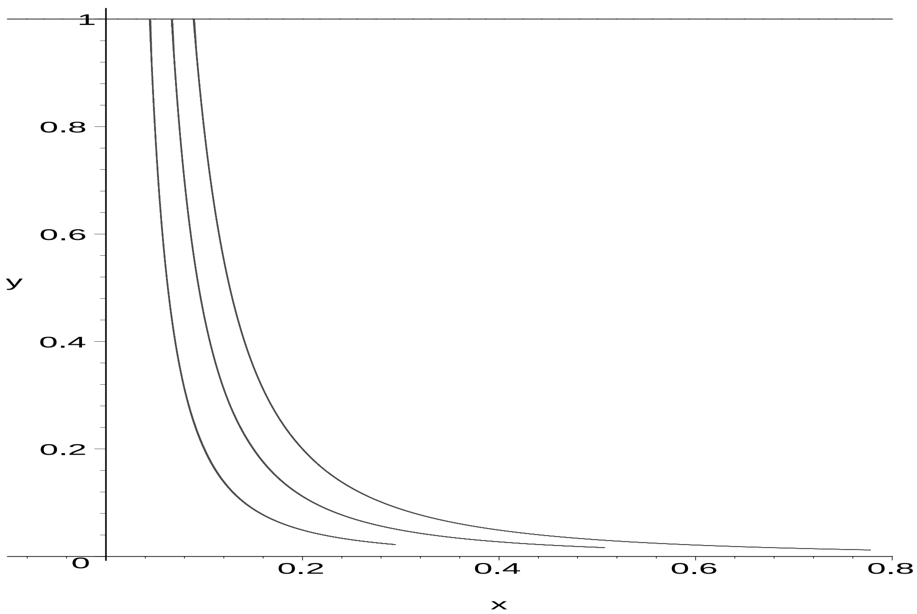


Figure 1: Curves of the family $x^2y = c$ in the region (30) for $c_1 = 0.002$, $c_2 = 0.0045$ and $c_3 = 0.008$

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